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ON DENY'S CHARACTERIZATION OF THE POTENTIAL KERNEL FOR A CONVOLUTION FELLER SEMI-GROUP (1)

by J. C. TAYLOR

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Introduction.

Let G be an abelian locally compact group and let $\check{\varkappa}$ be a positive Radon measure with the property that the kernel V defined by $Vf(x) = (f * \varkappa)(x) = \int f(xy^{-1})\varkappa(dx)$ satisfies the domination principle. In [1] Deny characterized those measures $\varkappa$ for which $V = \int_0^\infty P_t\, dt$ where (P_t) is a convolution semigroup such that $(x, t) \rightarrow P_t(x, \Phi)$ is continuous for all $\Phi \in C_c(G)$. In particular, if V satisfies the complete maximum principle, his result characterizes the convolution Feller semi-groups.

The purpose of this article is to extend Deny's result, when V is assumed to satisfy the complete maximum principle, to the case where G is replaced by a homogeneous space $E = G/K$ with G an arbitrary locally compact group and K a compact subgroup of G. Specifically, the following is proved (see theorem 3.10) :

THEOREM. — *Assume that* G *is* σ*-compact. Let* (P_t) *be a Feller semigroup on* E *that commutes with the action of* G

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on E. *Assume that for any compact set* $A \subset E$,

$$V1_A = \int_0^\infty P_t 1_A \, dt$$

is finite. Let $\varkappa$ *be the* K-*invariant measure on* E *defined by* $\langle \varkappa, \Phi \rangle = V\Phi(0)$.

Then $\varkappa$ *satisfies the following condition:*

D) *There is a base* $\mathcal{B}$ *for the neighbourhood filter of* 0 *such that for each* $B \in \mathcal{B}$ *there exists* $\sigma \in M^+(E)$ *with*

(1) $\sigma * \varkappa \leqslant \varkappa$;

(2) $\sigma * \varkappa \neq \varkappa$, $\sigma * \varkappa = \varkappa$ *on* $\complement B$; *and*

(3) $\lim\limits_{n \to \infty} \sigma * \varkappa^n = 0$.

Conversely, if $\varkappa$ *satisfies* D) *and the kernel* $Vf = f * \varkappa$ *satisfies the complete maximum principle then there is a unique convolution Feller semi-group* (P_t) *with*

$$V = \int_0^\infty P_t \, dt.$$

The condition of σ-compactness is not essential but for the sake of simplicity the detailed proofs are given under this assumption. The measure-theoretic complements needed to permit arguments to carry over in the general case are outlined in the appendix.

Let X be a locally compact space. Then $\mathcal{X}$ denotes the σ-ring generated by the compact subsets of X and $f \in \mathcal{X}^+$ if $\{f > 0\} = A \in \mathcal{X}$ and $f|A$ is measurable and non-negative relative to $\mathcal{X}|A$. The set of non-negative Radon measures is denoted by $M^+(X)$ and $C_c^+(X)$ (resp. $C_0^+(X)$) denotes the set of non-negative continuous functions with compact support (resp. vanishing at infinity).

A kernel is viewed as an operator on functions as in [2] rather than as an operator on measures as in [1].

1. The resolvent defined by a convolution kernel.

Let G be a locally compact group whose topology is σ-compact and denote by K a compact subgoup. Let E denote the locally compact quotient space G/K of right cosets and denote by π the projection of G onto E (let $\pi(t)$ be also denoted by $[t]$). Let $0 = [e]$, e the identity of G.

Denote by $\varkappa$ a positive Radon measure on E and let m be the left-invariant probability measure on K. Define the measure $\tilde{\varkappa}$ on G by setting

$$\langle \tilde{\varkappa}, f \rangle = \int \left[\int f(tx^{-1}) m\,(dx) \right] \varkappa\,(d[t]),$$

for $f \in \mathcal{G}^+$ (note that $t \to f^{\#}(t) = \int f(tx^{-1}) m\,(dx)$ is constant on each right coset since a compact group is unimodular).

Define the translation kernels T_t and S_t by the formulas $(T_t f)(x) = f(t^{-1}x)$ and $(S_t f)(x) = f(xt^{-1})$, $f \in G^+$. A Radon measure α on G is said to be K-*right-invariant* if

$$\langle \alpha, S_t f \rangle = \langle \alpha, f \rangle$$

for all $t \in K$ and $f \in \mathcal{G}^+$. The measure $\tilde{\varkappa}$ is then the unique K-right invariant measure α on G whose image $\pi(\alpha) = \varkappa$ and the map $\varkappa \to \tilde{\varkappa}$ identifies $M^+(E)$ with the set of K-right-invariant measures on G (note that $\langle \tilde{\varkappa}, f \rangle = \langle \varkappa, \overline{f} \rangle$, where $f^{\#} = \overline{f} \circ \pi$ and $(\overline{S_t f}) = \overline{f}$ if $t \in K$).

If $f \in \mathcal{E}^+$ let $\tilde{f} = f \circ \pi$. Then $g \in \mathcal{G}^+$ is of the form $g = \tilde{f}$, $f \in \mathcal{E}^+$, if and only if $S_t g = g$ for all $t \in K$. Consequently, if $g \in \mathcal{G}^+$ and $\varkappa \in M^+(E)$ the function h defined by $h(x) = (g * \tilde{\varkappa})(x) = \int g(xt^{-1}) \tilde{\varkappa}\,(dt)$ is of the form $h = \tilde{l}$, $l \in \mathcal{E}^+$. As a result, if $f \in \mathcal{E}^+$ there is a unique function $g \in \mathcal{E}^+$ with $\tilde{g} = \tilde{f} * \tilde{\varkappa}$. Define g to be $f * \varkappa$. Clearly $f \to f * \varkappa$ defines a kernel N such that $NT_t = T_t N$ for all $t \in G$ and $f \in \mathcal{E}^+$ (note that $T_t f([x]) = f([t^{-1}x])$). Such a kernel will be called a *convolution kernel*.

A measure μ on E is said to be K-*invariant* if

$$\langle \mu, f \rangle = \langle \mu, T_t f \rangle$$

for all $t \in K$ and $f \in \mathcal{E}^+$. This is equivalent to requiring that $\langle \tilde{\mu}, g \rangle = \langle \tilde{\mu}, S_t g \rangle = \langle \tilde{\mu}, T_t g \rangle$ for all $t \in K$ and $g \in \mathcal{G}^+$, i.e. $\tilde{\mu}$ is K-*bi-invariant*.

LEMMA 1.1. — *Let* N *be a convolution kernel on* E. *Then there exists a unique* K-*invariant measure* α *on* E *such* *that* $Nf = f * \alpha$ *for all* $f \in \mathcal{E}^+$. *In case* $Nf = f * \varkappa$ *the measure* $\alpha = \pi((\tilde{\beta})^\vee)$, *where* $\beta = \pi((\tilde{\varkappa})^\vee)$.

Proof. — Define $\langle \beta, f \rangle = Nf(0)$. Then, if $t \in K$,

$$\langle \beta, f \rangle = Nf(0) = (T_t Nf)(0) = N(T_t f)(0) = \langle \beta, T_t f \rangle.$$

Hence, β is K-invariant.

Clearly, $N([x], f) = \int \tilde{f}(xs)\tilde{\beta})\, ds)$ if $x \in G$ and $f \in \mathcal{E}^+$. Further, $\tilde{\beta}$ is K-biinvariant and so $\alpha = \pi((\tilde{\beta})^\vee)$ is K-invariant. Hence, $\tilde{\alpha} = (\tilde{\beta})^\vee$ and so

$$N([x], f) = (\tilde{f} * \tilde{\alpha})(x) = (f * \alpha)[x].$$

The uniqueness of α is clear as is the fact that $N = * \varkappa$ implies $\beta = \pi((\tilde{\varkappa})^\vee)$.

Let $\varkappa \in M^+(E)$ be such that the kernel V defined by $Vf = f * \varkappa$ satisfies the complete maximum principle (note that $\varkappa$ is not assumed to be K-invariant). Since $\varkappa$ is Radon, V is proper and so, as remarked in [3], it is reasonable to define $u \in \mathcal{E}^+$ as *excessive* if $u = \sup_n Vf_n$ with $(f_n) \subset \mathcal{E}^+$ and (Vf_n) increasing. Also, $u \in \mathcal{E}^+$ is said to be *supermedian* if, for all f and $g \in \mathcal{E}^+$, $u + Vf \geqslant Vg$ on $\{g > 0\}$ implies $u + Vf \geqslant Vg$.

If $\alpha, \beta \in M^+(G)$ and β is K-right invariant then an easy calculation shows that $\alpha * \beta$ is also K-right-invariant. Hence, if $\mu, \nu \in M^+(E)$ the Radon measure $\tilde{\mu} * \tilde{\nu}$ (when defined) equals $\tilde{\eta}$ where $\pi(\tilde{\mu} * \tilde{\nu}) = \eta \in M^+(E)$. The measure η is defined to be $\mu * \nu$.

Remark. — If N is a convolution kernel on E and

$$\mu \in M^+(E)$$

then $\mu N = \mu * \beta$ where $\beta = \pi((\tilde{\alpha})^\vee)$ if $Nf = f * \alpha$. In the case of a group the convolution kernels are associated with β rather than α so that the formula $\langle \mu N, f \rangle = \langle \mu, Nf \rangle$ holds.

Assume that the following condition is satisfied by $\varkappa$:

(D_1) there is a compact neighbourhood B of 0 and $\sigma \in M^+(E)$ such that

(1) $\sigma * \varkappa \leqslant \varkappa$;

(2) $\sigma * \varkappa = \varkappa$ on $\complement B$; and

(3) $\sigma^n * \varkappa$ tends to zero weakly (where σ^n is the n-fold convolution of σ with itself).

PROPOSITION 1.2. — *Let* $\Phi \in C_c^+(E)$, $x_0 \in E$ *and* $\varepsilon > 0$. *Then there exists an excessive function* s *and a compact set* $K \subset E$ *with*

(1) $s(x_0) < \varepsilon$; *and*

(2) $s \geqslant V\Phi$ *on* $\complement K$.

In other words, $V\Phi$ *vanishes at the natural boundary of* E *in the sence of* [3].

Proof. — If $\psi \in C_c^+(G)$ then there exists $\Phi \in C_c^+(E)$ with $\psi \leqslant \tilde{\Phi}$. Hence, in view of D_1) (3) it suffices to prove that, for each $n \geqslant 0$, for all $\Phi \in C_c^+(E)$ and for all $\varepsilon > 0$, there exists an excessive function $v = v(n, \Phi, \varepsilon)$ and a compact set $L_n = L_n(v, \Phi, \varepsilon)$ with (a) $\Phi * (\sigma^n * \varkappa) + v \geqslant \Phi * \varkappa$ on $\complement L_n$ and (b) $v(x_0) < \varepsilon$. Let $P(n)$ denote this statement.

First, let $n = 1$. From D_1) (2) it follows that if $\Phi \in C_c^+(E)$ then $\Phi * (\sigma * \varkappa) = \Phi * \varkappa$ on $\complement D$, $D = \pi(\tilde{A}\tilde{B})$, where

$$\tilde{A} = \pi^{-1}(\operatorname{supp} \Phi)$$

and $\tilde{B} = \pi^{-1}(B)$. Since D is compact, $P(1)$ is established with $v = 0$.

Assume $P(n)$. Let $\sigma = \sigma' + \tau$ where σ' has compact support and $(\Phi * (\tau * \varkappa))(x_0) < \varepsilon/2$. Then,

$$\Phi * (\sigma^{n+1} * \varkappa) \geqslant (\Phi * \sigma') * (\sigma^n * \varkappa)$$

and $\Phi * \sigma' \in C_c^+(E)$. If $w = v(n, \Phi * \sigma', \varepsilon/2)$ then

$$\Phi * (\sigma^{n+1} * \varkappa) + w \geqslant (\Phi * \sigma') * \varkappa$$

on $\complement L_n(v, \Phi * \sigma', \varepsilon/2) = \complement L_n$. Hence, if

$$v = w + \Phi * (\tau * \varkappa)$$

it follows that $v + \Phi * (\sigma^{n+1} * \varkappa) \geqslant \Phi * (\sigma * \varkappa)$ on $\complement L_n$ and $v(x_0) < \varepsilon$.

In view of $P(1)$ this establishes $P(n + 1)$.

Lemma 1.3. — *Let* V *and* T *be proper kernels on a measurable space* $(E, \mathscr{E})$ *such that* $VT = TV$. *If* $V = \lim_{\lambda \downarrow 0} V_\lambda$, *where* (V_λ) *is a sub-Markovian resolvent of kernels* V_λ, *then* $TV_\lambda = V_\lambda T$ *for all* $\lambda > 0$, *providing* $T1 < \infty$.

Proof. — Let $f \in \mathscr{E}^+$ be such that f, Vf, Tf and VTf are all finite. Now $V_\lambda f$ is the unique function h such that $(I + \lambda V)h = Vf$. Hence,

$$VTf = TVf = T(I + \lambda V)h = (I + \lambda V)Th$$

implies that $V_\lambda(Tf) = T(V_\lambda f)$. Since each $f \in \mathscr{E}^+$ is of the form $f = \sum_n f_n$, where each f_n satisfies the above hypotheses, the result follows.

Theorem 1.4. — *Let* V *be the kernel defined by* $Vf = f * \varkappa$, $\varkappa \in M^+(E)$. *Assume that* V *satisfies the complete maximum principle. If* $\varkappa$ *satisfies* D_1) *then there is a unique family* $(\varkappa_\lambda)$ *of* K-*invariant measures* $\varkappa_\lambda$ *such that the kernels*

$$V_\lambda f = f * \varkappa_\lambda$$

form a sub-Markovian resolvent (V_λ) *of kernels* V_λ *on* E *with* $V = \lim_{\lambda \downarrow 0} V_\lambda$.

Further, if $\tilde{V}$ *is the kernel defined by* $\tilde{V} g = g * \tilde{\varkappa}$ (*where* $\varkappa$ *also denotes the* K-*invariant measure for which* $Vf = f * \varkappa$), *the kernels* $\tilde{V}_\lambda$ *defined by* $\tilde{V}_\lambda g = g * \tilde{\varkappa}_\lambda$ *form the unique sub-Markovian resolvent* $(\tilde{V}_\lambda)$ *on* G *with* $\tilde{V} = \lim_{\lambda \downarrow 0} \tilde{V}$.

Proof. — From Proposition 1.1 and Theorem 2 in [3] it follows that there is a unique sub-Markovian resolvent (V_λ) with $V = \lim_{\lambda \downarrow 0} V_\lambda$. From Lemma 1.3 it follows that each V_λ is a convolution kernel. For all $\lambda \geqslant 0$, let $\varkappa_\lambda$ be the unique K-invariant measure on E such that $V_\lambda f = f * \varkappa_\lambda$, $f \in \mathscr{E}^+$.

The resolvent equation, $0 \geqslant \lambda \geqslant \mu$,

$$\varkappa_\lambda = \varkappa_\mu + (\mu - \lambda)\varkappa_\lambda * \varkappa_\mu = \varkappa_\mu + (\mu - \lambda)\varkappa_\mu * \varkappa_\lambda$$

holds when each measure η is replaced by $\tilde{\eta}$. Define

$$\tilde{V}_\lambda g = g * \tilde{\varkappa}_\lambda, \qquad g \in \mathscr{G}^+.$$

Then $(\tilde{V}_\lambda)$ is a sub-Markovian resolvent and $f \in \mathcal{E}^+$ implies $\tilde{V}_\lambda \tilde{f} = V_\lambda f)^\sim$. Also, $\tilde{V}g = g * \tilde{\varkappa} \geqslant \tilde{V}_\lambda g = g * \tilde{\varkappa}_\lambda$ for all $g \in \mathcal{G}^+$ and since $V = \lim_{\lambda \downarrow 0} V_\lambda$, $\tilde{V} = \lim_{\lambda \downarrow 0} \tilde{V}_\lambda$ (note that if $\psi \in C_c^+(G)$ there exists $\Phi \in C^+(E)$ with $\tilde{\Phi} \geqslant \psi$).

Remark. — Since $\varkappa$ is K-invariant it can be directly verified that $\tilde{V}$ satisfies the complete maximum principle (note that $\tilde{V}f = \tilde{V}f^\#$, for all $f \in \mathcal{G}^+$).

2. The existence of a Feller semigroup.

The measure $\varkappa$ on E will be assumed to satisfy the following condition:

D_2) there is a base $\mathcal{B}$ of compact neighbourhoods of 0 such that for each $B \in \mathcal{B}$ there exists $\sigma \in M^+(E)$ with

(1) $\sigma * \varkappa \leqslant \varkappa$;

(2) $\sigma * \varkappa \neq \varkappa$; and

(3) $\sigma * \varkappa = \varkappa$ on $\complement B$.

Remark. — If, in addition, one requires in D_2) that each $\sigma^n * \varkappa$ converge weakly to zero as $n \to \infty$ and that each σ is carried by $\complement \overline{B}$ then there is a family associated with $\varkappa$ in the sense of Deny [1].

Since the resolvent (V_λ) maps $C_0(E)$ into itself the Hille-Yosida theorem can be applied if $D = \overline{V_\lambda(C_0(E))} = C_0(E)$. This fact is established by the following sequence of lemmas and propositions.

LEMMA 2.1. — *Assume* $\alpha \leqslant \beta$. *Then* $\alpha = \beta$ *if*

$$(\Phi * \alpha)(0) = (\Phi * \beta)(0)$$

for all $\Phi \in C_c^+(E)$.

Proof. — $(\Phi * \alpha)(0) = (\Phi * \beta)(0)$ for all $\Phi \in C_c^+(E)$ implies that $\tilde{\alpha}(\tilde{A}^{-1}) = \tilde{\beta}(\tilde{A})^{-1})$ for every compact set $A \subset E$.

If $B \subset G$ is compact then $B^{-1} \subset \tilde{A}$ where $A = \pi(B^{-1})$ is compact. Hence, $B \subset \tilde{A}^{-1}$. Since $\tilde{\alpha} \leqslant \tilde{\beta}$ if follows that

$\tilde{\alpha}(B) = \tilde{\beta}(B)$ for all compact sets $B \subset G$. Consequently, $\alpha = \beta$.

LEMMA 2.2. — *If* $\sigma * \varkappa \leqslant \varkappa$ *then* $V(\Phi * \sigma) = \Phi * (\sigma * \varkappa)$ *is continuous and excessive whenever* $\Phi \in C_c^+(E)$.

Proof. — Let $\varepsilon > 0$, $x_0 \in E$ and $\Phi \in C_c^+(E)$. Let O be a compact neighbourhood of e such that $t \in O$ implies $\|T_t\Phi - \Phi\| < \varepsilon$. If $\pi(t_0) = x_0$ then $\pi(Ot_0)$ is a neighbourhood U of x_0.

Let $\psi \in C_c^+(G)$ be such that

$$\{\psi = 1\} \supset \bigcup_{t \in O} \{T_t\tilde{\Phi} \neq \tilde{\Phi}\}.$$

Then, if $x \in U$, where $x = [tt_0]$ with $t \in O$,

$$|V(\Phi * \sigma)(x) - V(\Phi * \sigma)(x_0)| \leqslant \int |\tilde{\Phi}((tt_0s^{-1}) - \tilde{\Phi}(t_0s^{-1})|(\tilde{\sigma} * \tilde{\varkappa})(ds) \leqslant \varepsilon \int \psi(t_0s^{-1})(\tilde{\sigma} * \tilde{\varkappa})(ds).$$

Since there exists $\theta \in C_c^+(E)$ with $\tilde{\theta}(s) \geqslant \psi(t_0s^{-1})$, for all $s \in G$, the last integral is finite.

PROPOSITION 2.3. — *Let* U *be a neighbourhood of* 0. *Then there exists* $\psi \in C_c^+(E)$ *such that:*

(1) $\psi = u - v$, u *and* v *both continuous excessive functions*;

(2) $0 \neq \psi(0) = \|\psi\|$; *and*

(3) $\operatorname{supp} \psi \subset U$.

Proof. — There exists a compact neighbourhood D of 0 such that $\tilde{D}^{-1}\tilde{D} \subset \tilde{U}$. Further, there exist compact neighbourhoods A and B of 0 with $A = \operatorname{supp} \psi$, $\psi \in C_c^+(E)$, $B \in \mathscr{B}$ and $\tilde{A}\tilde{B} \subset \tilde{D}$.

Let σ be a measure satisfying the conditions in D_2) relative to B. Then, if

$$X = \operatorname{supp}(\varkappa - \sigma * \varkappa), \qquad \Phi * \varkappa - \Phi * (\sigma * \varkappa) \in C_c^+(E)$$

(its support lies in $\pi(\tilde{A}\tilde{B})$) and attains its maximum at a point

$$x_0 \in \pi(\{\tilde{\Phi} > 0\}\tilde{X}) \subset \pi(\tilde{A}\tilde{B}) \subset D.$$

Choose $s_0 \in \{\tilde{\Phi} > 0\}\tilde{X}$ with $\pi(s_0) = x_0$ and let $\theta = T_{s_0^{-1}}\Phi$. Then $\psi = \theta * \varkappa - \theta * (\sigma * \varkappa)$ is a function that satisfies (1), (2) and (3) above.

Corollary 2.4. — *The functions* $V_\lambda\Phi, \lambda > 0$ *and* $\Phi \in C_c^+(E)$ *separate the points of* E.

Proof. — If u is lower semicontinuous and excessive then $u = \sup \{\lambda V_\lambda \Phi | \lambda > 0$ and $\Phi \in C_c^+(E)$ with $\Phi \leqslant u\}$. Hence, the functions $V_\lambda\Phi$ separate 0 from any other point $x \in E$. Since $V_\lambda T_s = T_s V_\lambda$, for all $s \in G$, the result follows.

. *Remark.* — As pointed out by Faraut and Harzallah, given Corollary 2.4. the theory of Ray semigroups can be applied (in the metrisable case) to give a proof of the fact that (V_λ) is the resolvent of a Feller semigroup. For example, Corollary 2.4 implies that the hypotheses of Theorem 1.7 in [4] are verified. Hence, (V_λ) is the resolvent of a semigroup (P_t) of kernels P_t. The set D of non-branching points is non-void (corollary 2.6 in [4]) and since one can show that, for all $s \in G$ and $t > 0$, $T_s P_t = P_t T_s$, $D = E$. From this it follows, since $C_0(E)$ is invariant under (P_t), that (P_t) is a Feller semigroup.

A direct proof of this fact (which does not use metrizability or σ-compactness) continues with the following result.

Corollary 2.5. — *If* U *is an open Baire neighbourhood of* 0 *then* $\lim_{\lambda \to \infty} \lambda V_\lambda(0, U) = 1$.

Proof. — Let $\psi \in C_c^+(E)$ satisfy conditions (1), (2) and (3) of Proposition 2.3. Then, since $\lim_{\lambda\to\infty} \lambda V_\lambda(0, \psi) = \psi(0)$ the result follows as $\lambda V_\lambda(0, \psi) \leqslant \lambda V_\lambda(0, U)\psi(0)$.

Corollary 2.6. — *Let* u *and* v *be two lower semicontinuous excessive functions. Then* $w = u \wedge v$ *is also excessive.*

Proof. — If $x_0 \in E$ and $\varepsilon > 0$ let $U = \{w > w(x_0) - \varepsilon\}$. Then, U is open and $\lim_{\lambda\to\infty} \lambda V_\lambda(x_0, U) = 1$. Hence,

$$\hat{w}(x_0) \geqslant w(x_0) - \varepsilon.$$

Proposition 2.7. — *Let* $A \subset E$ *be compact. Then there is a compact neighbourhood* O *of* A *and* $\lambda_0 > 0$ *such that, for* $\varepsilon > 0$,

$$\lambda V_\lambda(x, A) < \varepsilon \qquad \text{if} \qquad x \notin O \qquad \text{and} \qquad \lambda \geqslant \lambda_0.$$

Proof. — Let $\varepsilon > 0$ and let U be a compact neighbourhood of 0. Let $\lambda_0 > 0$ be such that

$$1 - \varepsilon < \lambda\, V_\lambda\,(0, U) = \lambda(1_U * \varkappa_\lambda)(0) \qquad \text{for} \qquad \lambda \geqslant \lambda_0.$$

Let $O = \pi(\tilde{A}\tilde{U})$.

Denote by β any one of the measures $\lambda\varkappa_\lambda$, $\lambda \geqslant \lambda_0$. Then, if $x = \pi(t)$

$$\begin{aligned}(1_A * \beta)(x) &= \int 1_{\tilde{A}}(ts^{-1})\tilde{\beta}\,(ds)\\ &= \int 1_{\tilde{A}}(ts^{-1})1_{\tilde{U}}(s)\tilde{\beta}\,(ds) + \int 1_{\tilde{A}}(ts^{-1})1_{\complement\tilde{U}}(s)\tilde{\beta}\,(ds)\\ &\leqslant \int 1_{\complement\tilde{U}}(s)\tilde{\beta}\,(ds) < \varepsilon, \quad \text{if} \quad t \notin \tilde{A}\tilde{U}.\end{aligned}$$

Corollary 2.8. — *Let* u, v, *be two continuous excessive functions on* E *with* $u - v \in C_c^+(E)$. *Then,*

$$\lim_{\lambda \to \infty} \|\lambda V_\lambda(u - v) - (u - v)\| = 0.$$

Proof. — Let $A = \operatorname{supp}(u - v)$ and let $\varepsilon > 0$. Denote by O a compact neighbourhood of A such that

$$\lambda V_\lambda(x, A) < \varepsilon \qquad \text{if} \qquad x \notin O \qquad \text{and} \qquad \lambda \geqslant \lambda_0.$$

Then $|\lambda V_\lambda(x,\ u - v)| \leqslant \varepsilon\|u - v\|$ if $x \notin O$. Since $\lambda V_\lambda u$ $\lambda V_\lambda v$ are lower semicontinuous, $\lambda V_\lambda(u - v)$ converges uniformly to $u - v$ on O. The result follows.

The above results imply that $\overline{V_\lambda(C_0(E))} = C_0(E)$ and hence the following result.

Theorem 2.9. — *Let* G *be a locally compact group (that is* σ*-compact) and let* $K \subset G$ *be a compact subgroup. Let* $V = *\varkappa$ *be a convolution kernel on the homogeneous space* $E = G/K$, $\varkappa \in M^+(E)$. *Assume that* V *satisfies the complete maximum principle.*

If $\varkappa$ *satisfies* D_1) *and* D_2) *then there is a unique Feller semigroup* (P_t) *on* E *with* $V = \int_0^{+\infty} P_t\, dt$.

Proof. — Let u_i, v_i for $i = 1, 2$ be continuous excessive functions such that $\psi_i = u_i - v_i \in C_c^+(E)$. Then

$$\psi_1 \wedge \psi_2 = (u_1 + v_2) \wedge (u_2 + v_1) - (v_1 + v_2)$$

is of the same form. Hence, the vector space generated by functions $\psi \in C_c^+(E)$, which are differences of continuous excessive functions, is dense in $C_0(E)$.

Corollary 2.8 implies that $D = \overline{V_\lambda(C_0(E))} = C_0(E)$. The result then follows from the Hille-Yosida theorem (c.f. [2]).

As an immediate corollary one has the following restricted version of a result of Deny [1].

COROLLARY 2.10. — *Let* G *be a locally compact abelian group (that is* σ*-compact) and let* $V = *\varkappa$ *be a convolution kernel on* G *that satisfies the complete maximum principle.*

Then, V *is the potential kernel of a Feller semigroup if the following condition is verified:*

D) *for a base* $\mathscr{B}$ *of compact neighbourhoods of the identity* e *of* G *there is, for each* $B \in \mathscr{B}$, *a measure* $\sigma \in M^+(E)$ *with*

(1) $\sigma * \varkappa \leqslant \varkappa$ *and* $\sigma * \varkappa \neq \varkappa$;

(2) $\sigma * \varkappa = \varkappa$ *on* $\complement B$; *and*

(3) $\lim\limits_{n\to\infty} (\sigma^n) * \varkappa = 0$ *(weakly).*

Remarks. — Deny's result is more general. He not only did not require G to be σ-compact (a hypothesis that can be removed from all the above results as indicated in the appendix) but also did not assume that the kernel $*\varkappa$ satisfied the complete maximum principle. Further, while in the commutative case it is immaterial whether one writes $\sigma * \varkappa$, or $\varkappa * \sigma$ it seems to be necessary in general to have $\sigma * \varkappa \leqslant \varkappa$ if the kernel V commutes with the left action of G on E.

3. The characterization of convolution Feller semi-groups.

Let (P_t) be a Feller semigroup on E that commutes with the action of G on E, i.e., if $s \in G$ and $t > 0$ then

$$T_sP_t = P_tT_s.$$

Further, assume that if $A \subset E$ is compact,

$$V1_A = \int_0^\infty P_t 1_A \, dt$$

is finite.

Denote by $\check{\varkappa}$ the unique K-invariant measure on E defined by $\langle \check{\varkappa}, \Phi \rangle = V\Phi(0)$. Then $Vf = f * \varkappa$ and $\mu V = \mu * \check{\varkappa}$ (note that $(\check{\varkappa})^{\sim}$ is K-biinvariant and so $((\check{\varkappa})^{\sim})^{\vee}$, being K-right invariant, is of the form $\tilde{\varkappa}$ for a unique $\varkappa \in M^+(E)$). It will be shown first that $\check{\varkappa}$ satisfies conditions D_1) and D_2).

Note that $\mu \to \mu P_t$, $\mu \in M_c^+(E)$, defines a continuous Hunt semigroup in the terminology of Deny [1]. Hence, all the results of paragraphs 3 and 4 in [1] hold.

To begin with it is proved that 1 is an excessive function.

LEMMA 3.1. — $\lim_{t \to 0} P_t 1 = 1$.

Proof. — Obviously, it suffices to show that $\lim_{t \to 0} P_t(0, 1) = 1$. Choose $\Phi \in C_c^+(E)$ with $\Phi(0) = 1$ and $\Phi \leqslant 1$. Then $1 = \lim_{t \to 0} P_t(0, \Phi) \leqslant \limsup_{t \to 0} P_t(0, 1) \leqslant 1$.

COROLLARY 3.2. — *Let* $\sigma \in M^+(E)$ *be such that* $\sigma * \check{\varkappa} \leqslant \check{\varkappa}$. *Then* $\langle \sigma, 1 \rangle \leqslant 1$.

Proof. — Since by Lemma 3.1 1 is excessive there exists $(f_n) \subset E$ with $(f_n * \varkappa)$ increasing to 1. Hence,

$$\langle \sigma, 1 \rangle = \lim_n \langle \sigma, f_n * \varkappa \rangle = \lim_n \langle \sigma * \check{\varkappa}, f_n \rangle$$
$$\leqslant \lim_n \langle \check{\varkappa}, f_n \rangle = \lim_n f_n * \varkappa(0) = 1.$$

LEMMA 3.3. — *Let* (α_i) *and* $(\beta_j) \subset M^+(E)$ *be two nets that converge weakly to* α *and* β *respectively. Assume*

$$\langle \alpha_i, 1 \rangle \leqslant 1 \quad \textit{and} \quad \langle \beta_j, 1 \rangle \leqslant 1$$

for all i *and* j. *In addition assume that each* β_j *is K-invariant. Then,*

$$\alpha * \beta = \lim_i \lim_j \alpha_i * \beta_j = \lim_j \lim_i \alpha_i * \beta_j.$$

Proof. — Let $\Phi \in C_c^+(E)$. Then $\langle \alpha_i * \beta_j, \Phi \rangle = \langle \alpha_i, \Phi * \check{\beta}_j \rangle$ implies $\lim_i \alpha_i * \beta_j = \alpha * \beta_j$. Further, since $(\tilde{\alpha}_i)^{\vee} * \tilde{\Phi} = \tilde{\Psi}$, with $\psi \in C_0(E)$, it follows from $\langle \alpha_i * \beta_j, \Phi \rangle = \langle \tilde{\beta}_j, \tilde{\psi} \rangle$ that $\lim_j \alpha_i * \beta_j = \alpha_i * \beta$. Applying both these arguments to $\alpha_i * \beta$ and $\alpha * \beta_j$ respectively gives the result.

Corollary 3.4. — *If* β *is* K-*invariant and* $\langle \beta, 1 \rangle \leqslant 1$ *then* $\lim_i \alpha_i * \beta = \alpha * \beta$. *If* $< \beta, 1 \rangle \leqslant 1$ *and each* α_i *is* K-*invariant then* $\lim_i \beta * \alpha_i = \beta * \alpha$.

Proof. — Let $\beta_j = \beta$ for all j.

Corollary 3.5. — *Let* μ *be a weak accumulation point of* $\{\sigma^n | n \in \mathbf{N}\}$, *where* $\sigma * \check{\varkappa} \leqslant \check{\varkappa}$ *and* σ *is* K-*invariant. Then* $\mu * \sigma = \sigma * \mu$.

Proof. — Let $\sigma^{n_i} = \alpha_i$ be a net converging to μ. Then

$$\mu * \sigma = \lim_i \alpha_i * \sigma = \lim_i \sigma * \alpha_i = \sigma * \mu.$$

A Radon measure ξ is said to be *excessive* if it is $\geqslant 0$ and $\xi * \lambda \varkappa_\lambda \leqslant \xi$ for all $\lambda > 0$. It is said to be a *potential* if $\xi = \gamma * \check{\varkappa}$ for some $\gamma \in M^+(E)$.

Proposition 3.6. — *Let* (ξ_i) *be a net of potentials*

$$\xi_i = \gamma_i * \check{\varkappa}$$

each dominated by a potential $\beta * \check{\varkappa}$ *with* $\langle \beta, 1 \rangle < \infty$. *Assume that* ξ *is the weak limit of* (ξ_i).

Then ξ *is a potential* $\gamma * \check{\varkappa}$ *and* $\gamma = \lim_i \gamma_i$ *if* $\langle \gamma_i, 1 \rangle \leqslant 1$ *for all* n.

Proof (cf. the proofs of Theorem 6.1 and Lemma 7.1 in [1]). — The measure ξ is excessive and since $\xi \leqslant \beta * \check{\varkappa}$ its invariant part is zero (see [1]). Let $\mu_\lambda = \lambda \xi * (\delta - \lambda \check{\varkappa}_\lambda)$.

Then,

$$\begin{aligned}\langle \mu_\lambda, 1 \rangle &\leqslant \lambda \langle \beta * \check{\varkappa} * (\delta - \lambda_\lambda \check{\varkappa}), 1 \rangle \\ &= \langle \beta * \lambda \check{\varkappa}_\lambda, 1 \rangle \leqslant \langle \beta, 1 \rangle < \infty.\end{aligned}$$

Hence, by Lemma 3.3, if γ is a weak accumulation point of $\{\mu_n | n > 0\}$ and equals $\lim_j \mu_{n_j}$, where $j \to \mu_{n_j}$ is a net, then $\lim_j \mu_{n_j} * \check{\varkappa}_\lambda = \gamma * \check{\varkappa}_\lambda$.

Deny's argument in [1] is now used to show $\xi = \gamma * \check{\varkappa}$ (see proof of his Theorem 6.1). Specifically, since for any $\lambda > 0$ $\lim_j \mu_\lambda * \check{\varkappa}_{n_j} = 0$ (the net $j \to n_j$ is unbounded) it follows that

$$\mu_\lambda * \check{\varkappa} = \lim_j \mu_\lambda * (\check{\varkappa} - \check{\varkappa}_{n_j}) = \lim_j \mu_{n_j} * (\check{\varkappa} - \check{\varkappa}_\lambda) = \xi - \gamma * \check{\varkappa}_\lambda,$$

since $\lim_{\lambda \to \infty} \lambda(\xi * \check{\varkappa}_\lambda) = \xi$ follows from the fact that for all $\Phi \in C_c(E)$ $\lim_{\lambda \to \infty} \lambda(\Phi * \varkappa_\lambda) = \Phi$.

Following Deny, let $\lambda \to 0$ in this identity. Since

$$\mu_\lambda * \check{\varkappa} = \xi * \lambda \check{\varkappa}_\lambda$$

implies $\lim_{\lambda \to 0} \mu_\lambda * \check{\varkappa} = 0$ (the invariant part of ξ is zero) it follows that $\xi = \gamma * \check{\varkappa}$.

It remains to show that $\gamma = \lim_i \gamma_i$. Since

$$\xi_i * \lambda \check{\varkappa}_\lambda = \xi_i - \gamma_i * \check{\varkappa}_\lambda,$$

by lemma 3.3, $\lim_i \gamma_i * \check{\varkappa}_\lambda$ exists and equals

$$\xi - \xi * \lambda \check{\varkappa}_\lambda = \gamma * \check{\varkappa}_\lambda.$$

Let $j \to \gamma_{n_j}$ be a net converging to α. Then

$$\alpha * \check{\varkappa}_\lambda = \lim_j \gamma_{n_j} * \check{\varkappa}_\lambda = \gamma * \check{\varkappa}_\lambda.$$

Hence, as $\overline{V_\lambda(C_c(E))} = C_0(E)$, $\alpha = \gamma$ and so (γ_i) converges weakly to γ.

Corollary 3.7. — *If* $U \subset E$ *is open and* $\beta \in M_b^+(E)$ *there exists a measure* $\beta' \in M^+(E)$ *with* (1) $\beta' * \check{\varkappa} \leqslant \beta * \check{\varkappa}$; (2) β' *carried by* $\overline{U}$ *and* (3) $\beta' * \check{\varkappa} = \beta * \check{\varkappa}$ *on* U.

Proof. — The argument used by Deny to prove Lemma 7.2 in [1] applies without change once it is noted that

$$\mu * \check{\varkappa} \leqslant \beta * \check{\varkappa} \quad \text{and} \quad \langle \beta, 1 \rangle = b$$

implies $\langle \mu, 1 \rangle \leqslant b$ (see the proof of Corollary 3.2).

COROLLARY 3.8. — *Assume* $\sigma * \check{\kappa} \leqslant \check{\kappa}$. *The excessive measure* $\xi = \lim_{n\to\infty} \sigma^n * \check{\kappa}$ *is a potential* $\mu * \check{\kappa}$ *and* $\mu = \lim_n \sigma^n$.

Proof. — Let $\xi_n = \sigma^n * \check{\kappa}$.

From these results one can quickly deduce the following key fact.

PROPOSITION 3.9. — *Let* $\sigma \in M^+(E)$ *be such that* $\sigma * \check{\kappa} \leqslant \check{\kappa}$ *and* $\sigma * \check{\kappa} \neq \check{\kappa}$. *Then,* $\lim_{n\to\infty} \sigma^n * \check{\kappa} = 0$.

Proof (cf. the proof of Theorem 7.1 in [1]). — Let

$$\xi = \lim_{n\to\infty} \sigma^n * \check{\kappa}.$$

Then $\sigma * \xi = \xi$ and $\xi = \mu * \check{\kappa}$ where $\mu = \lim_n \sigma^n$ (see Proposition 3.6). Hence,

$$\mu * \xi = \lim_{n\to\infty} \mu * \sigma^n * \check{\kappa} = \lim_{n\to\infty} \sigma^n * \mu * \check{\kappa} = \lim_{n\to\infty} \sigma^n * \xi = \xi$$

(note that the first equality holds by monotonicity).

Since $\sigma * \check{\kappa} \neq \check{\kappa}$ the positive measure $\check{\kappa} - \xi$ is not zero. Hence, $\mu * (\check{\kappa} - \xi) = 0$ implies $\mu = 0$ and so $\xi = 0$.

Deny's Proposition 3.3 in [1] states that if $\mu, \nu \in M^+(E)$ are such that $\mu * \check{\kappa}, \nu * \check{\kappa} \in M^+(E)$ and $\mu * \check{\kappa} = \nu * \check{\kappa}$ then $\mu = \nu$. Hence, Corollary 3.7 (applied to $\beta = \delta$) and Proposition 3.9 imply that $\eta = \check{\kappa}$ satisfies the following condition:

D) for a base $\mathcal{B}$ of compact neighbourhoods B of 0 there is, for each $B \in \mathcal{B}$, a measure $\sigma \in M^+(E)$ with

(1) $\sigma * \eta \leqslant \eta$ and $\sigma * \eta \neq \eta$;

(2) $\sigma * \eta = \eta$ on $\complement B$;

(3) $\lim_{n\to\infty} (\sigma^n) * \eta = 0$ (weakly).

One can now state and prove the following characterization of Feller semigroups on E whose potential kernel is proper and which commute with the action of G on E.

THEOREM 3.10. — *Let* G *be a locally compact group (that is* σ*-compact) and let* E *be the homogeneous space* G/K *of right cosets of* K, *a compact subgroup of* G. *Denote by* $\varkappa$ *a positive* K-*invariant Radon measure on* E.

The following conditions are equivalent:

(1) *there is a family* $(\alpha_t)t > 0$ *of* K-*invariant Radon measures* α_t *on* E *such that* $\varkappa = \int_0^\infty \alpha_t dt$ *and* $(*\,\alpha_t)_{t>0}$ *is a Feller semigroup*;

(2) *the kernel* $*\,\varkappa$ *satisfies the complete maximum principle and* $\varkappa$ *satisfies* D);

($2^\vee$) *the kernel* $*\,\check{\varkappa}$ *satisfies the complete maximum principle and* $\check{\varkappa}$ *satisfies* D).

Further, if D') *denotes the condition obtained from* D) *by reversing all the convolutions then* (1) *implies:*

(3) *the kernel* $*\,\varkappa$ *satisfies the complete maximum principle and* $\varkappa$ *satisfies* D'); *and*

($3^\vee$) *the analogue of* ($2^\vee$) *with* D) *replaced by* D').

Proof. — Theorem 2.9 states that (2) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2). As noted above the measure $\check{\varkappa}$ satisfies D). Further, if $\varkappa_\lambda = \int_0^\infty e^{-\lambda t}\alpha_t\, dt$, the family $(*\,\check{\varkappa}_\lambda)$ of convolution kernels is a sub-Markovian resolvent family. Lemma 3.11 shows that $*\,\check{\varkappa} = \lim_{\lambda\downarrow 0} *\,\check{\varkappa}_\lambda$ and so $*\,\check{\varkappa}$ satisfies the complete maximum principle. Hence, from Theorem 2.9 and the above remark $\varkappa = (\check{\varkappa})^\vee$ satisfies D).

The statement (1) is equivalent to the statement obtained by replacing each measure η by $\check{\eta}$. Hence, (1) $\Longleftrightarrow$ ($2^\vee$).

Lemma 3.11. — *Assume* $(*\,\varkappa_\lambda)$ *is a sub-Markovian resolvent family of convolution kernels* $V_\lambda = *\,\varkappa_\lambda$ *with each* $\varkappa_\lambda$ *a* K-*invariant measure on* E *and* $\lim_{\lambda\to 0} V_\lambda = *\,\varkappa$. *Then,*

$$*\,\varkappa = \lim_{\lambda\downarrow 0} *\,\varkappa_\lambda \Longleftrightarrow \varkappa = \lim_{\lambda\downarrow 0} \varkappa_\lambda .$$

Proof. — Since $\langle\beta,\, g\rangle = \langle\tilde{\beta},\, \tilde{g}\rangle$, it suffices to show that $*\,\varkappa = \lim_{\lambda\downarrow 0} *\,\varkappa_\lambda$ if for all $g \in \mathscr{G}^+$, $\lim_{\lambda\downarrow 0} \langle\tilde{\varkappa}_\lambda,\, g\rangle = \langle\tilde{\varkappa},\, g\rangle$.

One implication is obvious. Now assume that, for all $f \in \mathscr{E}^+$, $\lim_{\lambda\downarrow 0} f * \varkappa_\lambda = f * \varkappa$. Let $g_1 \in \mathscr{G}^+$ be bounded and vanish outside a compact set. Then there exists $\Phi \in C^+(E)$ with $(\tilde{\Phi})^\vee \geqslant \tilde{g}_1$. Since $\Phi * \varkappa_\lambda(0) = \langle \tilde{\varkappa}_\lambda, (\tilde{\Phi})^\vee \rangle$ and $\tilde{\varkappa}_\lambda \leqslant \tilde{\varkappa}$, for all $\lambda > 0$ if follows that $\lim\limits_{\lambda \downarrow 0} \langle \tilde{\varkappa}_\lambda, g_1 \rangle = \langle \tilde{\varkappa}, g_1 \rangle$. Since $\tilde{\varkappa}$ is a Radon measure this implies that $\lim\limits_{\lambda \downarrow 0} \langle \tilde{\varkappa}_\lambda, g \rangle = \langle \tilde{\varkappa}, g \rangle$ for all $g \in \mathcal{G}^+$.

Lemma 3.12. — *Let* $\sigma \in M^+(E)$ *and set*

$$\langle \nu, f \rangle = \int \langle \sigma, T_s f \rangle m\,(ds).$$

Then $\nu \in M^+(E)$ *is a* K-*invariant measure. Further, if*

$$\alpha \in M^+(E)$$

and $\alpha * \sigma \in M^+(E)$ *so too is* $\alpha * \nu$ *and* $\alpha * \nu = \alpha * \sigma$. *If, in addition,* α *is* K-*invariant then* $\nu * \alpha = \sigma * \alpha$ *when* $\sigma * \alpha \in M^+(E)$.

Proof. — Clearly ν is K-invariant. Let $f \in \mathcal{E}^+$. Then $\langle \nu, f \rangle = \langle \tilde{\nu}, \tilde{f} \rangle = \iint \tilde{f}(s^{-1}z)\tilde{\sigma}\,(dz) m\,(ds)$. Hence,

$$\langle \alpha * \nu, f \rangle = \langle \tilde{\alpha} * \tilde{\nu}, \tilde{f} \rangle$$

$$= \int \Big[\int \tilde{f}(xy)\tilde{\nu}\,(dy) \Big] \tilde{\alpha}\,(dx) = \int \Big[\iint \tilde{f}(xs^{-1}z)\tilde{\sigma}\,(dz) m\,(ds) \Big] \tilde{\alpha}\,(dx)$$

(because the function $y \to \tilde{f}(xy) = \tilde{g}(y),\ g \in \mathcal{E}^+$)

$$= \iint \Big[\int \tilde{f}(xs^{-1}z)\tilde{\alpha}\,(dx) \Big] \tilde{\sigma}\,(dz) m\,(ds)$$
$$= \iint \Big[\int \tilde{f}(xz)\tilde{\alpha}\,(dx) \Big] \tilde{\sigma}\,(dz) m\,(ds)$$

(because $s \in K$ and $\tilde{\alpha}$ is K-right invariant)

$$= \langle \tilde{\alpha} * \tilde{\sigma}, \tilde{f} \rangle = \langle \alpha * \sigma, f \rangle.$$

The calculation that proves $\nu * \alpha = \sigma * \alpha$ when α is K-invariant is entirely similar.

Corollary 3.13. — *Let* $\varkappa * \sigma \leqslant \varkappa$ *and* $\lim\limits_{n \to \infty} \varkappa * \sigma^n = 0$ *where* $\varkappa, \sigma \in M^+(E)$ *and* $\varkappa$ *is* K-*invariant. Then the* K-*invariant measure* ν *of Lemma* 3.12 *is such that* $\varkappa * \nu \leqslant \varkappa$ *and* $\lim\limits_{n \to \infty} \varkappa * \nu^n = 0$. *Further, if* $\varkappa * \sigma = \varkappa$ *on* A *then* $\varkappa * \nu = \varkappa$ *on* A.

The corresponding results hold if the convolutions are done in the reverse order.

Proof. — For the first statement if suffices to note that

$$\varkappa * \sigma^n = (\varkappa * \sigma^{n-1}) * \sigma = (\varkappa * \sigma^{n-1}) * \nu$$

and so $\varkappa * \sigma^n = \varkappa * \nu^n$. For the second one note that if

$$\nu^{n-1} * \varkappa = \sigma^{n-1} * \varkappa = \alpha$$

then α is K-invariant and so $\nu^n * \varkappa = \sigma * \alpha = \sigma^n * \varkappa$.

The proof of the theorem is now completed by the above lemmas and corollary.

Remarks. — The conditions (3) and ($3^\vee$) do not appear to imply condition (1). By considering the situation on the space F of left cosets one could show $(3) \Longrightarrow (1)$ providing that the kernel $\varkappa *$ on F satisfies the complete maximum principle. However one only knows that $\check{\varkappa} *$ has this property.

To prove the last statement it suffices to show that $\varkappa$ satisfies D′) whenever $\varkappa$ satisfies D).

First of all if $\mathscr{B}$ is a neighbourhood base for 0 satisfying D) the measures σ can, by corollary 3.13 below, be assumed to be K-invariant. Now $(\sigma * \varkappa)^\vee = \check{\varkappa} * \check{\sigma}$ and so since the sets of the form $\pi((\tilde{\mathrm{A}})^\vee)$, $\mathrm{B} \in \mathscr{B}$, also from a base for the neighbourhoods of 0 it follows that $\check{\varkappa}$ satisfies D′).

Appendix.

In the non σ-compact case the complications arise because theorem 2 of [4] no longer applies and has to be replaced by theorem 3 of [5]. In the terminology of [5] if $\mathrm{V} = * \varkappa$ then every Baire set is σ-bounded. This condition replaces the hypothesis that V is a proper kernel in the σ-compact case.

In proposition 1.2 « excessive » should be replaced by « supermedian » as defined in [5]. Now, as V is sub-Markovian, 1 is supermedian and so, in view of theorem 3 in [5], theorem 1.4 holds. Note that in lemma 1.3 « proper » should be replaced by « every Baire set is σ-bounded ».

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