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A GENERAL DEFINITION OF CAPACITY

by Makoto OHTSUKA

*Dédié à Monsieur M. Brelot à l'occasion
de son 70^e anniversaire.*

Introduction.

During the past 20 years the notion of extremal length proved its usefulness in many branches of analysis. Given a family Γ of locally rectifiable curves in the (x, y) -plane, the extremal length of Γ is defined to be the reciprocal of the infimum of $\iint \rho^2 dx dy$ for the family of Borel measurable functions $\rho \geq 0$ satisfying $\int_{\gamma} \rho ds \geq 1$ for every $\gamma \in \Gamma$.

There are also many definitions of capacity. One way to define the Newtonian capacity in R^n is to consider the class M of non-negative measures μ with finite energy. It is known that $\text{grad } U^\mu$ exists a.e., where U^μ denotes the Newtonian potential of μ . The Newtonian capacity of a compact set K is defined to be the infimum of $\int_{R^n} |\text{grad } U^\mu|^2 dx$ taken with respect to $\mu \in M$ satisfying $U^\mu \geq 1$ on K .

Recently, Meyers [2] defined $C_{k; \mu_0; p}(A)$ for $A \subset R^n$ by $\inf \int \rho^p d\mu_0$ taken with respect to $\rho \geq 0$ satisfying

$$\int k(x, y) \rho(y) d\mu_0(y) \geq 1$$

on A , where μ_0 is a non-negative measure and $k(x, y)$ is a positive lower semicontinuous function on $R^n \times R^n$.

In the present note we shall give a general definition of capacity which includes the above three quantities as special cases, and prove that this general capacity is continuous from the left.

1. General definition.

Let Ω be a space, and F be a family of non-negative functions defined on Ω such that $cf_1 \in F$ and $f_1 + f_2 \in F$ whenever $0 \leq c < \infty$ and $f_1, f_2 \in F$; we set $0 \cdot \infty = 0$ if $0 \cdot \infty$ happens for cf_1 . It follows that $f \equiv 0$ belongs to F . Let $\Phi \not\equiv \infty$ be a non-negative functional defined on F . Assume that there exist $p, q > 0$ such that $\Phi(cf) \leq c^p \Phi(f)$ for any constant $c \geq 0$ and $f \in F$, and

$$(\Phi(f_1 + f_2))^q \leq (\Phi(f_1))^q + (\Phi(f_2))^q$$

if $f_1, f_2 \in F$. In addition, we assume that, if $f_1, f_2, \dots \in F$ and $\Phi(f_{m+1} + \dots + f_n) \rightarrow 0$ as $n, m \rightarrow \infty$, then

$$f = \sum_{k=1}^{\infty} f_k \in F$$

and $\Phi\left(\sum_{k=1}^n f_k\right) \rightarrow \Phi(f)$ as $n \rightarrow \infty$. It follows that

$$(\Phi(\sum f_n))^q \leq \sum (\Phi(f_n))^q$$

for such $\{f_n\}$.

Let Γ_0 be another space, and G be a class of subsets of Γ_0 such that $\Gamma_1, \Gamma_2, \dots \in G$ implies $\bigcup_n \Gamma_n \in G$ and that $\Gamma \in G$ and $\Gamma' \subset \Gamma$ imply $\Gamma' \in G$. We shall say that a property holds G -a.e. on $\Gamma \subset \Gamma_0$ if the exceptional set belongs to G . For each $f \in F$ suppose a non-negative function $T_f(\gamma)$ is defined G -a.e. on Γ_0 , and assume that, for any $f_1, f_2 \in F$ and $c \geq 0$, $T_{cf_1} = cT_{f_1}$ and $T_{f_1+f_2} = T_{f_1} + T_{f_2}$ hold and $f_1 \leq f_2$ implies $T_{f_1} \leq T_{f_2}$, where all relations as to T_{f_1} and T_{f_2} are supposed to hold wherever they are defined.

We shall say that f is G -almost admissible (or simply G -alm. ad.) for $\Gamma \subset \Gamma_0$ when $f \in F$ and $T_f \geq 1$ G -a.e. on Γ . We set

$$G_G(\Gamma) = \inf_{G\text{-alm.ad.}f} \Phi(f)$$

if there is at least one G -alm. ad. f , and otherwise

$$C_G(\Gamma) = \infty.$$

Evidently $C_G(\Gamma) \leq C_G(\Gamma')$ if $\Gamma \subset \Gamma'$. We observe that $C_G(\Gamma) = 0$ for every $\Gamma \in G$ because $f \equiv 0$ is G -alm. ad. and $\Phi(0) = 0$.

We shall denote by L the family of functions $f \in F$ with finite $\Phi(f)$.

THEOREM 1. — $C_G(\Gamma) = 0$ if and only if there exists $f \in L$ such that $T_f = \infty$ G -a.e. on Γ .

Proof. — The if part follows from the definition of C_G and the properties of Φ and T_f . To prove the only-if part take $f_n \in F$ and $\Gamma_n \in G$ for each n so that $T_{f_n} \geq 1$ on $\Gamma - \Gamma_n$ and $(\Phi(f_n))^q \leq 2^{-n}$, and set $f = \sum_n f_n$. Then $f \in F$ and

$$(\Phi(f))^q \leq \Sigma(\Phi(f_n))^q \leq 1.$$

We have

$$T_f(\gamma) \geq \sum_{k=1}^m T_{f_k}(\gamma) \geq m$$

for every $\gamma \in \Gamma - \bigcup_n \Gamma_n$ and m so that $T_f(\gamma) = \infty$ for every $\gamma \in \Gamma - \bigcup_n \Gamma_n$. Since $\bigcup_n \Gamma_n \in G$, our theorem is proved.

LEMMA 1. — $(C_G(\bigcup_n \Gamma_n))^q \leq \Sigma(C_G(\Gamma_n))^q$.

Proof. — We may assume that $\Sigma(C_G(\Gamma_n))^q < \infty$. Given $\varepsilon > 0$, let f_n be G -alm. ad. for Γ_n such that

$$(\Phi(f_n))^q \leq (C_G(\Gamma_n))^q + \varepsilon 2^{-n}.$$

By our assumption on Φ , $f = \Sigma f_n \in F$ and

$$(\Phi(f))^q \leq \Sigma(\Phi(f_n))^q.$$

Evidently f is G -alm. ad. for $\bigcup_n \Gamma_n$ so that

$$(C_G(\bigcup_n \Gamma_n))^q \leq (\Phi(f))^q \leq \Sigma(\Phi(f_n))^q \leq \Sigma(C_G(\Gamma_n))^q + \varepsilon.$$

This gives the required inequality.

LEMMA 2. — Suppose $f \in F$ satisfies $T_f \geq 1$ on $\Gamma - \Gamma'$, where $C_G(\Gamma') = 0$. Then $C_G(\Gamma) \leq \Phi(f)$.

Proof. — By Theorem 1 there exist $f' \in L$ and $\Gamma'' \in G$ such that $T_{f'} = \infty$ on $\Gamma' - \Gamma''$. For any $\varepsilon > 0$ we have $T_{f+\varepsilon f'} \geq 1$ on $\Gamma - \Gamma''$, and hence

$$C_G(\Gamma) \leq \Phi(f + \varepsilon f') \leq \{(\Phi(f))^q + (\varepsilon^p \Phi(f'))^q\}^{\frac{1}{q}} \rightarrow \Phi(f)$$

as $\varepsilon \rightarrow 0$. Thus $C_G(\Gamma) \leq \Phi(f)$.

THEOREM 2. — Denote $\{\Gamma^* \subset \Gamma_0; C_G(\Gamma^*) = 0\}$ by G_0 . Then

$$C_{G_0}(\Gamma) = C_G(\Gamma)$$

for any $\Gamma \subset \Gamma_0$.

Proof. — We observe that $\Gamma_1, \Gamma_2, \dots \in G_0$ implies $\bigcup_n \Gamma_n \in G_0$ in virtue of Lemma 1 and that $\Gamma \in G_0$ and $\Gamma' \subset \Gamma$ imply $\Gamma' \in G_0$. Since $G \subset G_0$, $C_{G_0}(\Gamma) \leq C_G(\Gamma)$. Assume that $C_{G_0}(\Gamma) < \infty$, and take $f \in F$ such that $T_f \geq 1$ on $\Gamma - \Gamma'$ where $\Gamma' \in G_0$. By Lemma 2 $C_G(\Gamma) \leq \Phi(f)$. Because of the arbitrariness of f we derive

$$C_G(\Gamma) \leq C_{G_0}(\Gamma).$$

The equality now follows.

THEOREM 3 (cf. [2], Theorem 4). — Each of the following statements implies the succeeding one.

(i) $T_{f_n} \rightarrow T_f$ in C_G , namely, for any $a > 0$,

$$C_G(\{\gamma \in \Gamma_0 - \Gamma; |T_{f_n}(\gamma) - T_f(\gamma)| \geq a\}) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where $\Gamma \in G$ and all T_{f_n} and T_f are defined on $\Gamma_0 - \Gamma$; $\infty - \infty$ is set to be 0 if it happens for $T_{f_n} - T_f$.

(ii) We can find $\{f_{n_k}\}$ with the property that, given $\varepsilon > 0$, there exists $\Gamma' \subset \Gamma_0$ with $C_G(\Gamma') < \varepsilon$ such that $T_{f_{n_k}} - T_f \rightarrow 0$ uniformly on $\Gamma_0 - \Gamma'$.

(iii) For the sequence $\{f_{n_k}\}$ in (ii), $T_{f_{n_k}} \rightarrow T_f$ on $\Gamma_0 - \Gamma''$, where $C_G(\Gamma'') = 0$.

Proof. — (i) → (ii). There exist $\{f_{n_k}\}$ and $\{\Gamma_k\}$ in Γ_0 such that, for each k , $\Gamma_k \supset \Gamma$, $(C_G(\Gamma_k))^q \leq 2^{-k}$ and

$$|T_{f_{n_k}} - T_f| \leq \frac{1}{k} \quad \text{on } \Gamma_0 - \Gamma_k.$$

Given $\varepsilon > 0$, choose k_0 so that $2^{-k_0+1} < \varepsilon^q$. We see that

$T_{f_{n_k}} - T_f \rightarrow 0$ uniformly on $\Gamma_0 - \bigcup_{k=k_0}^{\infty} \Gamma_k$, and

$$\left(C_G \left(\bigcup_{k=k_0}^{\infty} \Gamma_k \right) \right)^q \leq \sum_{k=k_0}^{\infty} (C_G(\Gamma_k))^q \leq \varepsilon^q$$

by Lemma 1. This establishes (i) → (ii).

(ii) → (iii) is evident.

Now, let $\Psi(f, g)$ be a functional on $F \times F$ such that, for any $f_1, f_2 \in F$, there exists $\tilde{f} \in F$ satisfying $\Phi(\tilde{f}) \leq \Psi(f_1, f_2)$ and $|T_{f_1} - T_{f_2}| \leq T_{\tilde{f}}$ G-a.e. on Γ_0 . Then we have

THEOREM 4. — For all $f_1, f_2 \in F$ and $0 < a < \infty$ we have

$$C_G(\{\gamma \in \Gamma_0 - \Gamma; |T_{f_1}(\gamma) - T_{f_2}(\gamma)| \geq a\}) \leq a^{-p} \Psi(f_1, f_2),$$

where $\Gamma \in G$ is chosen so that both T_{f_1} and T_{f_2} are defined on $\Gamma_0 - \Gamma$.

Proof. — Denote $\{\gamma \in \Gamma_0 - \Gamma; |T_{f_1}(\gamma) - T_{f_2}(\gamma)| \geq a\}$ by Γ' . By our assumption there exists $\tilde{f} \in F$ such that $\Phi(\tilde{f}) \leq \Psi(f_1, f_2)$ and $|T_{f_1} - T_{f_2}| \leq T_{\tilde{f}}$ G-a.e. on Γ_0 . Evidently \tilde{f}/a is G-alm. ad. for Γ' so that

$$C_G(\Gamma') \leq \Phi \left(\frac{\tilde{f}}{a} \right) \leq a^{-p} \Psi(f_1, f_2).$$

Hereafter we assume that the relation

$$\limsup_{n, m \rightarrow \infty} \left\{ \frac{\Phi(f_n) + \Phi(f_m)}{2} - \Phi \left(\frac{f_n + f_m}{2} \right) \right\} \leq 0$$

for $\{f_n\} \subset F$ implies the existence of $f \in F$ such that $\Psi(f_n, f) \rightarrow 0$ and $\Phi(f_n) \rightarrow \Phi(f)$.

THEOREM 5. — *If $C_G(\Gamma) < \infty$, then there exist $f \in F$ and Γ' with $C_G(\Gamma') = 0$ such that $T_f \geq 1$ on $\Gamma - \Gamma'$ and $\Phi(f) = C_G(\Gamma)$.*

Proof. — Choose f_1, f_2, \dots G-alm. ad. for Γ so that $\Phi(f_n) \rightarrow C_G(\Gamma)$. Then $(f_n + f_m)/2$ is G-alm. ad. for Γ , and hence $\Phi((f_n + f_m)/2) \geq C_G(\Gamma)$. Therefore

$$\limsup_{n, m \rightarrow \infty} \left\{ \frac{\Phi(f_n) + \Phi(f_m)}{2} - \Phi\left(\frac{f_n + f_m}{2}\right) \right\} \leq 0.$$

By our assumption there exists $f \in F$ such that $\Phi(f_n, f) \rightarrow 0$ and $\Phi(f_n) \rightarrow \Phi(f)$ as $n \rightarrow \infty$. Choose $\Gamma' \in G$ so that $T_{f_n} \geq 1$ on $\Gamma - \Gamma'$ for every n . In view of (i) \rightarrow (iii) of Theorem 3 and Theorem 4 we find $\{f_{n_k}\}$ and $\Gamma'' \subset \Gamma$ with $C_G(\Gamma'') = 0$ such that

$$T_f = \lim_{k \rightarrow \infty} T_{f_{n_k}} \geq 1 \quad \text{on } \Gamma - \Gamma''.$$

We have $\Phi(f) = \lim_{n \rightarrow \infty} \Phi(f_n) = C_G(\Gamma)$.

THEOREM 6. — *If $\Gamma_n \uparrow \Gamma$, then $C_G(\Gamma_n) \uparrow C_G(\Gamma)$.*

Proof. — Denote $\lim_{n \rightarrow \infty} C_G(\Gamma_n)$ by C . Clearly $C \leq C_G(\Gamma)$. Hence it suffices to establish $C_G(\Gamma) \leq C$. We may assume that $C < \infty$. Choose f_n G-alm. ad. for Γ_n so that $\Phi(f_n) \rightarrow C$. If $m > n$, then f_m and hence $(f_n + f_m)/2$ is G-alm. ad. for Γ_n . Therefore $\Phi((f_n + f_m)/2) \geq C_G(\Gamma_n)$. As in the proof of Theorem 5 we find $f \in F$ and Γ' with $C_G(\Gamma') = 0$ so that $T_f(\gamma) \geq 1$ on $\Gamma - \Gamma'$ and $\Phi(f_n) \rightarrow \Phi(f)$ as $n \rightarrow \infty$. By Lemma 2 $C_G(\Gamma) \leq \Phi(f)$. Hence

$$C_G(\Gamma) \leq \Phi(f) = \lim_{n \rightarrow \infty} \Phi(f_n) = C.$$

From this theorem we derive immediately

THEOREM 7. — *If $C_G(\Gamma' \cup \Gamma'') \leq C_G(\Gamma') + C_G(\Gamma'')$ for any Γ', Γ'' , then*

$$(1) \quad C_G\left(\bigcup_n \Gamma_n\right) \leq \sum_n C_G(\Gamma_n).$$

Remark. — If $\max(f_1, f_2)$ belongs to F and

$$\Phi(\max(f_1, f_2)) \leq \Phi(f_1) + \Phi(f_2)$$

whenever $f_1, f_2 \in F$, then $C_G(\Gamma_1 \cup \Gamma_2) \leq C_G(\Gamma_1) + C_G(\Gamma_2)$. It suffices to show $C_G(\Gamma_1 \cup \Gamma_2) \leq \Phi(f_1) + \Phi(f_2)$ when each $C_G(\Gamma_i)$ is finite and f_i is G -alm. ad. for $\Gamma_i, i = 1, 2$. This is actually true because $f = \max(f_1, f_2)$ is G -alm. ad. for $\Gamma_1 \cup \Gamma_2$ so that

$$C_G(\Gamma_1 \cup \Gamma_2) \leq \Phi(f) \leq \Phi(f_1) + \Phi(f_2).$$

2. Examples.

Let Ω be a general space. Hereafter take as Γ_0 the class M_0 of all non-negative measures defined on a σ -field E of sets in Ω , and let F be a family of non-negative E -measurable functions on Ω such that $cf_1 \in F$ and $f_1 + f_2 \in F$ whenever $c \geq 0$ and $f_1, f_2 \in F$. For every $f \in F$ we define $T_f(\mu)$ to be $\int f d\mu$. We take $G = \emptyset$ and denote $C_G(\Gamma)$ by $C(M)$ for $\Gamma = M \subset M_0$.

Example 1. — Let F consist of all non-negative E -measurable functions. With a fixed $m \in M_0$ set

$$\Phi(f) = \int f^p dm \quad \text{for } f \in F$$

and $\Psi(f_1, f_2) = \int |f_1 - f_2|^p dm$ for $f_1, f_2 \in F$ if

$$\int f_1^p dm + \int f_2^p dm < \infty.$$

If $\int f_1^p dm + \int f_2^p dm = \infty$,

then define $\Psi(f_1, f_2)$ to be ∞ . Then Ψ satisfies the conditions required in § 1. We call $C(M)$ the module of M of order p . Its reciprocal is called the extremal length of M of order p . In this case we have $\sup_n f_n \in F$ for $f_1, f_2, \dots \in F$ and $\Phi(\sup_n f_n) \leq \sum \Phi(f_n)$. We obtain the subadditivity from this immediately without appealing to Theorem 7.

Example 2. — Let F, Φ and ψ be as above. Let $k(x, e) \geq 0$ be an E -measurable function of x on Ω for every fixed

$e \in E$, and a measure for every fixed $x \in \Omega$. Set

$$\nu_\mu(e) = \int k(x, e) d\mu(x) \quad \text{for } \mu \in M_0,$$

and $N_M = \{\nu_\mu; \mu \in M\}$. We may consider $C(N_M)$. This gives a generalization of $C_{k; \mu_0; p}(A)$ referred to in the introduction when $M = \{\varepsilon_x; x \in A\}$ and

$$k(x, e) = \int_e k(x, y) d\mu_0(y),$$

where A is a subset of R_n , ε_x is the unit point measure at x , $k(x, y) \geq 0$ is E -measurable for every fixed x and μ_0 is a fixed measure in M_0 . As in Example 1, (1) follows immediately.

Example 3. — Let Ω be an open set in R^n , E be the Borel class of sets in Ω , m be the Lebesgue measure, F consist of (some) non-negative p -precise functions f in Ω , take $\Phi(f) = \int |\text{grad } f|^p dm$, and define $\Psi(f_1, f_2)$ by

$$\int |\text{grad } (f_1 - f_2)|^p dm.$$

See [3] for p -precise functions and properties of these functions. In order to assure $\sum_n f_n \in F$ for every $\{f_n\} \subset F$ satisfying $\int |\text{grad } (f_{n+1} + \dots + f_m)|^p dm \rightarrow 0$, we assume that there is a family, with positive module of order p , of curves in Ω such that every $f \in F$ tends to 0 along p -a.e. curve of the family. Then all the conditions required in the beginning of § 1 are satisfied. If $\max(f_1, f_2) \in F$ for any $f_1, f_2 \in F$, then (1) holds because $|\text{grad } (\max(f_1, f_2))| = |\text{grad } f_1|$ or $|\text{grad } f_2|$ m -a.e. so that $\Phi(\max(f_1, f_2)) \leq \Phi(f_1) + \Phi(f_2)$. Consider the case when the module of order p of the family Λ of curves terminating at $\partial\Omega$ is positive, $M = \{\varepsilon_x; x \in A \subset \Omega\}$ and F consists of all non-negative p -precise functions in Ω tending to 0 along p -a.e. curve of Λ . Then the capacity is called the p -capacity of A (relative to Ω).

Example 4. — Let Ω be a topological space, and E be the Borel class of sets in Ω . Let F be as in the beginning of this section, and Φ be as in § 1. Denote by S the family of lower semicontinuous functions in Ω , and define $c(K)$

for every compact set K by $\inf \Phi(f)$ for $f \in F \cap S$ satisfying $f \geq 1$ on K . Let us see that $c(K_n) \downarrow c(K)$ if a sequence $\{K_n\}$ of compact sets decreases to K . Suppose $f \in F \cap S$ satisfies $f(x) \geq 1$ on K . Since f is lower semicontinuous, $f/(1 - 1/n) > 1$ for each $n \geq 2$ on an open set ω containing K . There exists m_0 such that $K_{m_0} \subset \omega$, and hence

$$\begin{aligned} c(K) &\leq \lim_{m \rightarrow \infty} c(K_m) \leq \Phi(f/(1 - 1/n)) \\ &\leq (1 - 1/n)^{-p} \Phi(f) \rightarrow \Phi(f) \end{aligned}$$

as $n \rightarrow \infty$. The arbitrariness of f yields $c(K) = \lim_{m \rightarrow \infty} c(K_m)$.

This together with Theorem 6 shows that $C(M_A)$ with $M_A = \{\varepsilon_x; x \in A\}$ is a true capacity if it is shown that

$$c(K) = C(M_K)$$

for every compact set K ; cf. [1; Part II, Chap. 1]. This is the case, for example, when $\Omega = \mathbb{R}^n (n \geq 3)$, F consists of all Newtonian potentials of non-negative measures with finite energy and $\Phi(f) = \int |\text{grad } f|^2 dm$; the 2-capacity is equal to the Newtonian outer capacity. Evidently all $f \in F$ are superharmonic.

Another example of such a case is found in [4]. See [3] too.

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