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## ON DEFINITIONS OF SUPERHARMONIC FUNCTIONS

by Seizô ITÔ

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*Dédié à Monsieur M. Brelot à l'occasion  
de son 70<sup>e</sup> anniversaire.*

### 1. Introduction.

The classical definition of superharmonic functions by F. Riesz [3] (see also M. Brelot [1]) can be generalized in natural way to the case of the elliptic differential operator  $A$  of second order with variable coefficients (§ 2 of the present paper). On the other hand, L. Schwartz [4] has defined the superharmonicity with respect to the general elliptic differential operator in view-point of the theory of distribution and given an elegant proof to Riesz decomposition theorem. One may easily prove that the superharmonicity with respect to  $A$  (abbreviated to *A-superharmonicity*) of the Riesz-Brelot sense implies that of Schwartz sense in case  $A$  is the ordinary Laplacian.

However, in the case of the elliptic differential operator  $A$  with variable coefficients, it seems not to be evident that the theory of distribution is applicable to  $A$ -superharmonic functions in the classical sense; in fact, even the local summability of an  $A$ -superharmonic function in the classical sense seems not to be trivial.

The purpose of the present paper is to prove that any  $A$ -superharmonic function in the Riesz-Brelot sense is locally

summable and satisfies the  $A$ -superharmonicity in the sense of Schwartz distribution. The  $A$ -superharmonicity in Schwartz sense implies the Riesz decomposition formula as shown in [4], while one may easily see that any function represented by the Riesz decomposition formula is  $A$ -superharmonic in the Riesz-Brelot sense. Thus we may conclude the equivalence of the  $A$ -superharmonicity in the Riesz-Brelot sense, that of Schwartz sense and the Riesz decomposition formula for arbitrary elliptic differential operator  $A$  of second order with variable coefficients.

## 2. Main results.

Let  $\Omega$  be a subdomain of an orientable  $m$ -dimensional  $C^\infty$ -manifold ( $m \geq 2$ ), and  $A$  be an elliptic differential operator of the form:

$$\begin{aligned} Au(x) &= \operatorname{div} [\nabla u(x)] + (\mathbf{b}(x) \cdot \nabla u(x)) + c(x)u(x) \\ &\equiv \sum_{i,j} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i} \left[ \sqrt{a(x)} a^{ij}(x) \frac{\partial u(x)}{\partial x^j} \right] + \sum_i b^i(x) \frac{\partial u(x)}{\partial x^i} + c(x)u(x), \end{aligned}$$

where  $\|a^{ij}(x)\|$  is a contravariant tensor of class  $C^2$  in  $\Omega$  and is symmetric and strictly positive-definite for any  $x \in \Omega$ ,  $a(x) = \det \|a_{ij}(x)\| = \det \|a^{ij}(x)\|^{-1}$ ,  $\mathbf{b}(x) \equiv \|b^i(x)\|$  is a contravariant vector of class  $C^2$  in  $\Omega$ , and  $c(x)$  is a Hölder-continuous function satisfying  $c(x) \leq 0$  in  $\Omega$ . We shall denote by  $dx$  and  $dS(x)$  respectively the volume element and the  $m - 1$  dimensional hypersurface element with respect to the Riemannian metric defined by the tensor  $\|a_{ij}(x)\|$ . The formally adjoint operator  $A^*$  of  $A$  is defined by

$$A^*u(x) = \operatorname{div} [\nabla u(x) - \mathbf{b}(x)u(x)] + c(x)u(x).$$

By definition, a function  $u(x)$  is said to be  $A$ -harmonic in  $\Omega$  if it satisfies  $Au = 0$  in  $\Omega$ , and is said to be  $A$ -superharmonic in  $\Omega$  if it satisfies the following three conditions:

- i)  $-\infty < u(x) \leq \infty$  and  $u(x) \not\equiv \infty$  in  $\Omega$ ,
- ii)  $u(x)$  is lower semi-continuous in  $\Omega$ ,

iii) if  $D$  is a domain with its compact closure  $\bar{D} \subset \Omega$ , and if  $\omega(x)$  is continuous on  $\bar{D}$ ,  $A$ -harmonic in  $D$  and satisfies  $\omega(x) \leq u(x)$  on  $\partial D$ , then  $\omega(x) \leq u(x)$  holds in  $D$ .

We shall prove the following two theorems in § 4.

**THEOREM 1.** — Any  $A$ -superharmonic function in  $\Omega$  is locally summable in  $\Omega$ .

**THEOREM 2.** — Any  $A$ -superharmonic function  $u(x)$  in  $\Omega$  satisfies  $Au \leq 0$  in  $\Omega$  in the sense of distribution.

### 3. Preliminary lemmas.

We shall use some properties of fundamental solutions of parabolic equations. The following facts are implied by the results of one of the author's previous papers [3].

For any subdomain  $D$  of  $\Omega$  with compact closure  $\bar{D} \subset \Omega$  and with boundary  $\partial D$  of class  $C^3$ , there exists one and only one fundamental solution  $U_D(t, x, y)$  of the initial-boundary value problem :

$$(3.1) \quad \frac{\partial u}{\partial t} = Au \text{ in } (0, \infty) \times D, u|_{t=0} = u_0, u|_{x \in \partial D} = \varphi.$$

The function  $U_D(t, x, y)$  satisfies that

$$(3.2) \quad \left\{ \begin{array}{l} U_D(t, x, y) \geq 0 \text{ for any} \\ \langle t, x, y \rangle \in (0, \infty) \times \bar{D} \times \bar{D}; \\ \text{the equality holds if and only if at least one} \\ \text{of } x \text{ and } y \text{ belongs to } \partial D \end{array} \right.$$

and that

$$(3.3) \quad \frac{\partial U_D(t, x, y)}{\partial \mathbf{n}(y)} \leq 0 \text{ for any } t > 0, y \in \partial D$$

and  $x \in D - \{y\}$  where  $\frac{\partial}{\partial \mathbf{n}(y)}$  denotes the exterior normal derivative at  $y \in \partial D$ . For any continuous functions  $u_0(x)$  on  $\bar{D}$  and  $\varphi(t, x)$  on  $[0, \infty) \times \partial D$ , there exists one and only

one solution  $u(t, x)$  of the initial boundary value problem (3.1) and it is given by

$$(3.4) \quad u(t, x) = \int_D U_D(t, x, y) u_0(y) dy - \int_0^t d\tau \int_{\partial D} \frac{\partial U_D(t - \tau, x, y)}{\partial \mathbf{n}(y)} \varphi(\tau, y) dS(y).$$

In particular, if  $\omega(x)$  satisfies  $A\omega = 0$  in  $D$  and  $\omega|_{\partial D} = \psi$  where  $\psi$  is continuous on  $\partial D$ , then

$$(3.5) \quad \omega(x) = \int_D U_D(t, x, y) \omega(y) dy - \int_0^t d\tau \int_{\partial D} \frac{\partial U_D(\tau, x, y)}{\partial \mathbf{n}_D(y)} \psi(y) dS(y).$$

LEMMA 1. — Let  $\Omega_0$  be a subdomain of  $\Omega$  with its compact closure  $\bar{\Omega}_0 \subset \Omega$  and with boundary  $\partial\Omega_0$  of class  $C^3$ ,  $u(x)$  be an  $A$ -superharmonic function on  $\Omega$  such that  $u(x) > 0$  on  $\bar{\Omega}_0$  and  $\nu(x)$  be a continuous function on  $\bar{\Omega}_0$  such that  $0 \leq \nu(x) < u(x)$  on  $\bar{\Omega}_0$ . Then  $\int_{\Omega_0} U_{\Omega_0}(t, x, y) \nu(y) dy < u(x)$  on  $(0, \infty) \times \bar{\Omega}_0$ .

*Proof.* — The function  $\nu(t, x) = \int_{\Omega_0} U_{\Omega_0}(t, x, y) \nu(y) dy$  is the solution of the initial-boundary value problem (3.1) with  $D = \Omega_0$ ,  $u_0 = \nu$  and  $\varphi = 0$ . Suppose that

$$\nu(t, x) \geq u(x)$$

at some point  $\langle t, x \rangle \in (0, \infty) \times \bar{\Omega}_0$ , and put

$$t_1 = \inf \{t; \nu(t, x) \geq u(x) \text{ for some } x \in \bar{\Omega}_0\}.$$

Then

$$(3.6) \quad 0 \leq \nu(\tau, x) < u(x) \quad \text{whenever } 0 < \tau < t_1$$

and  $x \in \bar{\Omega}_0$ . By means of the continuity of  $\nu(t, x)$ , lower semi-continuity of  $u(x)$  and by the fact:  $\nu(t, x) = 0$  for any  $x \in \partial\Omega_0$ , we may find a point  $x_1 \in \Omega_0$  such that

$$(3.7) \quad \nu(t_1, x_1) = u(x_1) < \infty.$$

Since  $u(x) - \nu(x)$  is positive and lower semi-continuous on

$\bar{\Omega}_0$ , there exists a positive number  $\delta$  such that

$$(3.8) \quad 0 < v(x) + 3\delta < u(x) \quad \text{on } \bar{\Omega}_0.$$

Further we may find a domain  $D$  with boundary  $\partial D$  of class  $C^3$  such that  $x_1 \in D \subset \Omega_0$  and that

$$v(x) < v(x_1) + \delta \quad \text{and} \quad u(x) > u(x_1) - \delta \quad \text{on } \bar{D}.$$

Combining these inequalities with (3.8), we get

$$(3.9) \quad v(x) + \delta < \inf_{x \in \bar{D}} u(x) \quad \text{on } \bar{D}.$$

Let  $\{u_n\}$  be a monotone increasing sequence of continuous functions on  $\partial D$  such that  $\lim_{n \rightarrow \infty} u_n(y) = u(y)$  on  $\partial D$ . Then we may easily show that

$$(3.10) \quad \lim_{n \rightarrow \infty} \left[ \inf_{y \in \partial D} u_n(y) \right] = \inf_{y \in \partial D} u(y).$$

Let  $\omega_n$  be the solution of elliptic boundary value problem:  $A\omega_n = 0$  in  $D$ ,  $\omega_n|_{\partial D} = u_n$ . Then  $\omega_n(x) \leq u(x)$  in  $D$  by means of the  $A$ -superharmonicity of  $u$ , and the sequence  $\{\omega_n\}$  is monotone increasing. Hence

$$\omega(x) = \lim_{n \rightarrow \infty} \omega_n(x) (\leq \infty)$$

exists and  $\omega(x) \leq u(x)$  in  $D$ . Since  $\omega_n(x) \geq \inf_{y \in \partial D} u_n(y)$  in  $D$ , we obtain from (3.10) and (3.9) that

$$(3.11) \quad \omega(x) \geq \inf_{y \in \partial D} u(y) \geq v(x) + \delta \quad \text{in } D.$$

On the other hand (cf. (3.5))

$$\begin{aligned} \omega_n(x) &= \int_D U_D(t, x, y) \omega_n(y) dy \\ &\quad + \int_0^t d\tau \int_{\partial D} \left\{ - \frac{\partial U_D(\tau, x, y)}{\partial \mathbf{n}(y)} \right\} u_n(y) dS(y). \end{aligned}$$

Let  $n \rightarrow \infty$ , and we obtain

$$(3.12) \quad \begin{aligned} u(x) \geq \omega(x) &= \int_D U_D(t, x, y) \omega(y) dy \\ &\quad + \int_0^t d\tau \int_{\partial D} \left\{ - \frac{\partial U_D(\tau, x, y)}{\partial \mathbf{n}(y)} \right\} u(y) dS(y). \end{aligned}$$

Applying (3.4) to  $\varphi(t, x)$  restricted on  $(0, \infty) \times \bar{D}$ , we get

$$\begin{aligned} \varphi(t, x) &= \int_{\bar{D}} U_D(t, x, y) \varphi(y) dy \\ &\quad + \int_0^t d\tau \int_{\partial D} \left\{ - \frac{\partial U_D(t - \tau, x, y)}{\partial \mathbf{n}(y)} \right\} \varphi(\tau, y) dS(y) \\ &\leq \int_{\bar{D}} U_D(t, x, y) [\varphi(y) - \delta] dy \\ &\quad + \int_0^t d\tau \int_{\partial D} \left\{ - \frac{\partial U_D(\tau, x, y)}{\partial \mathbf{n}(y)} \right\} u(y) dS(y) \\ &\hspace{15em} \text{(from (3.11) and (3.6))} \\ &\leq u(x) - \delta \int_{\bar{D}} U_D(t, x, y) dy \\ &\hspace{15em} \text{(from (3.12)).} \end{aligned}$$

In particular  $\varphi(t_1, x_1) \leq u(x_1) - \delta \int_{\bar{D}} U_D(t_1, x_1, y) dy$ ; this contradicts (3.7) since  $\int_{\bar{D}} U_D(t_1, x_1, y) dy > 0$  by (3.2).

*Remark.* — Even the fact that  $\{x | u(x) = \infty\}$  has no interior point is not guaranteed before Theorem 1 is proved. So, for instance, each term in (3.12) might be  $\infty$  (where we use the usual convention rule:  $\infty \geq \infty$ ,  $\infty >$  any real number). However we do not have to care for such situations in the above proof since  $U_D(t, x, y) \geq 0$  and  $-\frac{\partial U_D(\tau, x, y)}{\partial \mathbf{n}(y)} \geq 0$ .

LEMMA 2. — Let  $\Omega_0$  and  $u(x)$  be as in Lemma 1. Then

$$\int_{\Omega_0} U_{\Omega_0}(t, x, y) u(y) dy \leq u(x) \text{ on } (0, \infty) \times \bar{\Omega}_0.$$

*Proof.* — Let  $\gamma$  be a positive number less than  $\min_{x \in \bar{\Omega}_0} u(x)$ , and  $\{\varphi_n\}$  be a monotone increasing sequence of continuous functions on  $\bar{\Omega}_0$  such that  $\varphi_n(x) \geq \gamma$  ( $n = 1, 2, \dots$ ) and  $\lim_{n \rightarrow \infty} \varphi_n(x) = u(x)$  on  $\bar{\Omega}_0$ . For every  $n > \gamma^{-1}$ , we apply Lemma 1 to the function  $\varphi(x) = \varphi_n(x) - n^{-1}$  and obtain  $\int_{\Omega_0} U_{\Omega_0}(t, x, y) [\varphi_n(x) - n^{-1}] dy \leq u(x)$ . Let  $n \rightarrow \infty$ , and we get the conclusion of Lemma 2.

4. Proof of Theorems.

Let  $u(x)$  be an arbitrary A-superharmonic function on  $\Omega$ . For any given subdomain  $\Omega_0$  of  $\Omega$  with compact closure  $\bar{\Omega}_0 \subset \Omega$  and with boundary  $\partial\Omega_0$  of class  $C^3$ , we may assume in proofs of Theorems 1 and 2 that  $u(x) > 0$  on  $\bar{\Omega}_0$  because, if  $\inf_{x \in \bar{\Omega}_0} u(x) = \alpha \leq 0$ , we may replace  $u(x)$  by

$$u(x) + (1 - \alpha)u_0(x)$$

where  $u_0$  is the solution of the elliptic boundary value problem:  $Au_0 = 0$  in  $\Omega_0$ ,  $u_0|_{\partial\Omega_0} = 1$ .

*Proof of Theorem 1.* — Let  $D$  be an arbitrary subdomain of  $\Omega$  with compact closure  $\bar{D}$ . By the A-superharmonicity of  $u$ , we may find a point  $x_0 \in \Omega$  where  $u(x_0) < \infty$ . Let  $\Omega_0$  be a subdomain of  $\Omega$  such that  $\bar{D} \cup \{x_0\} \subset \Omega_0$ ,  $\bar{\Omega}_0$  is compact and  $\partial\Omega_0$  is of class  $C^3$ . Then, as we have noticed above, we may assume that  $u(x) > 0$  on  $\bar{\Omega}_0$ . Hence

$$(4.1) \quad u(x) \geq \int_{\Omega_0} U_{\Omega_0}(t, x, y)u(y) dy \text{ on } (0, \infty) \times \bar{\Omega}_0$$

by Lemma 2. We fix a positive number  $t_0$ . Then, since  $U_{\Omega_0}(t_0, x, y) > 0$  on  $\Omega_0 \times \Omega_0$  and  $\beta \equiv \min_{y \in \bar{D}} U_{\Omega_0}(t_0, x_0, y) > 0$  by (3.2), it follows from (4.1) that  $u(x_0) \geq \int_D \beta u(y) dy$ , which implies  $\int_D u(y) dy \leq u(x_0)/\beta < \infty$ , q.e.d.

*Proof of Theorem 2.* — Let  $\varphi(x)$  be an arbitrary non-negative valued function of class  $C^2$  and with compact support in  $\Omega$ , and  $\Omega_0$  be a subdomain of  $\Omega$  containing the support of  $\varphi$  and such that  $\bar{\Omega}_0$  is compact and  $\partial\Omega_0$  is of class  $C^3$ . It suffices to prove that

$$\int_{\Omega_0} u(x) \cdot A^* \varphi(x) dx \leq 0.$$

We may assume that  $u(x) > 0$  on  $\bar{\Omega}_0$  as we have noticed above. Hence we have by Lemma 2

$$(4.2) \quad \int_{\Omega_0} u(y) \left\{ \int_{\Omega_0} \varphi(x) U_{\Omega_0}(t, x, y) dx - \varphi(y) \right\} dy \\ = \int_{\Omega_0} \varphi(x) \left\{ \int_{\Omega_0} U_{\Omega_0}(t, x, y) u(y) dy - u(x) \right\} dx \leq 0.$$

On the other hand, since

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega_0} \varphi(x) U_{\Omega_0}(t, x, y) dx &= \int_{\Omega_0} \varphi(x) \frac{\partial U_{\Omega_0}(t, x, y)}{\partial t} dx \\ &= \int_{\Omega_0} \varphi(x) \cdot A_x U_{\Omega_0}(t, x, y) dx = \int_{\Omega_0} A^* \varphi(x) \cdot U_{\Omega_0}(t, x, y) dx \end{aligned}$$

(the subscript  $x$  to  $A$  indicates to operate  $A$  to  $U_{\Omega_0}(t, x, y)$  as a function of  $x$ ), we get

$$\lim_{t \downarrow 0} \frac{\partial}{\partial t} \int_{\Omega_0} \varphi(x) U_{\Omega_0}(t, x, y) dx = A^* \varphi(y)$$

boundedly in  $y \in \Omega_0$ ; accordingly

$$\begin{aligned} \lim_{t \downarrow 0} \frac{1}{t} \left\{ \int_{\Omega_0} \varphi(x) U_{\Omega_0}(t, x, y) dx - \varphi(y) \right\} \\ = \lim_{t \downarrow 0} \frac{1}{t} \int_0^t \left\{ \frac{\partial}{\partial \tau} \int_{\Omega_0} \varphi(x) U_{\Omega_0}(\tau, x, y) dx \right\} d\tau = A^* \varphi(y) \end{aligned}$$

boundedly in  $y$ . Combining this result with (4.2), we obtain

$$\int_{\Omega_0} A^* \varphi(y) \cdot u(y) dy \leq 0,$$

q.e.d.

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