

ANNALES DE L'INSTITUT FOURIER

MANUEL VALDIVIA

On B_r -completeness

Annales de l'institut Fourier, tome 25, n° 2 (1975), p. 235-248

http://www.numdam.org/item?id=AIF_1975__25_2_235_0

© Annales de l'institut Fourier, 1975, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

ON B_r -COMPLETENESS (*)

by Manuel VALDIVIA

Let E be a separated locally convex space and let E'_σ be its topological dual provided with the topology $\sigma(E', E)$ of the uniform convergence on the finite sets of E . E is said to be B_r -complete if every dense subspace Q of E'_σ such that $Q \cap A$ is $\sigma(E', E)$ -closed in A for each equicontinuous set A in E' , coincides with E' , [8]. In this paper we prove that if $\{E_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ are two sequences of infinite-dimensional Banach spaces then $H = \left(\bigoplus_{n=1}^\infty E_n \right) \times \prod_{n=1}^\infty F_n$ is not B_r -complete and if F coincides with $\prod_{n=1}^\infty F_n$ we have that $F \times F'[\mu(F', F)]$ is not B_r -complete, $\mu(F', F)$ being the topology of Mackey on the topological dual F' of F . We prove that if $\{E_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ are also reflexive spaces there is on H a separated locally convex topology \mathcal{F} coarser than the initial one, such that $H[\mathcal{F}]$ is a bornological barrelled space which is not an inductive limit of Baire spaces. We give also another results on B_r -completeness and bornological spaces.

The vector spaces we use here are non-zero and they are defined over the field K of the real or complex numbers. By "space" we mean "separated locally convex space". If $\langle E, F \rangle$ is a dual pair we denote by $\sigma(E, F)$ and $\mu(E, F)$ the weak and the Mackey topologies on E , respectively. If a space E has the topology \mathcal{F} and M is a subset of E , then $M[\mathcal{F}]$ is the set M , provided with the topology induced by \mathcal{F} . If A is a bounded closed absolutely convex subset of a space E , we mean by E_A the normed space over the linear hull of A , being A the closed unit ball of E_A . The topological dual of E is denoted by E' . If u is a continuous linear mapping from E into F , we denote by ${}^t u$ the mapping from F' into E' , transposed of u .

(*) Supported in part by the "Patronato para el Fomento de la Investigación en la Universidad".

In [15] we have proved the following result : a) *Let E be a separable space. Let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence of subspaces of E, with E as union. If there exists a bounded set A in E such that $A \not\subset E_n$, $n = 1, 2, \dots$ there exists a dense subspace F of E, $F \neq E$, such that $F \cap E_n$ is finite-dimensional for every positive integer n.*

LEMMA 1. — *Let E be an infinite-dimensional space such that in $E'[\sigma(E', E)]$ there is an equicontinuous total sequence. Let F be a space with a separable absolutely convex weakly compact total subset. If F is infinite-dimensional, there is a linear mapping u, continuous and injective, from E into F, such that $u(E)$ is separable, dense in F and $u(E) \neq F$.*

Proof. — Let $\{u_n\}_{n=1}^{\infty}$ be a total sequence in $E'[\sigma(E', E)]$, equicontinuous in E and linearly independent. By a method due to Klee, (see [7] p. 118), we can find a sequence $\{v_n\}_{n=1}^{\infty}$ in E' , such that its linear hull coincides with the linear hull of $\{u_n\}_{n=1}^{\infty}$, and a sequence $\{x_n\}_{n=1}^{\infty}$ in E, such that $\langle v_n, x_n \rangle = 1$, $\langle v_n, x_m \rangle = 0$, if $n \neq m$, $n, m = 1, 2, \dots$. If B is the closed absolutely convex hull of $\{u_n\}_{n=1}^{\infty}$, then B absorbs v_n and, therefore, we can take the sequence $\{v_n\}_{n=1}^{\infty}$ equicontinuous in E. Let A be a weakly compact separable absolutely convex subset of F which is total in F. We can take in A a linearly independent sequence $\{y_n\}_{n=1}^{\infty}$ which is total in F. Applying the method of Klee, ([7], p. 118), we can find a sequence $\{z_n\}_{n=1}^{\infty}$ in A, such that its linear hull coincides with the linear hull of $\{y_n\}_{n=1}^{\infty}$, and a sequence $\{w_n\}_{n=1}^{\infty}$ in F' such that $\langle w_n, z_n \rangle = 1$, $\langle w_m, z_m \rangle = 0$, $n \neq m$, $n, m = 1, 2, \dots$. Let u be the mapping from E into F defined by

$$u(x) = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n, x \rangle z_n, \quad \text{for } x \in E.$$

Let us see, first, that u is well defined. Since $\{v_n\}_{n=1}^{\infty}$ is equicontinuous in E there is a positive real number h, such that

$$|\langle v_n, x \rangle| \leq h, \quad n = 1, 2, \dots$$

Given a neighbourhood U of the origin in F, there is a positive number λ such that $\lambda A \subset U$. Since $\{(1/n)z_n\}_{n=1}^{\infty}$ converges to the origin in the Banach space F_A , there is a positive integer n_0 such that $(1/n)z_n \in (\lambda/h)A$, for every positive integer $n \geq n_0$, and since λA is convex and

$$1/2^n + 1/2^{n+1} + \dots + 1/2^{n+p} < 1, p \geq 0$$

we have that

$$\sum_{q=1}^{n+p} (1/q2^q) \langle v_q, x \rangle z_q \in \lambda A \subset U, n \geq n_0, p \geq 0$$

and, therefore, the sequence

$$\left\{ \sum_{n=1}^r (1/n2^n) \langle v_n, x \rangle z_n \right\}_{r=1}^{\infty}$$

is Cauchy in F . Since $z_n \in A$, it follows that the members of this sequence are contained in the weakly compact set hA and, therefore,

$$\sum_{n=1}^{\infty} (1/n2^n) \langle v_n, x \rangle z_n$$

is convergent in F . Obviously u is linear. If $x, y \in E, x \neq y$, there exists a positive integer n_1 such that $\langle v_{n_1}, x - y \rangle \neq 0$, since $\{v_n\}_{n=1}^{\infty}$ is total in $E'[\sigma(E', E)]$. Then

$$\begin{aligned} \langle w_{n_1}, u(x - y) \rangle &= \sum_{n=1}^{\infty} (1/n2^n) \langle v_n, x - y \rangle \langle z_n, w_{n_1} \rangle = \\ &= (1/n_1 2^{n_1}) \langle v_{n_1}, x - y \rangle \neq 0, \end{aligned}$$

and, therefore, u is injective. If V is a neighbourhood of the origin in F , we can find a positive number μ such that $\mu A \subset V$. If W is the set of E , polar of $\{v_1, v_2, \dots, v_n, \dots\}$ then μW is a neighbourhood of the origin in E and if $z \in \mu W$ we have that

$$u(z) = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n, z \rangle z_n \in \mu A \subset V$$

and, therefore, u is continuous. Since

$$u(x_p) = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n, x_p \rangle z_n = (1/p2^p) z_p$$

it follows that $u(E)$ is separable and dense in F . Finally, given any element $x \in E$ there is a positive number $\alpha > 0$ such that $\alpha x \in W$, hence $|\langle v_n, \alpha x \rangle| \leq 1, n = 1, 2, \dots$, and

$$u(\alpha x) = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n, \alpha x \rangle z_n$$

belongs to the closed absolutely convex hull $M \subset A$ of $\{(1/n) z_n\}_{n=1}^{\infty}$ and, therefore, $u(E)$ is contained in the linear hull of M . The set M is compact in the infinite-dimensional Banach space F_A and, therefore, applying the theorem of Riesz, (see [5], p. 155), it follows that M is not absorbing in F_A , hence $u(E) \neq F$.

q.e.d.

THEOREM 1. — *Let $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ be two sequence of infinite-dimensional spaces, such that, for every positive integer n , the following conditions hold :*

1) *There exists in E_n a separable weakly compact absolutely convex subsets which is total in E_n .*

2) *There exists in F_n $[\sigma(F'_n, F_n)]$ an equicontinuous total sequence.*

Then there is in $L = \left(\bigoplus_{n=1}^{\infty} E_n\right) \times \left(\prod_{n=1}^{\infty} F_n\right)$ a dense subspace G , different from L , which intersects every bounded and closed set of L in a closed set of L .

Proof. — Since for every E_n and F_n , the conditions of Lemma 1 hold there exists an injective linear continuous mapping u_n from F_n into E_n such that $u_n(F_n)$ is separable, dense in E_n and $u(F_n) \neq E_n$. We set

$$u = (u_1, u_2, \dots, u_n, \dots) \quad \text{and} \quad {}^t u = ({}^t u_1, {}^t u_2, \dots, {}^t u_n, \dots)$$

If $y = (y_1, y_2, \dots, y_n, \dots) \in \prod_{n=1}^{\infty} F_n$ and

$$x' = (x'_1, x'_2, \dots, x'_n, \dots) \in \bigoplus_{n=1}^{\infty} E'_n \text{ we put}$$

$$u(y) = (u_1(y_1), u_2(y_2), \dots, u_n(y_n), \dots) \quad \text{and}$$

$${}^t u(x') = ({}^t u_1(x'_1), {}^t u_2(x'_2), \dots, {}^t u_n(x'_n), \dots) .$$

If $x \in \bigoplus_{n=1}^{\infty} E_n$ we define the mapping f from L into $\prod_{n=1}^{\infty} E_n$ putting $f(x, y) = x + u(y)$. It is immediate that f is continuous and linear

and, therefore, ${}^t f$ is weakly continuous from $\overset{\infty}{\bigoplus}_{n=1} E'_n$ in

$$\left(\overset{\infty}{\prod}_{n=1} E'_n \right) \times \left(\overset{\infty}{\bigoplus}_{n=1} F'_n \right).$$

If $y' \in \overset{\infty}{\prod}_{n=1} E'_n$ and $z' \in \overset{\infty}{\bigoplus}_{n=1} F'_n$ are elements such that ${}^t f(x') = (y', z')$, we have that

$$\begin{aligned} \langle y', x \rangle + \langle z', y \rangle &= \langle (y', z'), (x, y) \rangle = \langle {}^t f(x'), (x, y) \rangle = \\ &= \langle x', f(x, y) \rangle = \langle x', x + u(y) \rangle = \langle x', x \rangle + \langle x', u(y) \rangle = \\ &= \langle x', x \rangle + \langle {}^t u(x'), y \rangle. \end{aligned}$$

then $\langle y', x \rangle + \langle z', y \rangle = \langle x', x \rangle + \langle {}^t u(x'), y \rangle$. In the last relation if we take $y = 0$ it follows that $y' = x'$, and if we take $x = 0$ it results that $z' = {}^t u(x')$. Therefore, ${}^t f(x') = (x', {}^t u(x'))$.

Let $M = \{(x', {}^t u(x')) : x' \in \overset{\infty}{\prod}_{n=1} E'_n\}$. Since, for every positive integer n , ${}^t u_n$ is weakly continuous from E'_n into F'_n we have that M is weakly closed in $\left(\overset{\infty}{\prod}_{n=1} E'_n \right) \times \left(\overset{\infty}{\prod}_{n=1} F'_n \right)$ and, therefore,

$$N = M \cap \left[\left(\overset{\infty}{\prod}_{n=1} E'_n \right) \times \left(\overset{\infty}{\bigoplus}_{n=1} F'_n \right) \right]$$

is weakly closed in $\left(\overset{\infty}{\prod}_{n=1} E'_n \right) \times \left(\overset{\infty}{\bigoplus}_{n=1} F'_n \right)$. On the other hand, if $(x', {}^t u(x')) \in N$, then ${}^t u(x') \in \overset{\infty}{\bigoplus}_{n=1} F'_n$ and, therefore, ${}^t u_n(x'_n)$ is zero for all indices except a finite number of them. Since ${}^t u_n$ is injective it follows that x'_n is zero for all indices except a finite number of them, hence $x' \in \overset{\infty}{\bigoplus}_{n=1} E'_n$ and, therefore, ${}^t f \left(\overset{\infty}{\bigoplus}_{n=1} E'_n \right) = N$. Since ${}^t f \left(\overset{\infty}{\bigoplus}_{n=1} E'_n \right)$ is weakly closed in $\left(\overset{\infty}{\prod}_{n=1} E'_n \right) \times \left(\overset{\infty}{\bigoplus}_{n=1} F'_n \right)$ we have that f is a topological homomorphism from $L[\sigma(L, L')]$ onto

$$H = f(L) \left[\sigma \left(f(L), \overset{\infty}{\bigoplus}_{n=1} E'_n \right) \right].$$

Let $L_p = \left(\bigoplus_{n=1}^p E_n \right) \times \left(\prod_{n=1}^{\infty} F_n \right)$ and let $H_p = f(L_p)$. Since $u_n(F_n)$ is separable and dense in E_n we have that H_1 is separable and dense in H . If z_n is an element of E_n such that $z_n \notin u_n(F_n)$ let

$$z^{(p)} = (z_1, z_2, \dots, z_p, 0, 0, \dots, 0, \dots).$$

The set $A = \{z^{(1)}, z^{(2)}, \dots, z^{(n)}, \dots\}$ is bounded in H and $z^{(p+1)} \notin H_p$. According to result a), there exists a dense subspace D of H , $D \neq H$, such that $D \cap H_p$ is finite-dimensional, $p = 1, 2, \dots$. If $G = f^{-1}(D)$, then $G \neq L$ and G is dense in L , since f is weakly open from L into H . Given in L a bounded closed set B such that $G \cap B$ is not closed, there is a point z in B , which is in the closure of $G \cap B$, with $z \notin G$. There exists a positive integer p_0 such that $B \subset L_{p_0}$. Since f is continuous, $f(z) \notin D$ and $f(z) \in \overline{f(G \cap B)} \subset \overline{D \cap f(B)}$. On the other hand, $D \cap f(B)$ is contained in the closed subspace $D \cap H_{p_0}$, hence $f(z)$, which belongs to D , is not in the closure of $D \cap f(B)$, which is a contradiction.

q.e.d.

THEOREM 2. — *If $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ are two sequences of arbitrary infinite-dimensional Banach spaces, then $\left(\bigoplus_{n=1}^{\infty} E_n \right) \times \left(\prod_{n=1}^{\infty} F_n \right)$ is not B_r -complete.*

Proof. — Let G_n and H_n be separable closed subspaces of infinite dimension of E_n and F_n , respectively. Since every closed subspace of a B_r -complete space is B_r -complete, [8], and $\left(\bigoplus_{n=1}^{\infty} G_n \right) \times \left(\prod_{n=1}^{\infty} H_n \right)$ is closed in $\left(\bigoplus_{n=1}^{\infty} E_n \right) \times \left(\prod_{n=1}^{\infty} F_n \right)$ it is enough to carry out the proof, taking E_n and F_n to be separable spaces, which will be supposed. If $\{x_p\}_{p=1}^{\infty}$ is a dense sequence in E_n , we can find a sequence $\{\alpha_p\}_{p=1}^{\infty}$ of non-zero numbers such that $\{\alpha_p x_p\}_{p=1}^{\infty}$ converges to the origin in E_n . The sequence $\{\alpha_p x_p\}_{p=1}^{\infty}$ is total in E_n , and it is equicontinuous in $E'_n[\mu(E'_n, E_n)]$. If V_n is the closed unit ball in F_n and V_n^0 is the polar set of V_n in F'_n , then V_n^0 is a separable weakly compact absolutely convex set which is total in $F'_n[\mu(F'_n, F_n)]$. Since $\{F'_n[\mu(F'_n, F_n)]\}_{n=1}^{\infty}$ and $\{E'_n[\mu(E'_n, E_n)]\}_{n=1}^{\infty}$ satisfy conditions 1 and 2, respectively, of Theorem 1, there exists in

$$L = \left(\bigoplus_{n=1}^{\infty} F'_n[\mu(F'_n, F_n)] \right) \times \left(\prod_{n=1}^{\infty} E'_n[\mu(E'_n, E_n)] \right)$$

a dense subspace G , $G \neq L$, such that G intersects every bounded closed subset of L in a closed subset of L and, therefore,

$$\left(\bigoplus_{n=1}^{\infty} E_n \right) \times \left(\prod_{n=1}^{\infty} F_n \right) \text{ is not } B_r\text{-complete.} \quad \text{q.e.d.}$$

THEOREM 3. — *If $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ are two sequences of infinite-dimensional Banach spaces, then*

$$\left(\prod_{n=1}^{\infty} E_n \right) \times \left(\bigoplus_{n=1}^{\infty} F'_n[\mu(F'_n, F_n)] \right)$$

is not B_r -complete.

Proof. — We take in E_n a separable closed subspace G_n , of infinite dimension. If V_n is the closed unit ball of F_n , let V_n^0 be the polar set of V_n in F'_n . We take in V_n^0 an infinite countable set B linearly independent. If H_n is the closed linear hull of B in $F'_n[\sigma(F'_n, F_n)]$ and A is the $\sigma(F'_n, F_n)$ -closed absolutely convex hull of B , then $H_n[\mu(H_n, H'_n)]$ has a separable weakly compact absolutely convex set A which is total. Reasoning in the same way than in Theorem 2 it is sufficient to carry out the proof when E_n is a separable space and $F'_n[\mu(F'_n, F_n)]$ is of the form $H_n[\mu(H_n, H'_n)]$. Then the sequences $\{E'_n[\mu(E'_n, E_n)]\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ satisfy the conditions of Theorem 1, hence it follows that the space $\left(\prod_{n=1}^{\infty} E_n \right) \times \left(\bigoplus_{n=1}^{\infty} F'_n[\mu(F'_n, F_n)] \right)$ is not B_r -complete. q.e.d.

COROLLARY 1.3. — *Let E be a product of countable infinitely many Banach spaces of infinite-dimension. Then $E \times E'[\mu(E', E)]$ is not B_r -complete.*

By analogous methods used in Theorems 2 and 3, we can obtain Theorems 4 and 5.

THEOREM 4. — *Let $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ be two sequences of Banach spaces of infinite dimension. If, for every positive integer n , E_n is separable, then $\left(\prod_{n=1}^{\infty} E'_n[\mu(E'_n, E_n)] \right) \times \left(\bigoplus_{n=1}^{\infty} F'_n[\mu(F'_n, F_n)] \right)$ is not B_r -complete.*

THEOREM 5. — *Let $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ be two sequences of Banach spaces of infinite dimension. If, for every positive integer n , F_n is separable, then $\left(\bigoplus_{n=1}^{\infty} E_n\right) \times \left(\prod_{n=1}^{\infty} F'_n[\mu(F'_n, F_n)]\right)$ is not B_r -complete.*

Note 1. — It is easy to show that Theorems 2, 3, 4 and 5 are valid changing the condition “Banach space” by “Fréchet space”, with the additional hypothesis : In Theorem 3, the topology of E_n will be defined by a family of norms ; in Theorems 2 and 4 the topology of F_n will be also defined by a family of norms. Let us suppose, now, given an infinite-dimensional nuclear Fréchet space F , its topology is defined by a family of norms. Since F is a Montel space then it is separable, [3], (see [5], p. 370). If we take $E_n = F_n = F$ and we apply the generalized Corollary 1.3 it results that $E = \prod_{n=1}^{\infty} E_n$ is a nuclear Fréchet space such that $E \times E'[\mu(E', E)]$ is not B_r -complete. If we apply the generalized Theorem 2 it results that $G = \left(\bigoplus_{n=1}^{\infty} E_n\right) \times \left(\prod_{n=1}^{\infty} F_n\right)$ is a nuclear strict (LF)-space which is not B_r -complete and, finally, if we apply the generalized Theorem 4 it follows that $G'[\mu(G', G)]$ is a countable product of complete (DF)-spaces which is not B_r -complete.

In ([1], p. 35) N. Bourbaki notices that it is not known if every bornological barrelled space is ultrabornological. In [9] we have obtained a wide class of bornological barrelled spaces which are not ultrabornological. In [10] we give an example of a bornological barrelled space which is not the inductive limit of Baire spaces. This example is not a metrizable space. In Theorem 6 we shall obtain a class \mathcal{A} of bornological barrelled spaces which are not inductive limits of Baire spaces, such that \mathcal{A} contains metrizable spaces.

In [10] we have given the following result : b) *Let E be a bornological barrelled space which has a family of subspaces $\{E_n\}_{n=1}^{\infty}$ such that the following conditions hold : 1) $\bigcup_{n=1}^{\infty} E_n = E$. 2) For every positive integer n , there is a topology \mathfrak{T}_n on E_n , finer than the initial one, such that $E_n[\mathfrak{T}_n]$ is a Fréchet space. 3) There is in E a bounded set A such that $A \not\subset E_n$, $n = 1, 2, \dots$. Then there is a bornological*

barrelled space F which is not an inductive limit of Baire space, such that E is a dense hyperplane of F .

THEOREM 6. — *If $\{G_i : i \in I\}$ is an infinite family of ultrabornological spaces, there is in $G = \Pi\{G_i : i \in I\}$ a dense subspace E , bornological and barrelled, which is not an inductive limit of Baire spaces, so that E contains an ultrabornological subspace F , of codimension one.*

Proof. — We take in I an infinite countable subset $\{i_1, i_2, \dots, i_n, \dots\}$. If G_{i_n} is of dimension one we put $G_{i_n} = K_n$. If G_{i_n} is not of dimension one we can take $G_{i_n} = K_n \oplus H_n$, being H_n a closed subspace of codimension one of G_{i_n} . The space G can be put in the form

$$\left(\prod_{n=1}^{\infty} K_n \right) \times \Pi\{L_j : j \in J\},$$

such that L_j is ultrabornological for every $j \in J$. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of infinite-dimensional separable Banach spaces and let $\{E_n\}_{n=1}^{\infty}$ be a sequence such that $E_n = \prod_{p=1}^{\infty} K_p$, $n = 1, 2, \dots$. The sequences $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ satisfy conditions of Theorem 1 and, therefore using the same notations as in Theorem 1 we have that $\mu\left(H, \bigoplus_{n=1}^{\infty} E'_n\right)$ can be identified with the topology induced in H by $\prod_{n=1}^{\infty} E_n$, since the last space is metrizable. Hence, $L/f^{-1}(0)$ can be identified with $H\left[\mu\left(H, \bigoplus_{n=1}^{\infty} E'_n\right)\right]$ and, therefore, there is on H_n a topology \mathfrak{T}_n , finer than the one induced by $\prod_{n=1}^{\infty} E_n$, such that $H_n[\mathfrak{T}_n]$ is a Fréchet space isomorphic to $L_n/(f^{-1}(0) \cap L_n)$. On the other hand, A is a bounded set of $H\left[\mu\left(H, \bigoplus_{n=1}^{\infty} E'_n\right)\right]$ such that $A \not\subset H_n$, $n = 1, 2, \dots$, whence it follows, applying result b), and since $\prod_{n=1}^{\infty} E_n$ is complete, that there is a point $x \in \prod_{n=1}^{\infty} E_n$, $x \notin H$, such that the linear hull S of $H \cup \{x\}$ is a dense subspace of $\prod_{n=1}^{\infty} E_n$, bornological and barrelled, which is not an inductive limit of Baire

spaces and $H \left[\mu \left(H, \bigoplus_{n=1}^{\infty} E'_n \right) \right]$ is an ultrabornological subspace of S , of codimension one. Since $\prod_{n=1}^{\infty} K_n$ is topologically isomorphic to $\prod_{n=1}^{\infty} E_n$ there is in $\prod_{n=1}^{\infty} K_n$ a dense subspace D which is bornological and barrelled, such that it is not an inductive limit of Baire spaces, and it has an ultrabornological subspace T , of codimension one. We take in $\{L_j : j \in J\}$ the subspace U such that $x \in U$ if, and only if, all the components of x are zero except a most a countable infinite number of them. The space U is ultrabornological, (see the proofs of Theorem 1 and Theorem 2 in [11]). If $E = D \times U$ and $F = T \times U$, then E and F hold the conditions of the theorem.

q.e.d.

In [12] and [15] we have given, respectively, the two following results : c) *If E is a reflexive strict (LF)-space, then $E'[\mu(E', E)]$ is ultrabornological.* d) *Let Ω be a non-empty open set in the n -dimensional euclidean space R^n . Let $\mathcal{O}'(\Omega)$ the space of distributions, with the strong topology. Then there is a topology \mathfrak{F} on $\mathcal{O}'(\Omega)$ coarser than the initial one, so that $\mathcal{O}'(\Omega) [\mathfrak{F}]$ is a bornological barrelled space which is not ultrabornological.* In Theorem 7 we extend the result d).

THEOREM 7. — *Let E be a reflexive strict (LF)-space. If $E'[\mu(E', E)]$ is not B_r -complete, then there exists in E' a topology \mathfrak{F} , coarser than $\mu(E', E)$, so that $E'[\mathfrak{F}]$ is a bornological barrelled space which is not an inductive limit of Baire spaces.*

Proof. — Let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence of Fréchet subspaces of E , such that E is the inductive limit of this sequence. Let G be a dense subspace of E , $G \neq E$, which intersects to every weakly compact absolutely convex subset of E in a closed set. Let $\mathfrak{F} = \mu(E', G)$. Obviously every closed subset of $G[\sigma(G, E')]$ is compact and, therefore, $E'[\mathfrak{F}]$ is barrelled. Let us see, now, that $E'[\mathfrak{F}]$ is bornological. By a theorem of Köthe, ([5], p. 386), we shall see that $G[\mathfrak{F}_{c_0}]$ is complete, \mathfrak{F}_{c_0} being the topology of the uniform convergence on every sequence of $E'[\mathfrak{F}]$ which converges to the origin in the Mackey sense. According to result c), we have that $E[\mu(E', E)_{c_0}]$ is complete. Since $E'[\mu(E', E)]$ is the Mackey dual of a (LF)-space, it follows that $E'[\mu(E', E)]$ is

complete and, therefore, $\mu(E', E)_{c_0}$ is compatible with the dual pair $\langle E, E' \rangle$. Since $G \cap E_n$ is closed in E , we have that $(G \cap E_n) [\mu(E', E)_{c_0}]$ is closed in $E[\mu(E', E)_{c_0}]$, hence it results that $(G \cap E_n) [\mu(E', E)_{c_0}]$ is complete and hence, applying a theorem of Bourbaki, ([5], p. 210) one deduces that $(G \cap E_n) [\mathfrak{T}_{c_0}]$ is complete. Let us suppose, now, that $G[\mathfrak{T}_{c_0}]$ is not complete. We take in the completion $\hat{G}[\hat{\mathfrak{T}}_{c_0}]$ of $G[\mathfrak{T}_{c_0}]$ an element x_0 which is not in G . Since $(G \cap E_n) [\mathfrak{T}_{c_0}]$ is complete, we have that $G \cap E_n$ is closed in $\hat{G}[\hat{\mathfrak{T}}_{c_0}]$, and we can find a continuous linear form u_n on $\hat{G}[\hat{\mathfrak{T}}_{c_0}]$, such that $\langle u_n, x_0 \rangle = 1$ and $\langle u_n, x \rangle = 0$, for every $x \in G \cap E_n$. Given any point $y_0 \in G$ there is a positive integer n_0 , such that $y_0 \in G \cap E_{n_0}$, and, therefore, $\langle nu_n, y_0 \rangle = 0$, for $n \geq n_0$, hence it deduces that $\{nu_n\}_{n=1}^\infty$ converges to the origin in $E'[\mu(E', G)]$, from here $\{u_n\}_{n=1}^\infty$ converges to the origin in $E'[\mu(E', G)]$ in the sense of Mackey and, therefore, $\{u_n\}_{n=1}^\infty$ is equicontinuous in $\hat{G}[\hat{\mathfrak{T}}_{c_0}]$. Since $\{\langle u_n, x \rangle\}_{n=1}^\infty$ converges to the origin for every $x \in G$, and G is dense in $\hat{G}[\hat{\mathfrak{T}}_{c_0}]$ it follows that $\{\langle u_n, x \rangle\}_{n=1}^\infty$ converges to the origin, for every $x \in \hat{G}[\hat{\mathfrak{T}}_{c_0}]$, which is a contradiction since $\langle u_n, x_0 \rangle = 1, n = 1, 2, \dots$. Thus, $G[\mathfrak{T}_{c_0}]$ is complete. Finally, if f is the identity mapping from $E'[\mu(E', E)]$ onto $E'[\mathfrak{T}]$, then f is continuous and f^{-1} is not continuous. Applying the closed graph theorem in the form given by De Wilde, [2], we can derive that $E'[\mathfrak{T}]$ is not an inductive limit of Baire spaces.

q.e.d.

THEOREM 8. — *If $\{E_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ are two sequences of infinite-dimensional reflexive Banach spaces there is on*

$$E = \left(\bigoplus_{n=1}^\infty E_n \right) \times \left(\prod_{n=1}^\infty F_n \right)$$

a topology \mathfrak{T} , coarser than the initial one, so that $E[\mathfrak{T}]$ is a bornological barrelled space which is not an inductive limit of Baire spaces.

Proof. — It is immediate consequence from Theorem 2 and Theorem 7.

Note 2. — In part, the method followed in the proof of Theorem 7 suggest to us the following short proof of the well-known result that if E is the strict inductive limit of an increasing sequence $\{E_n\}_{n=1}^\infty$ of complete spaces, then E is complete, [6], ([5], p. 224-225) : Suppose that E is not complete and let x_0 be a point of the completion \hat{E} of E , $x_0 \notin E$. Since E_n is closed in \hat{E} we can find $u_n \in (\hat{E})'$ such that $\langle u_n, x_0 \rangle = 1$, $\langle u_n, x \rangle = 0$ for every $x \in E_n$. The set of restrictions of $A = \{u_1, u_2, \dots, u_n, \dots\}$ to E_n is a finite set which is, therefore, equicontinuous in E_n . Hence A is equicontinuous in E and, therefore, A is equicontinuous in \hat{E} , hence A is relatively compact in $(\hat{E})'[\sigma((\hat{E})', \hat{E})]$. If u is a cluster point of the sequence $\{u_n\}_{n=1}^\infty$ in $(\hat{E})'[\sigma((\hat{E})', \hat{E})]$ then it is immediate that u is zero on E and $\langle u, x_0 \rangle = 1$, which is a contradiction since E is dense in \hat{E} .

J. Dieudonné has proved in [4] the following theorem : e) *If F is a subspace of finite codimension of a bornological space E , then F is bornological.* We have proved in [13] the following result : f) *If F is a subspace of finite codimension of a quasi-barrelled space E , then F is quasi-barrelled.*

In e) and f) it is not possible to change “finite codimension” by “infinite countable codimension”. In [10] we have given a example of a bornological space E which has a subspace F , of infinite countable codimension, which is not quasi-barrelled. In this example F is not dense in E . In Theorem 9, using in part the method followed in [10] we shall give a class \mathcal{Q} of bornological spaces, such that if $E \in \mathcal{Q}$ there is a dense subspace F of E , of infinite countable codimension, which is not quasi-barrelled.

We shall need the following results given in [10] and [14], respectively : g) *Let E be the strict inductive limit of an increasing sequence of metrizable spaces. Let F be a sequentially dense subspace of E . If E is a barrelled then F is bornological.* h) *Let E be a barrelled space. If $\{E_n\}_{n=1}^\infty$ is an increasing sequence of subspaces of E , such that $E = \bigcup_{n=1}^\infty E_n$, then E is the strict inductive limit of $\{E_n\}_{n=1}^\infty$.*

THEOREM 9. — *If $\{E_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ are two sequences of infinite-dimensional separable Banach spaces, there is in*

$$L = \left(\bigoplus_{n=1}^\infty E_n \right) \times \left(\prod_{n=1}^\infty F_n \right)$$

a bornological dense subspace E , such that E contains a dense subspace F , of infinite countable codimension, which is not quasi-barrelled.

Proof. — The sequences $\{E_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ satisfy the conditions of Theorem 1 and, therefore, there is in L a dense subspace F , $F \neq L$, which intersects to every bounded closed subset of L in a closed subset of L . Let A_n and B_n be respectively two countable dense subsets of E_n and F_n , considered as subspaces of L . Let H be the linear hull of $\bigcup_{n=1}^\infty (A_n \cup B_n)$. If E is the linear hull of $H \cup F$, with the topology induced by the topology of L , then E is bornological, according to result g), since $E \cap \left(\bigoplus_{n=1}^p E_n \right) \times \left(\prod_{n=1}^\infty F_n \right)$ is dense in

$$\left(\bigoplus_{n=1}^p E_n \right) \times \left(\prod_{n=1}^\infty F_n \right).$$

Suppose that F is quasi-barrelled. Then F is barrelled, since F is quasi-complete and, therefore, by result h), F is the inductive limit of the sequence of complete spaces

$$\left\{ F \cap \left(\bigoplus_{n=1}^p E_n \right) \times \left(\prod_{n=1}^\infty F_n \right) \right\}_{p=1}^\infty$$

and, therefore, F is complete, hence $F = L$, which is a contradiction. Thus, F is not quasi-barrelled. Since H has countable dimension, then F is of countable codimension in E and, by result f), F is of infinite countable codimension in E .

q.e.d.

BIBLIOGRAPHY

- [1] N. BOURBAKI, *Eléments de Mathématiques, Livre V : Espaces vectoriels topologiques*, (Ch. III, Ch. IV, Ch. V), Paris, (1964).
- [2] M. De WILDE, Réseaux dans les espaces linéaires à semi-normes, *Mém. Soc. Royale des Sc. de Liège*, 5^e, série, XVIII, 2, (1969).

- [3] J. DIEUDONNE, Sur les espaces de Montel metrizable, *C.R. Acad. Sci. Paris*, 238, (1954), 194-195.
- [4] J. DIEUDONNE, Sur les propriétés de permanence de certains espaces vectoriels topologiques, *Ann. Soc. Polon. Math.*, 25 (1952), 50-55.
- [5] G. KÖTHE, Topological Vector Spaces I, Berlin-Heidelberg-New York, Springer (1969).
- [6] G. KÖTHE, Über die Vollständigkeit einer Klasse lokalconvexer Räume, *Math. Nachr. Z.*, 52.(1950), 627-630.
- [7] J.T. MARTI, Introduction to the theory of Bases, Berlin-Heidelberg-New York, Springer (1969).
- [8] V. PTAK, Completeness and the open mapping theorem, *Bull. Soc. Math. France.*, 86 (1958), 41-47.
- [9] M. VALDIVIA, A class of bornological barrelled spaces which are not ultrabornological, *Math. Ann.*, 194 (1971), 43-51.
- [10] M. VALDIVIA, Some examples on quasi-barrelled spaces, *Ann. Inst. Fourier*, 22 (1972), 21-26.
- [11] M. VALDIVIA, On nonbornological barrelled spaces, *Ann. Inst. Fourier*, 22 (1972), 27-30.
- [12] M. VALDIVIA, On countable strict inductive limits, *Manuscripta Mat.*, 11 (1971), 339-343.
- [13] M. VALDIVIA, A hereditary property in locally convex spaces, *Ann. Inst. Fourier*, 21 (1971), 1-2.
- [14] M. VALDIVIA, Absolutely convex sets in barrelled spaces, *Ann. Inst. Fourier*, 21 (1971), 3-13.
- [15] M. VALDIVIA, The space of distributions $\mathcal{D}'(\Omega)$ is not B_r -complete, *Math. Ann.*, 211 (1974), 145-149.

Manuscrit reçu le 22 juin 1974
accepté par J. Dieudonné.

Manuel VALDIVIA,
Facultad de Ciencias
Paseo Valencia Al Mar, 13
Valencia (España).