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A REMARK ON THURSTON'S STABILITY THEOREM

by Richard SACKSTEDER

Let L be a compact leaf of a smooth transversally oriented foliation of codimension one. Thurston [4] has generalized Reeb's stability theorem by showing that if $H_1(L, \mathbb{R}) = 0$, then all nearby leaves are diffeomorphic to L . His theorem answers, for oriented foliations, a question posed by Reeb [2]. If it were true, as has been erroneously asserted in the literature [3, p. 96], that $H_1(L, \mathbb{R}) = 0$ implies that $H_1(L', \mathbb{R}) = 0$ when L' is a 2-fold cover of L , then Thurston's assumption of transversal orientability would be unnecessary. However the assertion is false (cf. [1, p. 410]), as has been pointed out to the author painfully often.

In fact, the example below shows that Thurston's theorem cannot be generalized to non-oriented foliations, since in the example there is a compact leaf L with $H_1(L, \mathbb{R}) = 0$, but which has a neighborhood in which all leaves are non-compact.

The universal covering of L is $S^2 \times \mathbb{R}^1$ and $\pi = \pi_1(L)$ is the semi-direct product of $\mathbb{Z}_2 = \{-1, +1\}$ and the integers \mathbb{Z} , where \mathbb{Z}_2 acts on \mathbb{Z} in the obvious way. The product of elements of π is given by

$$(w_1, n_1) \cdot (w_2, n_2) = (w_1 w_2, w_1 n_2 + n_1),$$

where w_i is in \mathbb{Z}_2 and n_i is in \mathbb{Z} . The action ϕ of π on $S^2 \times \mathbb{R}$ is given by

$$\phi((w, n); (s, r)) = (ws, wr + n), \quad \text{where } s \rightarrow -s$$

is the antipodal map of S^2 . The quotient L is an oriented manifold that is easily seen to have the properties that $H_1(L, \mathbb{Z}) = \mathbb{Z}_2$, hence $H_1(L, \mathbb{R}) = 0$, and $S^2 \times S^1$ is a 2-fold cover of L .

To define a foliation of a neighborhood of L it suffices to define a representation ψ of π by C^∞ diffeomorphisms of neighborhoods of $0 \in \mathbb{R}$. Let f be any C^∞ diffeomorphism of \mathbb{R} satisfying :

$$f(0) = 0, f'(0) = 1, f^{(n)}(0) = 0 \quad \text{if } n > 1, f(x) < x$$

for $x \neq 0$, and

$$(1) f(x) = -f^{-1}(-x), \text{ hence } f^n(x) = -f^{-n}(-x) \text{ for } n = 0, \pm 1, \dots$$

An f satisfying these conditions is easily defined for $x \geq 0$ and can be extended to $x < 0$ by (1). It is easy to check that the derivatives match at 0 so the extended map is C^∞ . The second half of (1) shows that $\psi(w, n)(x) = f^n(wx)$ defines a representation of π with the desired properties. The leaves, other than L itself, of the foliation defined by ψ are non-compact, since $f^n(wx) = x$ can only occur if $(w, n) = (1, 0)$, or $x = 0$.

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