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INVARIANT SUBSPACES ON OPEN RIEMANN SURFACES

by Morisuke HASUMI

1. Introduction.

The purpose of the present paper is to classify completely the closed invariant subspaces of the L^p spaces with respect to a harmonic measure on the Martin boundary of a certain hyperbolic Riemann surface. Our problem has its origin in a famous paper [1] of Beurling, where he characterized, among others, the closed shift-invariant subspaces of the Hardy class H^2 on the unit disk. He showed that such a subspace is generated by a single inner function. In recent years, efforts have been directed to extending this result to multiply connected regions. We now know what happens for any bordered compact Riemann surface, due to works by Voichick [15, 16], Forelli [4] and the author [5]. Very recently, in his thesis [6] (see also [7]), Neville has studied extensively the invariant subspaces of the Hardy classes on certain infinitely connected plane regions called Blaschke regions and has obtained quite remarkable results. In a very long forthcoming paper [8], he has generalized his thesis results further to a class of Riemann surfaces including all Blaschke regions. The main result of the present paper will be general enough to imply all these previous results.

In this paper, we shall deal with a class of hyperbolic Riemann surfaces satisfying conditions (A), (B) and (C). Our conditions are almost the same as those discussed by Neville [8] and will be stated in Section 5. In order to prove our main result (Theorem 7.1), we shall follow the program developed by Neville [6]. Namely, we shall first prove a generalized Cauchy's theorem and its converse formulated in terms of the Martin boundary. At one delicate point, we shall employ the Brelot-Choquet theory of Green lines [2]. Once we get Cauchy theorems, it will not be so hard to determine the closed (weakly* closed, if $p = \infty$) invariant subspaces of L^p on the Martin boundary of our surface. The results concerning the Hardy classes can then be deduced rather quickly.

Now we sketch the contents of this paper. In Section 2, we shall list some basic facts, taken from Neville [6, 8], about the inner-outer factorization of certain meromorphic functions on a hyperbolic Riemann surface R and also about the Hardy classes $H^p(R)$. In Section 3, we shall give the integral representation of functions in certain classes $h^p(R)$ of harmonic functions on R and study the duality of such spaces. After proving a Cauchy theorem in its weaker form in Section 4, we shall establish in Section 5 direct and inverse Cauchy theorems for R satisfying the conditions (A), (B) and (C) (Theorems 5.3 and 5.12). Section 6 will contain further properties of the lifting operation from the surface R to its universal covering surface. Finally in Section 7, we shall determine the closed $H^\infty(R)$-submodules of the spaces L^p on the Martin boundary of R and prove, as a special case, the characterization theorem of the closed $H^\infty(R)$-submodules of $H^p(R)$ (Corollary 7.2).

The present paper came out of our efforts to answer some open questions posed in Neville's thesis [6]. After the first draft of this paper was written, we were informed that Neville himself had already found the same direct and inverse Cauchy theorems as well as the same characterization of the closed invariant subspaces of the Hardy classes prior to our discovery. His results will appear in [8]. But the two works look different in techniques. His discussion is based on the Hayashi boundary, whereas ours on the Martin boundary. By using the Martin boundary, we shall be able to give a much shorter exposition of the main results in [8]. Furthermore, our techniques will allow us to classify the closed invariant subspaces of the L^p spaces on the Martin boundary of our surface, which we believe is new. On the other hand, H. Widom has informed us that our condition (B) implies the condition (C) for any Riemann surface, independently of (A). So the conditions (A) and (B) alone will imply all our results. But we leave our conditions unchanged, in the hope that the conditions may be weakened in some way or other.

We were benefited in every way from Neville's thesis [6] and its influence on the present paper is quite evident. We wish to thank Professor Lee A. Rubel for having allowed us to see this very interesting thesis as soon as it was completed. Our thanks are also due to Professor Harold Widom for supplying us the valuable remark.

2. Definitions and some basic facts.

This section contains a brief sketch of some basic results in Neville [6, 8]. Let R be a hyperbolic Riemann surface, which will be fixed throughout this section. For any domain D on R, HP(D) will denote the real vector space of functions on D which can be expressed as the difference of two positive harmonic functions on D. Let $u_i \in \mathrm{HP}(\mathrm{R} \sim \mathrm{Z}_i)$, $i = 1, 2$, where Z_1 and Z_2 are discrete subsets of R. We identify u_1 and u_2 if there is a discrete subset Z_3 of R such that $\mathrm{Z}_1 \cup \mathrm{Z}_2 \subseteq \mathrm{Z}_3$ and $u_1 = u_2$ on $\mathrm{R} \sim \mathrm{Z}_3$. The union of the sets $\mathrm{HP}(\mathrm{R} \sim \mathrm{Z})$, with discrete $\mathrm{Z} \subseteq \mathrm{R}$, after the above identification, is denoted by SP(R). If $u \in \mathrm{HP}(\mathrm{R} \sim \mathrm{Z})$ with discrete $\mathrm{Z} \subseteq \mathrm{R}$, then every point a in Z is seen to be either a logarithmic singularity of u or a removable one.

PROPOSITION 2.1 ([8; Theorem 2.2.1]). — SP(R) *is a vector lattice with respect to the pointwise operations. It is order complete in the sense that, if* $\{u_\lambda\} \subseteq \mathrm{SP}(\mathrm{R})$ *and if there exists an element* $u \in \mathrm{SP}(\mathrm{R})$ *with* $u_\lambda \leqslant u$ *for all* λ, $\vee u_\lambda$ *exists in* SP(R).

For each $u \in \mathrm{SP}(\mathrm{R})$, we put $\|u\| = u \vee (-u)$. For each subset A of SP(R), we define $\mathrm{A}^\perp$ to be the set of all u in SP(R) such that $\|u\| \wedge \|v\| = 0$ for any $v \in \mathrm{A}$. We put $\mathrm{I}(\mathrm{R}) = \{1\}^\perp$ and $\mathrm{Q}(\mathrm{R}) = \mathrm{I}(\mathrm{R})^\perp$. A function in I(R) (resp., Q(R)) is called inner (resp., outer or quasibounded).

PROPOSITION 2.2 ([8; Theorem 2.2.2]). — *Both* I(R) *and* Q(R) *are bands of* SP(R) *and* $\mathrm{SP}(\mathrm{R}) = \mathrm{I}(\mathrm{R}) \oplus \mathrm{Q}(\mathrm{R})$. *The* *projection maps* p_{I} *and* p_{Q} *associated with this decomposition are positive.*

For any $u \in \mathrm{SP}(\mathrm{R})$, $p_{\mathrm{I}}(u)$ and $p_{\mathrm{Q}}(u)$ are called the inner and the outer parts of u, respectively. They are also denoted as u_{I} and u_{Q}, respectively. The following two facts are easily seen.

PROPOSITION 2.3. — *For any* $u \in \mathrm{SP}(\mathrm{R})$, *its outer parts* u_{Q} *has no irremovable singularities, so that* u *and* u_{I} *have the same singularities.*

PROPOSITION 2.4. — *For any* $u \in \mathrm{SP}(\mathrm{R})$, *we have*

$$u_{\mathrm{Q}} = \lim_{m \to \infty} \lim_{n \to \infty} [(-m) \vee (n \wedge u)].$$

Now let f be a meromorphic function of bounded characteristic on R, i.e., $f = f_1/f_2$ with bounded analytic functions f_1 and f_2 on R. Then, $\log |f| = \log |f_1| - \log |f_2|$ is contained in $\mathrm{SP}(\mathrm{R})$, so that $\log |f|$ ($= u$, say) is decomposed into its inner and outer parts u_{I} and u_{Q}. We put $f_{\mathrm{I}} = \exp(u_{\mathrm{I}} + i(u_{\mathrm{I}})_*)$ and $f_{\mathrm{Q}} = \exp(u_{\mathrm{Q}} + i(u_{\mathrm{Q}})_*)$, where the asterisk denotes the harmonic conjugate normalized in some fixed way. Then, f_{I} and f_{Q} are multiplicative meromorphic functions of bounded characteristic and $|f| = |f_{\mathrm{I}}||f_{\mathrm{Q}}|$, where f_{Q} is analytic in view of Proposition 2.3. Here, multiplicativity of a (multiple valued) meromorphic function h on R means the following. Let $\mathrm{H}_1(\mathrm{R}; \mathbf{Z})$ be the first singular homology group of R with integral coefficients and let Π be the group of multiplicative characters of $\mathrm{H}_1(\mathrm{R}; \mathbf{Z})$. Then, the multiplicativity of h means that, if h_1 is any function element of h at a point $a \in \mathrm{R}$ and if h_2 denotes the function element of h at the same point a which is obtained by the analytic continuation of h_1 along the path $\alpha \in \mathrm{H}_1(\mathrm{R}; \mathbf{Z})$ issuing from a, we have $h_2 = \theta(\alpha)h_1$, where θ is an element of Π determined uniquely by h. The character θ is called the character of h and denoted as $\theta(h)$. We call a nonnegative extended real-valued function u on R a *locally meromorphic modulus* (l.m.m.) if there exists a multiplicative meromorphic function f on R with $u = |f|$. If this f is of bounded characteristic, then u is said to be of bounded characteristic. If f is analytic, then u is called a *locally analytic modulus* (l.a.m.). Clearly, u is an l.m.m. of bounded characteristic if and only if $\log u \in \mathrm{SP(R)}$. An l.m.m. u of bounded characteristic is called *inner* (resp., *outer*) if $\log u \in \mathrm{I(R)}$ (resp., $\mathrm{Q(R)}$).

PROPOSITION 2.5 ([8; Theorem 2.3.1]). — *Every* l.m.m. u *of bounded characteristic can be factored uniquely into the product of an inner* l.m.m. u_{I} *and an outer* l.a.m. u_{Q}, *where* $u_{\mathrm{I}} = \exp(p_{\mathrm{I}}(\log u))$ *and* $u_{\mathrm{Q}} = \exp(p_{\mathrm{Q}}(\log u))$.

Next we shall define Hardy classes on R in the sense of Rudin. For $0 < p < \infty$, $\mathrm{H}^p(\mathrm{R})$ will denote the set of analytic functions f on R for which $|f|^p$ has a harmonic majorant. $\mathrm{H}^\infty(\mathrm{R})$ will denote the set of bounded analytic functions on R. Let $a_0 \in \mathrm{R}$ be fixed. For $f \in \mathrm{H}^p(\mathrm{R})$ with $0 < p < \infty$, we put $\|f\|_p = ((\mathrm{L.H.M.}(|f|^p)(a_0))^{1/p}$, where L.H.M. stands for the least harmonic majorant. For $f \in \mathrm{H}^\infty(\mathrm{R})$,

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathrm{R}\}.$$

Then it is well known that, for $1 \leqslant p \leqslant \infty$, the space $\mathrm{H}^p(\mathrm{R})$ is a complex Banach space with respect to the pointwise operations and the norm $\|\,.\,\|_p$ and that $\mathrm{H}^\infty(\mathrm{R})$ is a Banach algebra. Each $\mathrm{H}^p(\mathrm{R})$ with $1 \leqslant p \leqslant \infty$ is a topological $\mathrm{H}^\infty(\mathrm{R})$-module.

As is well known, the open unit disk, U, can be viewed as a universal covering Riemann surface of R. Let φ be the conformal covering map from U onto R such that $\varphi(0) = a_0$. Let T be the group of covering transformations for φ, i.e., the group of fractional linear transformations τ of U onto itself such that $\varphi \circ \tau = \varphi$. Put

$$\mathrm{SP_T} = \{s \in \mathrm{SP(U)} : s \circ \tau = s \text{ for any } \tau \in \mathrm{T}\}.$$

PROPOSITION 2.6 ([8; Theorem 2.4.1]). — *The mapping* $s \to s \circ \varphi$ *gives a vector lattice isomorphism of* $\mathrm{SP(R)}$ *onto* $\mathrm{SP_T}$. *For* $u \in \mathrm{SP(R)}$, $u \in \mathrm{I(R)}$ (*resp.,* $\mathrm{Q(R)}$) *if and only if* $u \circ \varphi \in \mathrm{I(U)}$ (*resp.,* $\mathrm{Q(U)}$).

We note that, for any l.a.m. u such that u^p has a harmonic majorant, there exists an analytic function f on U such that $u \circ \varphi = |f|$. In this case, $|f|^p$ has a harmonic majorant on U, so that f is in $H^p(U)$. If g is an analytic function on U and if $g \circ \tau = g$ for all $\tau \in T$, then there exists an analytic function h on R such that $h \circ \varphi = g$.

PROPOSITION 2.7 ([8; Theorem 2.5.3]). — *Let* u *be an* l.a.m. *on* R *such that either* u *is bounded or* u^p *has a harmonic majorant for some* $1 \leqslant p < \infty$. *Then,*

(*a*) $\mathrm{L.H.M.}(u^p) \in Q(R)$ *if* $p \neq \infty$;

(*b*) u *is of bounded characteristic*;

(*c*) $(\log u) \vee 0 \in Q(R)$;

(*d*) u_I *is a bounded* l.a.m. *and* $\|u_I\|_\infty = 1$.

3. Martin boundary and integral representation.

In this section we shall interprete some results in Neville [6] in terms of the Martin compactification theory found, for instance, in Constantinescu and Cornea [3]. Let R be a hyperbolic Riemann surface, R^* its Martin compactification, and $\Delta = R^* \sim R$ the Martin ideal boundary. Let $G(a, z) = G_a(z)$ be the Green function for R with pole at a point $a \in R$. We shall denote by k_b, $b \in R^*$, the Martin function with pole at b, which is defined as follows. Take a point a_0 in R, which is fixed throughout the discussion, and let α_0 be a fixed positive number so large that

$$\{z \in R : G(a_0, z) \geqslant \alpha_0\}$$

is a parametric disk on R. Let Φ be an indefinitely differentiable real function on $[-\infty, +\infty]$ such that $\Phi(t) \leqslant t$, $\Phi(t) = t$ for $t \leqslant 0$, Φ is constant for $t \geqslant 1$, and

$$d^2\Phi/dt^2 \leqslant 0.$$

We put $\Phi_0(t) = \Phi(t - \alpha_0) + \alpha_0$. Then, we define

$$k_b(z) = G(b, z)/\Phi_0(G(b, a_0))$$

for $b, z \in R$. The function $b \to k_b$, $b \in R$, is then extended by continuity to R^* and we get the Martin functions k_b for $b \in R^*$. Let Δ_1 be the set of points $b \in \Delta$ such that k_b is a minimal harmonic functions on R. Then, Δ_1 is a G_δ subset of Δ. The fundamental role of Δ_1 in the integral representation of harmonic functions on R is given by the following

Proposition 3.1 ([3: Folgesatz 13.1]). — *There exists a unique vector lattice isomorphism* $u \to \mu_u$ *of* HP(R) *onto the space* $M(\Delta_1)$ *of finite real regular Borel measures on* Δ_1 *such that*

$$u = \int_{\Delta_1} k_b \, d\mu_u(b).$$

Let χ *denote the measure corresponding to the constant function* 1. *Then,* $u \in HP(R)$ *is outer (resp., inner) if and only if the measure* μ_u *is absolutely continuous (resp., singular) with respect to* χ.

We note that χ is the harmonic measure on Δ_1 for the point a_0. We say that a function on Δ_1 is measurable (resp., integrable) if it is so with respect to χ, and that a property holds a.e. on Δ_1 if it holds on Δ_1 a.e. with respect to χ.

Next we shall define the boundary values of a function defined on R. For a positive superharmonic function s on R and a closed subset F of R, we define s_F to be the greatest lower bound of the positive superharmonic functions which are not smaller than s quasi-everywhere on the set F. Now let $b \in \Delta_1$. We shall denote by $\mathcal{G}_b = \mathcal{G}_b(R)$ the family of nonempty open subsets D of R such that $k_b \neq (k_b)_{R \sim D}$. Then, $\mathcal{G}_b$ is seen to be a filter base for each $b \in \Delta_1$.

Let f be any function from R into the complex sphere Ω. For $b \in \Delta_1$, we put $f^\wedge(b) = \cap\{\mathrm{Cl}\, f(D) : D \in \mathcal{G}_b\}$. Clearly, $f^\wedge(b)$ depends only on the values of f taken on the outside of any compact set in R. So the same definition can be made when f is defined only off some compact subset of R. Let $\mathcal{D}(f)$ be the set of $b \in \Delta_1$ for which $f^\wedge(b)$ is a singleton. We define $\hat{f}(b)$ for each $b \in \mathcal{D}(f)$ by the condition

$$\{\hat{f}(b)\} = f^\wedge(b)$$

and call $\hat{f}$ the *boundary function* for f. Then we have the following

Proposition 3.2 ([3; Hilfssätze 14.1, 14.2]). — *Suppose that a function* $f\colon R \to \Omega$ *is continuous outside a compact* *subset of* R. *Then:*

a) For any open neighborhood D′ *of* $\hat{f}(b)$ *in* Ω, $f^{-1}(D')$ *contains a set of* $\mathcal{G}_b$.

b) The function $\hat{f} : \mathcal{D}(f) \to \Omega$ *is measurable. In particular,* $\mathcal{D}(f)$ *is a Borel subset of* Δ_1.

As for harmonic functions on R, we have the following

PROPOSITION 3.3 ([3; Folgesatz 14.2]). — *If* $u \in HP(R)$, *then* $\hat{u}$ *exists* a.e. *on* Δ_1 *and the outer part of* u *is given by*

$$\int_{\Delta_1} \hat{u}(b) k_b \, d\chi(b)$$

In particular, if $u \in HP(R)$ is outer, then the measure $d\mu_u$ given by Proposition 3.1 is equal to $\hat{u}\, d\chi$. So we have

COROLLARY 3.4. — *If* u^* *is a real integrable function on* Δ_1, *then*

$$u = \int_{\Delta_1} u^*(b) k_b \, d\chi(b) \tag{1}$$

is an outer harmonic function in HP(R) *and* $\hat{u} = u^*$ a.e. *on* Δ_1.

Let $h^p(R)$, $1 \leqslant p < \infty$, be the space of complex-valued harmonic functions f on R such that $|f|^p$ has a harmonic majorant, and $h^\infty(R)$ the space of complex-valued bounded harmonic functions on R. We define the norm $\|\,.\,\|_p$ in $h^p(R)$, $1 \leqslant p < \infty$ by setting $\|f\|_p = ((\mathrm{L.H.M.}(|f|^p))(a_0))^{1/p}$ and the norm $\|\,.\,\|_\infty$ in $h^\infty(R)$ by $\|f\|_\infty = \sup \{|f(z)| : z \in R\}$. We shall denote by the symbol $h[u^*]$ the right-hand member of (1).

THEOREM 3.5. — *Let* $1 \leqslant p \leqslant \infty$. *For each* $f \in h^p(R)$, *the boundary function* $\hat{f}$ *is defined* a.e. *on* Δ_1 *and belongs to* $L^p(d\chi)$. *Put* $Sf = \hat{f}$. *Then,* S *is a linear map of* $h^p(R)$ *into* $L^p(d\chi)$ *such that.*

a) S *is isometric and surjective for* $1 < p \leqslant \infty$,

b) S *is norm-decreasing for* $p = 1$, *and is isometric as well as surjective on the space* $h^1_Q(R)$ *of all outer functions in* $h^1(R)$. S *is isometric and surjective on* $H^1(R)$.

Proof. — Consider the universal covering surface (U, φ) of R such that $\varphi(0) = a_0$. For $f \in h^p(R)$ with $1 \leqslant p \leqslant \infty$ (or $f \in H^1(R)$), we have $f \circ \varphi \in h^p(U)$ (or $f \circ \varphi \in H^1(U)$). We know that $f \circ \varphi$ is outer and so, by Proposition 2.6, f is outer, too. By Proposition 3.3, $\hat{f}$ exists a.e. on Δ_1, belongs to $L^1(d\chi)$ and

$$f = \int_{\Delta_1} \hat{f}(b) k_b \, d\chi(b).$$

Suppose first that $1 < p < \infty$ and $f \in h^p(R)$. Put

$$u = \text{L.H.M.}(|f|^p).$$

Since $|f|^p \leqslant u$, it follows that $|\hat{f}|^p \leqslant \hat{u}$ a.e. on Δ_1 and so $\hat{f} \in L^p(d\chi)$. The Hölder inequality then shows that

$$|f(z)|^p = \left| \int_{\Delta_1} \hat{f}(b) k_b(z) \, d\chi(b) \right|^p \leqslant \int_{\Delta_1} |\hat{f}(b)|^p k_b(z) \, d\chi(b).$$

Namely, $|f|^p \leqslant h[|\hat{f}|^p]$ and therefore $u \leqslant h[|\hat{f}|^p]$. Since $Q(R)$ is an order ideal, u is outer. So, $h[|\hat{f}|^p] \leqslant h[\hat{u}] = u$. Hence, we have $u = h[|\hat{f}|^p]$. Consequently we have

$$\begin{aligned} \|f\|_p^p = u(a_0) = h[|\hat{f}|^p](a_0) &= \int_{\Delta_1} |\hat{f}(b)|^p k_b(a_0) \, d\chi(b) \\ &= \int_{\Delta_1} |\hat{f}(b)|^p \, d\chi(b) = \|Sf\|_p^p. \end{aligned}$$

Thus, S is isometric. Surjectivity of S is obvious.

The case $p = \infty$ can be treated similarly.

Finally, let $f \in h^1(R)$. Then, by Proposition 3.3,

$$f_Q = h[\hat{f}] = \int_{\Delta_1} \hat{f}(b) k_b \, d\chi(b).$$

Let $u = \text{L.H.M.}(|f_Q|)$. Then we have $u = h[|\hat{f}_Q|] = h[|\hat{f}|]$, so that $S : h_Q^1(R) \to L^1(d\chi)$ is isometric and surjective. Next, let $v = \text{L.H.M.}(|f|)$ and let $v = v_I + v_Q$ be the inner-outer decomposition of v. Then, $|\hat{f}| \leqslant \hat{v} = \hat{v}_Q$ a.e. on Δ_1. So, $\|\hat{f}\|_1 = v_Q(a_0) \leqslant v(a_0) = \|f\|_1$. Thus, $S : h^1(R) \to L^1(d\chi)$ is norm-decreasing. The result for $H^1(R)$ comes from the fact $H^1(R) \subseteq h_Q^1(R)$. Q.E.D.

Now we introduce the notion of β topology (or strict topology) in a space H of bounded functions on R as follows. Let $C_0(R)$ be the space of continuous complex functions f on R such that $\{z \in R : |f(z)| \geqslant \varepsilon\}$ is compact for any $\varepsilon > 0$. Then, a net $\{h_\lambda\}$ in H is defined to converge to an $h \in \mathrm{H}$ with respect to the β topology if $(h_\lambda - h)f \to 0$ uniformly for each $f \in \mathrm{C}_0(\mathrm{R})$. This topology has been studied extensively for the spaces of bounded analytic functions by Rubel and Shields [11] and Neville [8; Chapter 4, Section 5].

Theorem 3.6. — *For $1 < p < \infty$, the Banach space dual of $h^p(\mathrm{R})$ is isometrically isomorphic with $\mathrm{L}^{p'}(d\chi)$ with*

$$p^{-1} + p'^{-1} = 1,$$

where the duality is given by

$$\langle f, g^* \rangle = \int_{\Delta_1} (\mathrm{S}f)(b) g^*(b)\, d\chi(b)$$

for $f \in h^p(\mathrm{R})$ and $g^* \in \mathrm{L}^{p'}(d\chi)$. For $p = 1$, the Banach space dual of $h^1_{\mathrm{Q}}(\mathrm{R})$ is isometrically isomorphic with $\mathrm{L}^\infty(d\chi)$. The dual of the space $h^\infty(\mathrm{R})$ equipped with the β topology is identified with $\mathrm{L}^1(d\chi)$.

Proof. — The last statement is a direct consequence of the theory of the β topology. Other assertions are also simple consequences of Theorem 3.5 and the duality theory of L^p spaces.

4. A preliminary Cauchy theorem.

We again consider a hyperbolic Riemann surface R and use the notations in the preceding section. Let f be a real continuous function defined on $\mathrm{R} \sim \mathrm{K}$, where K is any compact subset of R. Let $\overline{\mathscr{W}}[f]$ (resp., $\underline{\mathscr{W}}[f]$) be the class of superharmonic (resp., subharmonic) functions s on R for which there exists a compact subset K_s of R with $s \geqslant f$ (resp., $s \leqslant f$) on $\mathrm{R} \sim (\mathrm{K} \cup \mathrm{K}_s)$. If neither $\overline{\mathscr{W}}[f]$ nor $\underline{\mathscr{W}}[f]$ is empty, put $\overline{\mathrm{W}}[f](z) = \inf \{s(z) : s \in \overline{\mathscr{W}}[f]\}$ and $\underline{\mathrm{W}}[f](z) = \sup \{s(z) : s \in \underline{\mathscr{W}}[f]\}$ for $z \in \mathrm{R}$. Then, both $\overline{\mathrm{W}}[f]$ and $\underline{\mathrm{W}}[f]$ are harmonic functions on R and

$$\underline{\mathrm{W}}[f] \leqslant \overline{\mathrm{W}}[f].$$

If these functions coincide, then we denote the common function by $\mathrm{W}[f]$.

Suppose that the surface R is regular in the sense of potential theory, i.e., the set

$$\{z \in \mathrm{R} : \mathrm{G}(a, z) \geqslant \varepsilon\}$$

is compact for any $a \in \mathrm{R}$ and any $\varepsilon > 0$. Let $a \in \mathrm{R}$. Since the set of critical points of G_a is at most countable, we can find a monotonically decreasing sequence $\{\varepsilon_n\}$ of positive numbers converging to zero, in such a way that

$$\mathrm{R}_n = \{z \in \mathrm{R} : \mathrm{G}(a, z) > \varepsilon_n\},\ n = 1, 2, \ldots,$$

are Jordan regions, $\mathrm{Cl}\,\mathrm{R}_n \subseteq \mathrm{R}_{n+1}$ for $n = 1, 2, \ldots,$

$$\bigcup_{n=1}^{\infty} \mathrm{R}_n = \mathrm{R},$$

and $\delta\mathrm{G}(a, z) = 2\,\partial_z \mathrm{G}(a, z)\, dz$ is non-vanishing on each $\partial \mathrm{R}_n$, where $\partial_z = \dfrac{1}{2}(\partial_x - i\partial_y)$ denotes the partial differentiation with respect to $z = x + iy$ for any local coordinate. We call such an exhaustion $\{\mathrm{R}_n\}$ of R a *regular exhaustion* of R with center a. Now we show the following

LEMMA 4.1. — *Let* K *be a compact subset of* R *and* F *a positive continuous Wiener function on* $\mathrm{R} \sim \mathrm{K}$, *in the sense of* [3; p. 55], *such that there exists an outer harmonic function* u *on* R *with* $0 \leqslant \mathrm{F} \leqslant u$ *on* $\mathrm{R} \sim \mathrm{K}$. *Then, the boundary function* $\hat{\mathrm{F}}$ *for* F *exists* a.e. *on* Δ_1 *and is integrable.*

Suppose further that R *is regular. Let* $a \in \mathrm{R}$ *and* $\{\mathrm{R}_n\}$ *a regular exhaustion of* R *with center* a. *Then, we have*

$$-\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial \mathrm{R}_n} \mathrm{F}(z)\, \delta\mathrm{G}(a, z) = \int_{\Delta_1} \hat{\mathrm{F}}(b) k_b(a)\, d\chi(b).$$

Proof. — Since F is a Wiener function on $\mathrm{R} \sim \mathrm{K}$, it follows from [3; Hilfssatz 14.3] and Proposition 3.2 that $\hat{\mathrm{F}}$ exists a.e. on Δ_1 and is measurable. Since we have

$$0 \leqslant \mathrm{F} \leqslant u$$

on $\mathrm{R} \sim \mathrm{K}$, we have $0 \leqslant \hat{\mathrm{F}} \leqslant \hat{u}$ a.e. on Δ_1. Since $\hat{u}$ is integrable, so is $\hat{\mathrm{F}}$.

Now we suppose R to be regular. To show the convergence of the integrals, we first assume that $0 \leqslant F \leqslant 1$. Then, by [3; Satz 14.2], $W[F]$ exists and is given by

$$W[F] = \int_{\Delta_1} \hat{F}(b) k_b \, d\chi(b).$$

We also know (cf. [3; Hilfssatz 6.1]) that there exists a potential p on R such that p is finite everywhere on R and, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq R$ with $W[F] - \varepsilon p \leqslant F \leqslant W[F] + \varepsilon p$ on $R \sim K_\varepsilon$. Take n so large that $K_\varepsilon \subseteq R_n$ and integrate this inequality with respect to $d\mu_n$ which is the restriction of $-\frac{1}{2\pi i} \delta G(a, z)$ to ∂R_n. Since we have

$$W[F](a) = \int_{\partial R_n} W[F](z) \, d\mu_n(z) = \int_{\Delta_1} \hat{F}(b) k_b(a) \, d\chi(b)$$

and

$$\int_{\partial R_n} p(z) \, d\mu_n(z) \leqslant p(a),$$

we conclude that

$$\left| \int_{\partial R_n} F(z) \, d\mu_n(z) - \int_{\Delta_1} \hat{F}(b) k_b(a) \, d\chi(b) \right| \leqslant \varepsilon p(a).$$

So the desired result follows in this case.

Next we consider the general case. Put $F_m = \min\{F, m\}$ for $m = 1, 2, \ldots$. It is known that F_m are Wiener functions on $R \sim K$ and $\hat{F}_m = \min\{\hat{F}, m\}$ a.e. on Δ_1 (cf. [3]). By what we have shown in the preceding paragraph, there exists, for any m and any $\varepsilon > 0$, a number $n_0 = n_0(m, \varepsilon)$ such that

$$\left| \int_{\partial R_n} F_m(z) \, d\mu_n(z) - \int_{\Delta_1} \hat{F}_m(b) k_b(a) \, d\chi(b) \right| < \varepsilon \qquad \text{for} \qquad n \geqslant n_0.$$

Since $\hat{F}$ is integrable and $\hat{F}_m \to \hat{F}$ a.e., there exists, for any $\varepsilon > 0$, a number $m_0 = m_0(\varepsilon)$ such that

$$\int_{\Delta_1} \hat{F}(b) k_b(a) \, d\chi(b) < \int_{\Delta_1} \hat{F}_m(b) k_b(a) \, d\chi(b) + \varepsilon \qquad \text{for} \qquad m \geqslant m_0.$$

Since $0 \leqslant F \leqslant u$, we have $F - F_m \leqslant u - u_m$ on $R \sim K$, where $u_m = \min(u, m)$. If $K \subseteq R_n$, then we thus have

$$\begin{aligned} 0 &\leqslant \int_{\partial R_n} F(z)\, d\mu_n(z) - \int_{\partial R_n} F_m(z)\, d\mu_n(z) \\ &\leqslant \int_{\partial R_n} u(z)\, d\mu_n(z) - \int_{\partial R_n} u_m(z)\, d\mu_n(z) \\ &\leqslant u(a) - (u \wedge m)(a). \end{aligned}$$

If we take $m \geqslant m_0(\varepsilon)$ and $n \geqslant n_0(m, \varepsilon)$, then we have

$$\left|\int_{\Delta_1} \hat{F}(b) k_b(a)\, d\chi(b) - \int_{\partial R_n} F(z)\, d\mu_n(z)\right| \leqslant 2\varepsilon + u(a) - (u \wedge m)(a).$$

Since u is outer, $(u \wedge m)(a) \to u(a)$, so that we are done.

THEOREM 4.2. — *Suppose that* R *is regular and let* $a \in R$ *be fixed. Let* $z_1, \ldots, z_l$ *be* l *distinct critical points of the function* G_a *and let* c_j, $j = 1, 2, \ldots, l$, *be the multiplicity of* z_j. *Put* $g(z) = \exp\left(-\sum_{j=1}^{l} c_j G(z_j, z)\right)$. *If* f *is a meromorphic function on* R *such that* $|f|g$ *has a harmonic majorant on* R, *then* $\hat{f}$ *exists* a.e. *on* Δ_1, *is integrable and*

$$f(a) = \int_{\Delta_1} \hat{f}(b) k_b(a)\, d\chi(b).$$

Proof. — Since $|f|g$ has a harmonic majorant, Proposition 2.7 (a) shows that its least harmonic majorant, u, is outer. Since R is regular, there exist a compact set K in R and a constant $c > 0$ such that the interior of K contains $z_1, \ldots, z_l$ and $g \geqslant c$ on $R \sim K$. So we have $|f| \leqslant c^{-1}u$ on $R \sim K$. Since both $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic on $R \sim K$ and majorized there in modulus by the outer harmonic function $c^{-1}u$, they are Wiener functions on $R \sim K$. So, by [3; Hilfssatz 14.3], $\hat{f}$ exists a.e. on Δ_1 and is measurable. Moroever, we have $|\hat{f}| \leqslant c^{-1}\hat{u}$ a.e. on Δ_1. Hence, $\hat{f} \in L^1(d\chi)$.

Let $\{R_n\}$ be a regular exhaustion of R with center a. Then, $G_a - \varepsilon_n$ is the Green function for R_n with pole at a. We may assume without loss of generality that K is contained in R_1. For each n, $f(z)\delta G(a, z)$ is a meromorphic differential in z on $\mathrm{Cl}\, R_n$ with only one pole at a, whose residue is equal to $-2\pi i f(a)$. Thus we have

$$f(a) = -\frac{1}{2\pi i}\int_{\partial R_n} f(z)\,\delta G(a, z).$$

By applying Lemma 4.1, we get the desired result.

5. Direct and inverse Cauchy theorems.

Let R be a hyperbolic Riemann surface and a_0 the point in R which is used for defining the Martin functions. We consider the following three conditions (A), (B) and (C):

(A) R is regular.

(B) Let Π be the group of multiplicative characters of the group $H_1(R; \mathbf{Z})$. There exists a family of outer l.a.m.'s $\{\delta(\theta) : \theta \in \Pi\}$, such that (*a*) $\delta(1) = 1$; (*b*) $\delta(\theta)$ has character θ for each $\theta \in \Pi$; (*c*) $0 < \delta(\theta) \leqslant 1$ for each $\theta \in \Pi$; (*d*) if a sequence of the form $\{\delta(\theta_n) : n = 1, 2, \dots\}$ is pointwise convergent to a function of the form $|f|$ with $f \in H^\infty(R)$, then f is β exterior in the sense that $fH^\infty(R)$ is β dense in $H^\infty(R)$.

In order to state the condition (C), we denote, for each $a \in R$, by $Z(a) = \{z_j = z_j(a) : j = 1, 2, \dots\}$ a univalent enumeration of the critical points of G_a and by $c_j = c_j(a)$ the multiplicity of z_j. And we put

$$g^{(a)}(z) = \exp\left(-\sum_j c_j G(z_j, z)\right). \tag{2}$$

(C) There exists a point $a \in R$ for which $\sum_j c_j G(z_j, z) < \infty$ on $R \sim Z(a)$.

Remark. — H. Widom [18] observed that (C) holds (if and) only if $\sum_j c_j(a) G(z_j(a), z) < \infty$ on $R \sim Z(a)$ for every $a \in R$, provided that R satisfies (A). According to a recent private communication from him, the results in [18] will show that the condition (B) (or less: there only has to be a $\delta(\theta)$ for each $\theta \in \Pi$ such that $\delta(\theta) \leqslant 1$ and $\delta(\theta) \not\equiv 0$) implies the condition (C) for any Riemann surface, independently of (A). Thus, the condition (C) can be suppressed without changing our main results. For an interesting class of Riemann surfaces satisfying the conditions (A), (B) and (C), we refer the reader to Neville [8; Chapters 5 and 8]. See also Widom [17].

Our main objective of this section is to prove a Cauchy theorem and its converse for any surface R satisfying (A), (B) and (C). These theorems have also been found by Neville [8]. We shall begin with

LEMMA 5.1. — *Let* R *be a hyperbolic Riemann surface for which* (C) *holds. Then,* $\hat{g}^{(a)}$ *exists* a.e. *on* Δ_1 *and is equal to* 1 a.e. *on* Δ_1.

Proof. — Put $s(z) = \sum_j c_j G(z_j, z)$. Then, our hypothesis shows that s is a positive superharmonic function on R. It is therefore a Wiener function (cf. [3; p. 56]). By [3; Satz 14.2], $\hat{s}$ exists a.e. on Δ_1 and the outer part of $W[s]$ is equal to $\int \hat{s}(b) k_b \, d\chi(b)$. For $n = 1, 2, \ldots$, we put $s_n(z) = \sum_{j=1}^{n} c_j G(z_j, z)$ and $s'_n = s - s_n$. Since s_n is a potential, we have $W[s_n] = 0$ and so $W[s] = W[s'_n]$ for $n = 1, 2, \ldots$. Thus, $W[s] \leqslant s'_n$ for all n. Since $\sum_j c_j G(z_j, z)$ is convergent on $R \sim Z(a)$, $\{s'_n : n = 1, 2, \ldots\}$ converges to zero on R. So $W[s] = 0$ and therefore $\hat{s} = 0$ a.e. on Δ_1. Q.E.D.

LEMMA 5.2. — *Let* R *be a hyperbolic Riemann surface for which* (B) *and* (C) *hold. Then there exists a sequence*

$$\{B_j : j = 1, 2, \ldots\},$$

of functions in $H^\infty(R)$ *and a strictly increasing sequence of integers* $\{\nu(j) : j = 1, 2, \ldots\}$ *such that, for each* j, *the inner factor of* $|B_j|$ *is* $\exp\left(- \sum_{i \geqslant \nu(j)} c_i G(z_i, z)\right)$ *and such that*

$$\lim_{j \to \infty} B_j \; (= B, \textit{say})$$

exists in the β *topology and is* β *exterior.*

Proof. — We put $C_j(z) = \exp\left(-\sum_{i\geqslant j} c_i G(z_i, z)\right)$, $j = 1, 2, \ldots$. By (C), $\sum_{i\geqslant j} c_i G(z_i, z)$ is finite on $R \sim \{z_i : i \geqslant j\}$. Since each $G(z_i, z)$ belongs to $I(R)$, the order completeness of $I(R)$ implies that $\sum_{i\geqslant j} c_i G(z_i, z)$ belongs to $I(R)$, so that each C_j is an inner l.a.m. on R. Let θ_j be the character of C_j. Then there exists an $F_j \in H^\infty(R)$ such that

$$|F_j| = C_j \delta(\theta_j^{-1}).$$

Since $|F_j| \leqslant 1$, $j = 1, 2, \ldots$, there exists a β convergent subsequence $\{F_{\nu(j)} : j = 1, 2, \ldots\}$ of $\{F_j\}$. We put $B_j = F_{\nu(j)}$, $j = 1, 2, \ldots$, and let B be the β limit of

$$\{B_j : j = 1, 2, \ldots\}.$$

Since $\sum_j c_j G(z_j, z)$ converges uniformly on compact subsets of $R \sim Z(a)$, we see that $\lim_{j\to\infty} \sum_{i\geqslant j} c_i G(z_i, z) = 0$ uniformly on compact subsets of R. So C_j tend to 1 uniformly on compact subsets of R. Thus,

$$\delta(\theta_{\nu(j)}{}^{-1}) = |F_{\nu(j)}|/C_{\nu(j)} \to |B|$$

with respect to the β topology. Hence, by (B), the function B is β exterior.

Now we are in the position to prove our Cauchy theorem.

THEOREM 5.3. — *Let* R *be a hyperbolic Riemann surface for which* (A), (B) *and* (C) *hold. Let* $a \in R$ *be fixed. Let* f *be a meromorphic function on* R *such that* $|f|\, g^{(a)}$ *has a harmonic majorant. Then,* $\hat{f}$ *exists* a.e. *on* Δ_1, *is integrable, and*

$$f(a) = \int_{\Delta_1} \hat{f}(b) k_b(a)\, d\chi(b).$$

Proof. — We know that $\log g^{(a)} \in I(R) \subseteq SP(R)$. So, $\log|f| = \log(|f|\, g^{(a)}) - \log g^{(a)}$ belongs to $SP(R)$, too. We denote by Ψ the function $z \to \log|z|$ on the complex sphere Ω. Ψ is non-constant and continuous on Ω and the composite $\Psi \circ f$ $(= \log|f|)$ is a Wiener function on R, since it is in $SP(R)$. So, by [3; Folgesatz 10.1 and Satz 14.4],

$\hat{f}$ exists a.e. on Δ_1. By Lemma 5.1, we have $\hat{g}^{(a)} = 1$ a.e. on Δ_1. So, $|\hat{f}| = |\hat{f}|\hat{g}^{(a)} \leqslant \hat{u}$ a.e. on Δ_1, where u denotes the least harmonic majorant of $|f|g^{(a)}$ on R. Since $\hat{u} \in L^1(d\chi)$, we have $\hat{f} \in L^1(d\chi)$, too.

Now we use the notations in the proof of Lemma 5.2 and put $g_j(z) = \exp\left(-\sum_{i=1}^{\nu(j)-1} c_i G(z_i, z)\right)$, $j = 2, 3, \ldots$. Then, for any $s \in H^\infty(R)$, fsB_j is meromorphic on R and

$$|fsB_j|\, g_j = |s|\,|f|\, g^{(a)}\delta(\theta(C_{\nu(j)})^{-1}),$$

the latter having a harmonic majorant in view of our assumption. Applying Theorem 4.2, we have

$$(fsB_j)(a) = \int_{\Delta_1} \hat{f}(b)\hat{s}(b)\hat{B}_j(b)k_b(a)\, d\chi(b). \tag{3}$$

By Lemma 5.2, $B_j \to B$ in $H^\infty(R)$ with respect to the β topology. In view of Theorem 3.6, we have $\hat{B}_j \to \hat{B}$ in $L^\infty(d\chi)$ with respect to the weak topology $\sigma(L^\infty(d\chi), L^1(d\chi))$. Letting $j \to \infty$ in (3), we get

$$(fsB)(a) = \int_{\Delta_1} \hat{f}(b)\hat{s}(b)\hat{B}(b)k_b(a)\, d\chi(b).$$

Since B is β exterior, there exists a net $\{s_\lambda\} \subseteq H^\infty(R)$ such that $s_\lambda B \to 1$ with respect to the β topology. So $\hat{s}_\lambda\hat{B} \to 1$ with respect to $\sigma(L^\infty(d\chi), L^1(d\chi))$ and consequently

$$\begin{aligned} f(a) &= \lim_\lambda (fs_\lambda B)(a) = \lim_\lambda \int_{\Delta_1} \hat{f}(b)\hat{s}_\lambda(b)\hat{B}(b)k_b(a)\, d\chi(b) \\ &= \int_{\Delta_1} \hat{f}(b)k_b(a)\, d\chi(b), \end{aligned}$$

as was to be proved.

We proceed to prove an inverse Cauchy theorem, which will generalize previous results by Read [9], Royden [10] and Neville [6], and which has also been found by Neville [8]. Here we shall follow Neville's method in [6]. In order to do so, however, we have something to settle in advance, which we now describe.

For any two points $a, a' \in R$, we set

$$P(a, a'; z) = \delta G(a', z)/\delta G(a, z) \qquad \text{for} \qquad z \in R.$$

Then, $P(a, a'; z)$ is a meromorphic function on R. If $a \neq a'$, then it vanishes at a and has poles in the set $Z(a) \cup \{a'\}$.

Lemma 5.4. — *Let* R *be a hyperbolic Riemann surface for which* (A), (B) *and* (C) *hold. Let* $a, a' \in R$ *be fixed. Then,* $\hat{P}(a, a'; b)$ *exists* a.e. *on* Δ_1 *and is equal to* $k_b(a')/k_b(a)$ a.e. *on* Δ_1.

Of course, we have only to consider the case $a \neq a'$. The proof is rather long. We first prove the existence of the boundary function and will evaluate the function after some discussion about Green lines. We shall assume throughout the conditions (A), (B) and (C) even when we do not need the full strength of the conditions.

Existence of the boundary function. — Let $\{R_n\}$ be a regular exhaustion of R with center a and let $G^{(n)}(a', z)$ be the Green function for R_n with pole at a'. Since $G_a - \varepsilon_n$ is the Green function for R_n with pole at a, the Harnack inequality shows that there exists a constant c, depending only on a, a' and R, such that $0 < \delta G^{(n)}(a', z)/\delta G(a, z) < c$ on ∂R_n.

Put $u(a, a'; z) = g^{(a)}(z) \exp(-G(a', z))$, where $g^{(a)}$ was defined by (2). Then, the condition (C) implies that $u(a, a'; z)$ is a nontrivial inner l.a.m. on R. Since

$$u(a, a'; z) \leqslant 1$$

on R, we have

$$0 \leqslant u(a, a'; z)(\delta G^{(n)}(a', z)/\delta G(a, z)) \leqslant c \quad \text{on} \quad \partial R_n.$$

Since $u(a, a'; z)|\delta G^{(n)}(a', z)/\delta G(a, z)|$ is an l.a.m. on $\mathrm{Cl}\, R_n$, the maximum principle implies that

$$u(a, a'; z)|\delta G^{(n)}(a', z)/\delta G(a, z)| \leqslant c \quad \text{on} \quad \mathrm{Cl} R_n.$$

Since $\delta G^{(n)}(a', z)$ converge to $\delta G(a', z)$ almost uniformly on $R \sim \{a'\}$, we have

$$u(a, a'; z)|P(a, a'; z)| \leqslant c \quad \text{on} \quad R. \tag{4}$$

In particular, we have $\log|P(a, a'; z)| \in SP(R)$. By [3; Folgesatz 10.1 and Satz 14.4], $\hat{P}(a, a'; b)$ exists a.e. on Δ_1, as was to be proved.

Some properties of Green lines. — In order to evaluate $\hat{P}(a, a'; b)$ on Δ_1, we need the concept of Green lines. Let $a \in R$ be fixed and define $r(z)$ and $\omega(z)$ by the equations $dr(z)/r(z) = -dG(a, z)$ and $d\omega(z) = -{}^*dG(a, z)$. The first equation is solved by $r(z) = \exp(-G(a, z))$, which we shall use in what follows. Put

$$R(\rho) = \{z \in R : G(a, z) > \rho\} = \{z \in R : r(z) < e^{-\rho}\}$$

for ρ with $0 < \rho < \infty$. We call $R(\rho)$ regular if $\delta G_a \neq 0$ on the boundary of $R(\rho)$. An open arc on R is called a Green arc for G_a if it is a level arc of the function ω on which $d\omega(z) \neq 0$ and $\omega(z)$ is constant. A maximal Green arc is called a Green line. We shall denote by $\mathbf{G} = \mathbf{G}(R; a)$ the totality of Green lines L for G_a issuing from the point a. For a sufficiently large $\rho > 0$, $R(\rho)$ is regular and

$$w = f(z) = e^{\rho} r(z) \exp(i\omega(z))$$

is a conformal mapping from $\mathrm{Cl}\, R(\rho)$ onto the unit disk $\{w \in \mathbf{C} : |w| \leqslant 1\}$. We fix such a ρ $(= \rho_0$, say) and put $\mathbf{J} = \partial R(\rho_0)$. The function $z = f^{-1}(w)$ maps $\{w : |w| \leqslant 1\}$ onto $R(\rho_0) \cup \mathbf{J}$, so that each point z on $\mathbf{J}$ is represented by a real number $\omega \in [0, 2\pi)$ where $z = f^{-1}(e^{i\omega})$. So, every $L \in \mathbf{G}$ can be parametrized with ω as $L = L_\omega$ where ω represents the point in $L \cap \mathbf{J}$. We define a measure m, called the *Green measure* on $\mathbf{G}$ (or, more exactly, on $\mathbf{J}$), by

$$dm\,(L) = dm\,(\omega) = d\omega/2\pi \quad \text{with} \quad L = L_\omega.$$

We also put $\mathbf{E}_0 = \mathbf{E}_0(R; a) = \{L \in \mathbf{G} : \mathrm{Cl}\, L$ is compact in $R\}$. Clearly, $L \in \mathbf{E}_0$ (if and) only if L ends in a point of $Z(a)$. It follows that $\mathbf{E}_0$ is countable. Since R is regular, we see that $\sup\{r(z) : z \in L\} = 1$ for $L \in \mathbf{G}$ if and only if $L \notin \mathbf{E}_0$. If we take the branch of $\omega(z)$ at $z \in L_\omega$ with $\omega(z) = \omega$, then we can use the single-valued function $r(z)e^{i\omega(z)} = re^{i\omega}$ as a global coordinate on the star region

$$\mathbf{G}' = \mathbf{G}'(R; a) = \cup\{L : L \in \mathbf{G}\} \cup \{a\}.$$

Thus, if $R(\rho)$ is regular and if u is a harmonic function on $R(\rho)$, continuous on $\mathrm{Cl}\, R(\rho)$, then the usual Green formula states the following:

$$u(a) = -\frac{1}{2\pi}\int_{\partial R(\rho)} u(z)\,{}^*dG(a, z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\omega})\,d\omega$$
$$= \int_0^{2\pi} u(re^{i\omega})\,dm(L_\omega) \quad \text{with} \quad r = e^{-\rho}.$$

Let f be a function on R. We say that f has a *radial limit* a.e. on $\mathbf{G}$ if

$$f(L_\omega) = \lim_{r\to 1} f(re^{i\omega}) = \lim\,\{f(z) : z \in L_\omega, \quad r(z) \to 1\}$$

exists m-a.e. on $\mathbf{G} \sim \mathbf{E}_0$. Then, we have the following

LEMMA 5.5. — *a) Every bounded analytic function f on $\mathbf{G}' = \mathbf{G}'(R; a)$ possesses a radial limit* a.e. *on* $\mathbf{G} = \mathbf{G}(R; a)$ *and the limit function $f(L)$ is m-measurable on $\mathbf{G}$. If $f \neq 0$, then $f(L) \neq 0$ m*-a.e. *on $\mathbf{G}$. This is true of every meromorphic function f of bounded characteristic.*

b) Suppose that R is regular and let $1 \leqslant p < \infty$. If an analytic function f on $\mathbf{G}'$ is such that $|f|^p$ is majorized on $\mathbf{G}'$ by a harmonic function $u \in Q(R)$, then f has a radial limit a.e. *on $\mathbf{G}$ and the function $L \to f(L)$ belongs to $L^p(dm)$.*

Proof. — Part *a*) is essentially contained in [12; Chapter III, Theorems 6D and 6I]. So, we shall prove *b*).

Let f satisfy the condition in *b*). Then, by Proposition 2.7, $|f|$ is of bounded characteristic, so that f itself is of bounded characteristic on $\mathbf{G}'$ because $\mathbf{G}'$ is simply connected. By *a*), there exists a measurable subset $\mathbf{M}$ of $\mathbf{G} \sim \mathbf{E}_0$ such that $m(\mathbf{G} \sim (\mathbf{E}_0 \cup \mathbf{M})) = 0$ and $f(L)$ exists for every $L \in \mathbf{M}$. Let $0 < \rho < \infty$ be such that $R(\rho)$ is regular. Then,

$$\int_0^{2\pi} |f(re^{i\omega})|^p\,dm(L_\omega) \leqslant \int_0^{2\pi} u(re^{i\omega})\,dm(L_\omega) = u(a) \quad \text{with} \quad r = e^{-\rho}.$$

Take any decreasing sequence $\{\rho_n : n = 1, 2, \ldots\}$ with $\rho_n \to 0$ such that $R(\rho_n)$ is regular for each n. Put

$$\varphi_n(L_\omega) = |f(r_n e^{i\omega})|^p$$

Added in proof. — The suggestion made for the proof of Lemma 5.5 (a) is inexact; however, the lemma itself is true and follows from the conditions (A) and (C).

with $r_n = \exp(-\rho_n)$. Then, $\lim_{n\to\infty} \varphi_n(\mathrm{L})$ $(= \varphi(\mathrm{L})$, say) exists for every $\mathrm{L} \in \mathbf{M}$ and $\varphi(\mathrm{L}) = |f(\mathrm{L})|^p$. So, by the Fatou lemma, we see that φ is m-integrable and

$$\int \varphi(\mathrm{L})\, dm(\mathrm{L}) = \lim_{n\to\infty} \int \varphi_n(\mathrm{L})\, dm(\mathrm{L}) \leqslant u(a).$$

This is what we wished to show. Q.E.D.

Returning to our case, we see by the condition (B) that there exists a function $\mathrm{F} \in \mathrm{H}^\infty(\mathrm{R})$ with

$$|\mathrm{F}| = u(a, a'; .)\delta(\theta(u(a, a'; .))^{-1}).$$

Put $f(z) = \mathrm{P}(a, a'; z)\mathrm{F}(z)$. In view of (4), we have $f \in \mathrm{H}^\infty(\mathrm{R})$. By Lemma 5.5, $\mathrm{F(L)}$ and $f(\mathrm{L})$ exist m-a.e. on $\mathbf{G} \sim \mathbf{E}_0$. Since $\mathrm{F} \neq 0$, we have $\mathrm{F(L)} \neq 0$ m-a.e. on $\mathbf{G}$. It follows that $\mathrm{P}(a, a'; \mathrm{L})$ exist and is finite m-a.e. on $\mathbf{G}$.

Let $\mathrm{L} \in \mathbf{G} \sim \mathbf{E}_0$ and let e_L be the end of L, i.e.,

$$e_\mathrm{L} = \mathrm{Cl(L)} \sim (\mathrm{L} \cup \{a\})$$

in R^*. Thus, e_L is a non-void subset of Δ. We want to evaluate $\mathrm{P}(a, a'; \mathrm{L})$ when it exists. At each $z \in \mathrm{L}$, we take a local coordinate $z = x + iy$ such that $dx = d\mathrm{G}(a, z)$ and $dy = {}^*d\mathrm{G}(a, z)$. Along L, we then have

$$\delta \mathrm{G}(a, z) = \partial_x \mathrm{G}(a, z)\, dx = dx$$

and

$$\delta \mathrm{G}(a', z) = [\partial_x \mathrm{G}(a', z) + i\partial_x \mathrm{G}(a', z)_*]\, dx,$$

where $\mathrm{G}(a', z)_*$ denotes the harmonic conjugate of $\mathrm{G}(a', z)$. We may assume that $x = \mathrm{G}(a, z)$ and $y = y_0 =$ constant along L. Then we have on L

$$(5) \quad \mathrm{Re}\,(\mathrm{P}(a, a'; z)) = (d\mathrm{G}\,(a', x + iy_0)/dx)/(d\mathrm{G}\,(a, x + iy_0)/dx).$$

Suppose that $\mathrm{P}(a, a'; \mathrm{L})$ exists. Then, (5) has a limit as x tends to zero. Since both $\mathrm{G}(a, x + iy_0)$ and $\mathrm{G}(a', x + iy_0)$ tend to zero as x tends to zero in view of the condition (A), l'Hospital's rule shows that

$$\begin{aligned} \mathrm{Re}\,(\mathrm{P}(a, a'; \mathrm{L})) &= \lim_{x\to 0} (d\mathrm{G}\,(a', x + iy_0)/dx)/(d\mathrm{G}\,(a, x + iy_0)/dx) \\ &= \lim_{x\to 0} \mathrm{G}(a', x + iy_0)/\mathrm{G}(a, x + iy_0). \end{aligned}$$

Let $b \in e_L$. Then, there exists a sequence of points z_n in L with the coordinates $x_n + iy_0$ such that $x_n \to 0$ and $z_n \to b$. So, the final member is equal to $k_b(a')/k_b(a)$. Since b is arbitrary in e_L, $k_b(a')/k_b(a)$ is constant on e_L as a function in b.

Now let a' run over a countable dense subset A of R. Then we see that there exists a measurable subset $\mathbf{A}$ of $\mathbf{G} \sim \mathbf{E}_0$ such that (i) $m(\mathbf{G} \sim (\mathbf{E}_0 \cup \mathbf{A})) = 0$; (ii) for each $a' \in$ A and $L \in \mathbf{A}$, F(L) and f(L) exist and are finite, and $F(L) \neq 0$. So, for each $a' \in$ A and $L \in \mathbf{A}$, $P(a, a'; L)$ exists. Therefore, if $b, b' \in e_L$ with $L \in \mathbf{A}$, then

$$k_b(a')/k_b(a) = k_{b'}(a')/k_{b'}(a)$$

for every $a' \in$ A. Since the function k_b with $b \in \Delta$ is continuous on R, the density of A in R implies that k_b and $k_{b'}$ are proportional. Hence we have $b = b'$ by the fundamental property of the Martin functions (cf. [3; pp. 135-136]). Namely, e_L consists of a single point for each $L \in \mathbf{A}$.

We next prove

LEMMA 5.6. — *Let* (V, ψ) *be a parametric disk in* R *and put* $V(1/4) = \psi^{-1}(U(0; 1/4))$, *where* $U(w; r)$ *denotes the open disk in* **C** *with center* w *and radius* r. *Let* $a \in R$ *be fixed. Then, there exists a constant* C *such that*

$$|P(a, a'; z) - P(a, a''; z)|\, g^{(a)}(z) \leqslant C|\psi(a') - \psi(a'')|$$

for any $a', a'' \in V(1/4)$ *and any* $z \in R \sim \mathrm{Cl}\, V$.

Proof. — Let $\{R_n\}$ be a regular exhaustion with center a such that $\mathrm{Cl}\, V \subseteq R_1$. Let $a', a'' \in V(1/4)$ and let $G^{(n)}(a', z)$ and $G^{(n)}(a'', z)$ be the Green functions for R_n with poles at a' and a'', respectively. Then, for any real outer harmonic function h on R_n, we have

$$(6)\quad h(a') - h(a'') = -\frac{1}{2\pi i}\int_{\partial R_n} h(z)\left(\frac{\delta G^{(n)}(a', z)}{\delta G\,(a, z)} - \frac{\delta G^{(n)}(a'', z)}{\delta G\,(a, z)}\right)\delta G(a, z).$$

Put $h^+ = h \vee 0$ and $h^- = (-h) \vee 0$. Then, h^+ and h^- are positive outer harmonic functions on R_n. By the Harnack inequality, we have $c(r)^{-1} \leqslant h^{+}(a')/h^{+}(a'') \leqslant c(r)$, where

$$r = |\psi(a') - \psi(a'')|$$

and $c(r) = (3 + 4r)/(3 - 4r)$. The same is true of the function h^{-}. So,

$$\begin{aligned}(7)\quad |h(a')-h(a'')| &\leqslant |h^{+}(a') - h^{+}(a'')| + |h^{-}(a') - h^{-}(a'')| \\ &\leqslant 8r(h^{+}(a'') + h^{-}(a'')) \\ &= 8r\left(-\frac{1}{2\pi i}\right)\int_{\partial R_n} |h(z)| \frac{\delta G^{(n)}(a'', z)}{\delta G(a, z)} \delta G(a, z) \\ &\leqslant 8r\lambda\left(-\frac{1}{2\pi i}\right)\int_{\partial R_n} |h(z)| \delta G(a, z),\end{aligned}$$

where λ is a constant depending only on a, V and R, and not on n. Combining (6) and (7), we have

$$\left|\frac{\delta G^{(n)}(a', z)}{\delta G(a, z)} - \frac{\delta G^{(n)}(a'', z)}{\delta G(a, z)}\right| \leqslant 8\lambda r \quad \text{on } \partial R_n.$$

We put $\varphi(z) = g^{(a)}(z) \exp(- G(a', z) - G(a'', z))$. Then, $\varphi(z)$ is an inner l.a.m. on R and is $\leqslant 1$. So,

$$(8)\qquad \left|\frac{\delta G^{(n)}(a', z)}{\delta G(a, z)} - \frac{\delta G^{(n)}(a'', z)}{\delta G(a, z)}\right| \varphi(z) \leqslant 8\lambda r \quad \text{on } \partial R_n.$$

Since the left-hand member of (8) is an l.a.m. on $\mathrm{Cl}\, R_n$ so that the inequality sign remains to hold when z runs over R_n. Letting $n \to \infty$, we have $|P(a, a'; z) - P(a, a''; z)|\varphi(z) \leqslant 8\lambda r$ on R. Since $\mathrm{Cl}\, V(1/4)$ is a compact subset of $V \subseteq \mathrm{Cl}\, V$, the set of functions $\exp(G_{a'} + G_{a''})$ with $a', a'' \in V(1/4)$ form a uniformly bounded family of functions on $R \sim \mathrm{Cl}\, V$. Hence, there exists a desired constant C. Q.E.D.

Now let $a'' \in R$ be any point and take a sequence $\{a_n\}$ in A which converges to a''. We may assume that $\{a_n\}$ is contained in $V(1/4)$, where (V, ψ) is a fixed parametric disk centered at a''. By the preceding lemma, we have, for $z = x + iy_0 \in L$ in **A**,

$$|P(a, a_n; x + iy_0) - P(a, a_m; x + iy_0)||F(x + iy_0)| \leqslant C|\psi(a_n) - \psi(a_m)|$$

for $n, m = 1, 2, \ldots,$ and also

$$|P(a, a''; x + iy_0) - P(a, a_n; x + iy_0)||F(x + iy_0)| \leqslant C|\psi(a'') - \psi(a_n)|$$

for all x sufficiently near zero. It follows at once from these inequalities that $P(a, a''; L)$ exists for $L \in \mathbf{A}$. Summing up, we have the following.

Proposition 5.7. — *Let $a \in R$ be fixed. Then, there exists a measurable subset $\mathbf{A}$ of $\mathbf{G}(R; a) \sim \mathbf{E}_0(R; a)$ with*

$$m(\mathbf{G} \sim (\mathbf{E}_0 \cup \mathbf{A})) = 0$$

such that $F(L)$ $(\neq 0)$ and $P(a, a'; L)$ exist and are finite for every $a' \in R$ and every $L \in \mathbf{A}$. Furthermore, e_L consists of a single point, b_L, for every $L \in \mathbf{A}$ and

$$\mathrm{R}e\,(P(a, a'; L)) = k_b(a')/k_b(a)$$

for every $a' \in R$ and every $b = b_L$ with $L \in \mathbf{A}$.

We can thus apply the Brelot-Choquet theory of Green lines [2] to our problem. We know that the Martin compactification is metrizable and resolutive (cf. [3; Satz 13.4]). For each point $b \in \Delta_1$, let $\mathscr{F}_b$ be the filter of all sets of the form $R \cap W$ where W varies over the fine neighborhoods of b in R^*. As was shown by L. Naïm [19], there exists a measurable subset $\Delta' \subseteq \Delta_1$ such that $\chi(\Delta') = 1$ and the family $\mathfrak{F} = \{\mathscr{F}_b : b \in \Delta'\}$ satisfies Brelot-Choquet's conditions A and B, where

A : If h is subharmonic and bounded above and if we have limsup $h \leqslant 0$ along any $\mathscr{F} \in \mathfrak{F}$, then $h \leqslant 0$;

B : For each $\mathscr{F}_b \in \mathfrak{F}$, there exist an open neighborhood W of b in R^* and a superharmonic function $v > 0$ on $W \cap R$ such that $\lim v = 0$ along $\mathscr{F}_b$ and, for any neighborhood V of b, $\inf \{v(z) : z \in (W \cap R) \sim V\} > 0$.

Moreover, Proposition 5.7 shows that almost all Green lines in $\mathbf{G}(R; a) \sim \mathbf{E}_0(R; a)$ converge in R^*. Hence, the Brelot-Choquet theorey [2] implies the following

Proposition 5.8. — *Let $a \in R$ be fixed and let $\Lambda = \Lambda(a)$ be the set of Green lines $L \in \mathbf{G}(R; a) \sim \mathbf{E}_0(R; a)$ for which the end e_L consists of a single point, say b_L. Then, the following hold:*

a) $m(\mathbf{G} \sim (\mathbf{E}_0 \cup \Lambda)) = 0$;

b) The function $L \to b_L$ from Λ into Δ is measurable *and is measure preserving with respect to the measure* dm *on* Λ *and the harmonic measure* $k_b(a)\,d\chi(b)$ *on* Δ *corresponding to the point* a. *In particular, the points in* Δ *which are not in the image of* Λ *under the above mapping form a null set in* Δ.

c) Let f^* *be a bounded measurable function on* Δ_1 *and let* $f = h[f^*]$ *be the solution of the Dirichlet problem for* R *with the boundary values* f^* (cf. Section 3). *Then,* f *has a radial limit* a.e. *on* **G** *and* $f(\mathrm{L}) = f^*(b_{\mathrm{L}})$ m-a.e. *on* Λ.

Combining this with Corollary 3.4, we have the following

Corollary 5.9. — *Let* f^* *be a bounded measurable function on* Δ_1 *and* $f = h[f^*]$. *Let* $a \in \mathrm{R}$. *Then,* f *has a radial limit* a.e. *on* $\mathbf{G}(\mathrm{R}\,;\, a)$ *and*

$$(9) \qquad \hat{f}(b) = f^*(b) = f(\mathrm{L}) \quad \text{a.e. } on\ \Delta_1$$

where $b = b_{\mathrm{L}}$ *with* $\mathrm{L} \in \Lambda(a)$.

By Proposition 5.7 and Corollary 5.9, we conclude this:

Corollary 5.10. — *Let* $a, a' \in \mathrm{R}$ *be fixed. Then,*

$$(10) \qquad \mathrm{Re}\,(\hat{\mathrm{P}}(a, a'\,;\, b)) = k_b(a')/k_b(a) \quad \text{a.e. } on\ \Delta_1.$$

Proof. — Using the notations defined after the proof of Lemma 5.5, we have $\mathrm{P}(a, a'\,;\, z) = f(z)/\mathrm{F}(z)$ and it is clear that (9) is valid for both f and F. From this the desired result follows at once.

Completion of the proof of Lemma 5.4. — Let now a, a' and a'' be any pairwise distinct three points in R. Then, $\mathrm{P}(a, a'\,;\, z) = \mathrm{P}(a, a''\,;\, z)\mathrm{P}(a'', a'\,;\, z)$. So, if $\hat{\mathrm{P}}(a, a''\,;\, b)$ and $\hat{\mathrm{P}}(a'', a'\,;\, b)$ exist and are finite for some $b \in \Delta_1$, then $\hat{\mathrm{P}}(a, a'\,;\, b)$ exists and $\hat{\mathrm{P}}(a, a'\,;\, b) = \hat{\mathrm{P}}(a, a''\,;\, b)\hat{\mathrm{P}}(a'', a'\,;\, b)$. By Corollary 5.10, we see that, for almost all $b \in \Delta_1$,

$$\begin{aligned}\mathrm{Re}\,(\hat{\mathrm{P}}(a, a''\,;\, b)\hat{\mathrm{P}}(a'', a'\,;\, b)) &= \mathrm{Re}\,(\hat{\mathrm{P}}(a, a'\,;\, b)) = k_b(a')/k_b(a)\\ &= \mathrm{Re}\,(\hat{\mathrm{P}}(a, a''\,;\, b))\,\mathrm{Re}\,(\hat{\mathrm{P}}(a'', a'\,;\, b)).\end{aligned}$$

For such $b \in \Delta_1$, either $\hat{\mathrm{P}}(a, a''\,;\, b)$ or $\hat{\mathrm{P}}(a'', a'\,;\, b)$ should be real.

Finally we fix two distinct points $a, a' \in \mathrm{R}$ and suppose, on the contrary, that there exists a measurable subset Δ'

of Δ_1 with $\chi(\Delta') > 0$ such that, for each $b \in \Delta'$, $\mathrm{P}(a, a'; b)$ exists, satisfies (10) and is non-real. Take a sequence of points $a_n (\neq a')$ in R converging to a'. Then, there exists a measurable subset Δ'' of Δ' with $\chi(\Delta'') > 0$ such that $\hat{\mathrm{P}}(a, a_n; b)$ exist for all n and all $b \in \Delta''$. Since $\hat{\mathrm{P}}(a, a'; b)$ is non-real for any $b \in \Delta'$, we may assume, in view of the above observation, $\hat{\mathrm{P}}(a, a_n; b)$ exists and is equal to $k_b(a_n)/k_b(a)$ for all n and all $b \in \Delta''$. By the Harnack inequality, we see that $\lambda_n^{-1} \leqslant |\hat{\mathrm{P}}(a', a_n; b)| \leqslant \lambda_n$ a.e. on Δ'' and therefore

$$(11) \quad \lambda_n^{-1} \leqslant |\hat{\mathrm{P}}(a, a'; b)/\hat{\mathrm{P}}(a, a_n; b)| \leqslant \lambda_n \text{ a.e. on } \Delta'',$$

where $\{\lambda_n\}$ is a sequence of positive numbers tending decreasingly to 1. There exists a point b in Δ'' for which (11) holds for all n. For such b, we have

$$\begin{aligned} 1 = \lim_{n\to\infty} |\hat{\mathrm{P}}(a, a'; b)/\hat{\mathrm{P}}(a, a_n; b)| &= \lim_{n\to\infty} |\hat{\mathrm{P}}(a, a'; b)| k_b(a)/k_b(a_n) \\ &= |\hat{\mathrm{P}}(a, a'; b)| k_b(a)/k_b(a'). \end{aligned}$$

Since (10) holds for this b, we should have

$$\hat{\mathrm{P}}(a, a'; b) = k_b(a')/k_b(a),$$

which is real. This contradiction shows that $\hat{\mathrm{P}}(a, a'; b)$ is real a.e. on Δ_1. In view of Corollary 5.10, this completes the proof of Lemma 5.4.

LEMMA 5.11. — *Let* R *be a hyperbolic Riemann surface for which* (A), (B) *and* (C) *hold. Let* $a \in \mathrm{R}$ *be fixed,* (V, ψ) *a parametric disk in* $\mathrm{R} \sim \mathrm{Z}(a)$, *and* J *any closed rectifiable curve contained in* $\mathrm{V}(1/4)$. *Put*

$$\mathrm{P_J}(z) = \int_{\psi(\mathrm{J})} \mathrm{P}(a, \psi'(\xi); z)\, d\xi \quad \textit{for} \quad z \in \mathrm{R} \sim (\mathrm{Z}(a) \cup \mathrm{Cl}\,\mathrm{V}),$$

where ψ' *denotes the inverse map of* ψ. *Then,* $\mathrm{P_J}$ *is regular analytic on* $\mathrm{R} \sim (\mathrm{Z}(a) \cup \mathrm{Cl}\,\mathrm{V})$ *and can be extended analytically to* $\mathrm{R} \sim \mathrm{Z}(a)$. $\mathrm{P_J}(a) = 0$, $\mathrm{P_J}$ *is meromorphic, the set of poles of* $\mathrm{P_J}$ *is contained in* $\mathrm{Z}(a)$ *and, for each* $z_j \in \mathrm{Z}(a)$, *the order of pole of* $\mathrm{P_J}$ *at* z_j *is not larger than* c_j. *Moreover,* $|\mathrm{P_J}| g^{(a)}$ *is bounded,* $\hat{\mathrm{P}}_{\mathrm{J}}(b)$ *exists* a.e. *on* Δ_1 *and*

$$(12) \qquad \hat{\mathrm{P}}_{\mathrm{J}}(b) = \int_{\psi(\mathrm{J})} (k_b(\psi'(\xi))/k_b(a))\, d\xi \quad \text{a.e. } \textit{on } \Delta_1.$$

Proof. — Since the poles of $P(a, a'; z)$ are contained in $Z(a) \cup \{a'\}$, the function P_J is analytic on

$$R \sim (Z(a) \cup \mathrm{Cl}\, V(1/4)).$$

If $\xi, \xi' \in U = \psi(V)$, then

$$G(\psi'(\xi), \ \psi'(\xi')) = -\log|\xi - \xi'| + h(\xi, \xi') \quad \text{for} \quad \xi \neq \xi',$$

where $h(\xi, \xi')$ is symmetric in ξ and ξ', is harmonic in ξ' and has a removable singularity at $\xi' = \xi$. So, we have

$$\delta G(\psi'(\xi), \psi'(\xi')) = -(\xi - \xi')^{-1}\, d\xi' + \delta_{\xi'} h(\xi, \xi'),$$

where $\delta_{\xi'} h(\xi, \xi')$ is an analytic differential in $\xi' \in U$. For $\xi' \in U$ with $1/4 < |\xi'| < 1$, we have

$$\int_{\psi(J)} P(a, \psi'(\xi); \psi'(\xi'))\, d\xi = \int_{\psi(J)} \left(\frac{\delta_{\xi'} h(\xi, \xi')}{d\xi'} \bigg/ \frac{\delta G(a, \psi'(\xi'))}{d\xi'}\right) d\xi,$$

the right-hand member being analytic throughout U. Hence, the function P_J can be continued analytically to the whole V, so that P_J can be regarded as analytic on $R \sim Z(a)$.

Since $\delta G(a, z)$ has a pole at a, $P_J(a) = 0$. The poles of P_J are contained in $Z(a)$ and have the asserted orders. Since J is compact, the Harnack inequality shows that there exists a constant c depending only on a, J and R with

$$|P_J(z)|\, g^{(a)}(z) \leqslant c$$

on R. Thus, $\log|P_J|$ belongs to SP(R) and therefore $\hat{P}_J$ exists a.e. on Δ_1.

Finally we shall prove (12). Let $\gamma : [0, 1) \to \psi(J)$ be a fixed parametrization of the curve $\psi(J)$. Since

$$a' \to P(a, a'; z)$$

is continuous on J for any fixed $z \in R \sim (Z(a) \cup \mathrm{Cl}\, V)$, we have for such z

$$\begin{aligned} P_J(z) &= \int_{\psi(J)} P(a, \psi'(\xi); z)\, d\xi \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} P(a, \psi'(\xi_{n,j}); z)(\xi_{n,j} - \xi_{n,j-1}) \end{aligned}$$

where $\xi_{n,j} = \gamma(j/n)$, $j = 0, 1, \ldots, n-1$, and $\xi_{n,n} = \xi_{n,0}$.

Let Δ' be a measurable subset of Δ_1 with

$$\chi(\Delta_1 \sim \Delta') = 0$$

such that, for $b \in \Delta'$, $\hat{g}^{(a)}(b) = 1$ and

$$\hat{P}(a, \psi'(\xi_{n,j}); b) = k_b(\psi'(\xi_{n,j}))/k_b(a)$$

for every n and j. Such a set Δ' exists in view of Lemmas 5.1 and 5.4.

Take any $b \in \Delta'$. Then, for any $0 < \varepsilon < 1$ and any n, there exists an open set $D_n \in \mathcal{G}_b$ such that $D_n \subseteq R \sim \mathrm{Cl}\, V$ and

$$\begin{aligned}\mathrm{Cl}\, \{g^{(a)}(z)P(a, \psi'(\xi_{n,j}); z) : z \in D_n\} \subseteq U(\hat{P}(a, \psi'(\xi_{n,j}); b); \varepsilon) \\ = U(k_b(\psi'(\xi_{n,j}))/k_b(a); \varepsilon)\end{aligned}$$

for $j = 1, 2, \ldots, n$. Thus, for $z \in D_n$,

$$\begin{aligned}&\Big|\sum_{j=1}^{n} g^{(a)}(z)P(a, \psi'(\xi_{n,j}); z)(\xi_{n,j} - \xi_{n,j-1}) \\ &\qquad - \sum_{j=1}^{n} \frac{k_b(\psi'(\xi_{n,j}))}{k_b(a)} (\xi_{n,j} - \xi_{n,j-1})\Big| \\ &\leqslant \sum_{j=1}^{n} \left| g^{(a)}(z)P(a, \psi'(\xi_{n,j})); z) - \frac{k_b(\psi'(\xi_{n,j}))}{k_b(a)} \right| |\xi_{n,j} - \xi_{n,j-1}| \\ &\leqslant \varepsilon \text{ length } (\psi(J)).\end{aligned}$$

We take n_0 in such a way that $\gamma([(j-1)/n, j/n])$ is contained in a disk of diameter ε for each $n \geqslant n_0$ and $j = 1, 2, \ldots, n$. Put $J_{n,j} = \gamma([(j-1)/n, j/n])$, $j = 1, 2, \ldots, n$. Let $z \in D_n$ with $n \geqslant n_0$. Since $|\xi - \xi_{n,j}| < \varepsilon$ for each $\xi \in J_{n,j}$, we have, in view of Lemma 5.6,

$$\begin{aligned}&\Big|\int_{\psi(J)} g^{(a)}(z)P(z, \psi'(\xi); z)\, d\xi \\ &\qquad - \sum_{j=1}^{n} g^{(a)}(z)P(a, \psi'(\xi_{n,j}); z)(\xi_{n,j} - \xi_{n,j-1})\Big| \\ &\qquad \leqslant \sum_{j=1}^{n} \Big|\int_{J_{n,j}} \{g^{(a)}(z)P(a, \psi'(\xi); z) \\ &\qquad - g^{(a)}(z)P(a, \psi'(\xi_{n,j}); z)\}\, d\xi\Big| \\ &\qquad \leqslant C\varepsilon.\ \text{length}\, (\psi(J)).\end{aligned}$$

Since $a' \to k_b(a')$ is continuous on R, there exists an n_1

such that, for $n \geqslant n_1$,

$$\left|\sum_{j=1}^{n} \frac{k_b(\psi'(\xi_{n,j}))}{k_b(a)}(\xi_{n,j}-\xi_{n,j-1}) - \int_{\psi(J)} \frac{k_b(\psi'(\xi))}{k_b(a)}\, d\xi\right| < \varepsilon.$$

Hence, for $n \geqslant \max(n_0, n_1)$ and for $z \in D_n$, we have

$$\left|g^{(a)}(z)P_J(z) - \int_{\psi(J)} (k_b(\psi'(\xi))/k_b(a))\, d\xi\right| < \varepsilon.\text{length}\,(\psi(J)) + C\varepsilon.\text{length}\,(\psi(J)) + \varepsilon.$$

Thus we have shown that the boundary function for $g^{(a)}P_J$ exists a.e. on Δ_1 and is equal to $\int_{\psi(J)} (k_b(\psi'(\xi))/k_b(a))\, d\xi$. By Lemma 5.1, $\hat{g}^{(a)} = 1$ a.e. on Δ_1, so that we obtain the desired result. Q.E.D.

THEOREM 5.12. — *Let* R *be a hyperbolic Riemann surface for which* (A), (B) *and* (C) *hold. Let* $a \in R$ *be fixed. Let*

$$u^* \in L^1(d\chi)$$

and suppose that

$$\int_{\Delta_1} \hat{h}(b)u^*(b)k_b(a)\, d\chi(b) = 0$$

for each function h, *meromorphic on* R, *such that* $|h|g^{(a)}$ *is bounded on* R *and* $h(a) = 0$. *Then, there exists an* f *in* $H^1(R)$ *such that* $\hat{f} = u^*$ a.e. *on* Δ_1.

Proof. — We put

$$f(z) = \int_{\Delta_1} u^*(b)k_b(z)\, d\chi(b) \quad \text{for} \quad z \in R.$$

Then, f is an outer harmonic function on R. Let (V, ψ) be any parametric disk contained in $R \sim Z(a)$ and let J be any closed rectifiable curve in $V(1/4)$. Then, the Fubini theorem and Lemma 5.11 show that

$$\int_{\psi(J)} f(\psi'(\xi))\, d\xi = \int_{\Delta_1}\left(\int_{\psi(J)} \frac{k_b(\psi'(\xi))}{k_b(a)}\, d\xi\right)u^*(b)k_b(a)\, d\chi(b) = \int_{\Delta_1} \hat{P}_J(b)u^*(b)k_b(a)\, d\chi(b) = 0.$$

By the Morera theorem, f is analytic on $R \sim Z(a)$. Since f is continuous on R, every point in $Z(a)$ is a removable singularity and indeed f is analytic everywhere on R. Clearly, $|f|$ has a harmonic majorant, so that $f \in H^1(R)$. $\hat{f} = u^*$ a.e. on Δ_1 by Corollary 3.4. This completes the proof.

6. Further properties of the lifting.

We again consider a hyperbolic Riemann surface R and its universal covering Riemann surface (U, φ), where U is the open unit disk and φ is a conformal mapping of U onto R with $\varphi(0) = a_0$. We know that the Martin compactification of U is the usual closed unit disk, the Martin boundary is the usual circumference ∂U, and the harmonic measure for the origin is exactly the normalized Lebesgue measure on ∂U, which we shall denote by $d\sigma(\omega) = \frac{1}{2\pi} d\omega$. Further T will denote the group of covering transformations for φ. Since both U and R are hyperbolic, the boundary function $\hat{\varphi}$ for φ is defined a.e. on ∂U with values in the Martin compactification R^* of R by [3; Satz 10.2 and Satz 14.4]. Put $\nu(z) = \exp(-G(a_0, \varphi(z)))$ for $z \in U$. Then, ν is an inner l.a.m. on U, so that, by Lemma 5.1., $\hat{\nu}$ exists and is equal to 1 a.e. on ∂U. By using the notation defined before Proposition 3.2, we set

$$\mathscr{D} = \{w \in \partial U : w \in \mathscr{D}(\nu) \cap \mathscr{D}(\varphi) \text{ and } \hat{\nu}(w) = 1\}.$$

Lemma 6.1. — *$\mathscr{D}$ is a* T*-invariant Borel subset of* ∂U *with* $\sigma(\mathscr{D}) = 1$. *Further,* $\hat{\varphi}$ *maps* $\mathscr{D}$ *into* Δ.

Proof. — The first half is obvious. So, let $w \in \mathscr{D}$. By Proposition 3.2, we see that, for any $\varepsilon > 0$, $\nu^{-1}((1-\varepsilon, 1+\varepsilon))$ belongs to $\mathscr{G}_w(U)$. Suppose, on the contrary, that $\hat{\varphi}(w) \in R$. Then, there exist an open neighborhood W of $\hat{\varphi}(w)$ in R and a constant $c > 0$ such that $G(a_0, a) \geqslant c$ for every $a \in W$. Again by Proposition 3.2, $\varphi^{-1}(W) \in \mathscr{G}_w(U)$. It follows that $1 - \varepsilon < \nu(z) = \exp(-G(a_0, \varphi(z)) \leqslant e^{-c}$ for any z in the set $\nu^{-1}((1-\varepsilon, 1+\varepsilon)) \cap \varphi^{-1}(W)$. As ε is arbitrary, this gives a desired contradiction. Q.E.D.

In what follows, we regard $\hat{\varphi}$ as defined not on $\mathscr{D}(\varphi)$ but on $\mathscr{D}$.

LEMMA 6.2. — *Let* f^* *be any real or complex continuous function on* Δ *and put* $f = h[f^*] \circ \varphi$. *Then, for any* $w \in \mathcal{D}$, $\hat{f}(w)$ *exists and is equal to* $f^* \circ \hat{\varphi}(w)$. *In particular,*

$$h[f^*] \circ \varphi = h[f^* \circ \hat{\varphi}] \tag{13}$$

on U, *where the right-hand member of* (13) *is of course the solution of the Dirichlet problem for* U *with the boundary values* $f^* \circ \hat{\varphi}$.

Proof. — We put $h = h[f^*]$ on R and $= f^*$ on Δ. Then, h is continuous on R^*. Let $w \in \mathcal{D}$ and put $b = \hat{\varphi}(w)$. Since h is continuous, $h^{-1}(\mathrm{U}(f^*(b);\ \varepsilon))$ is an open neighborhood of b for any $\varepsilon > 0$. So, by Proposition 3.2,

$$\varphi^{-1}(h^{-1}(\mathrm{U}(f^*(b);\ \varepsilon)))\ (= \mathrm{D}_\varepsilon, \text{ say})$$

belongs to $\mathcal{G}_w(\mathrm{U})$. This implies that $f(\mathrm{D}_\varepsilon)$ is contained in $\mathrm{U}(f^*(b);\ \varepsilon)$. As ε is arbitrary, we see that $\hat{f}(w)$ exists and is equal to $f^*(b) = (f^* \circ \hat{\varphi})(w)$. Since f is bounded and harmonic on U, Proposition 3.3. shows that $f = h[f^* \circ \hat{\varphi}]$, as was to be proved.

LEMMA 6.3. — *The formula* (13) *is true of any bounded measurable function* f^* *on* Δ.

Proof. — We suppose first that f^* is a real function defined everywhere on Δ. Suppose moreover that f^* is lower semicontinuous and let $\{f^*_\lambda\}$ be the set of real continuous functions on Δ majorized by f^*. Then, $f^* = \sup\limits_\lambda f^*_\lambda$. In view of the vector lattice isomorphism given by Proposition 3.1, we have $\bigvee\limits_\lambda h[f^*_\lambda] = h\,[\sup\limits_\lambda f^*_\lambda] = h[f^*]$. Next, we regard $f^* \circ \hat{\varphi}$ and $f^*_\lambda \circ \hat{\varphi}$ as defined everywhere on $\partial\mathrm{U}$ by continuing them to be zero on $\partial\mathrm{U} \sim \mathcal{D}$. Then, they are bounded measurable on $\partial\mathrm{U}$ and $f^* \circ \hat{\varphi} = \sup\limits_\lambda (f^*_\lambda \circ \hat{\varphi})$. So, again using Proposition 3.1 now for U, we have

$$h[f^* \circ \hat{\varphi}] = \bigvee_\lambda h[f^*_\lambda \circ \hat{\varphi}].$$

By Lemma 6.2, we see $h[f_\lambda^* \circ \hat{\varphi}] = h[f_\lambda^*] \circ \varphi$. It follows from Proposition 2.6 that

$$h[f^*] \circ \varphi = \Big(\bigvee_\lambda h[f_\lambda^*]\Big) \circ \varphi = \bigvee_\lambda (h[f_\lambda^*] \circ \varphi)$$
$$= \bigvee_\lambda h[f_\lambda^* \circ \hat{\varphi}] = h[f^* \circ \hat{\varphi}].$$

The formula (13) is thus true of any lower semi-continuous f^* and also of any upper semi-continuous f^*.

Suppose now that f^* is just measurable. Then, there exist an increasing sequence $\{g_n^*\}$ of bounded upper semi-continuous functions and an decreasing sequence $\{h_n^*\}$ of bounded lower semi-continuous functions on Δ such that $g_n^* \leqslant f_n^* \leqslant h_n^*$ for all n and

$$\lim_{n\to\infty} \int g_n^*(b)\, d\chi(b) = \int f^*(b)\, d\chi\,(b) = \lim_{n\to\infty} \int h_n^*(b)\, d\chi\,(b).$$

It follows that $\{h[g_n^*] : n = 1, 2, \ldots\}$ is increasing, $\{h[h_n^*] : n = 1, 2, \ldots\}$ is decreasing, and $h[g_n^*] \leqslant h[f^*] \leqslant h[h_n^*]$ for all n. Moreover,

$$h[h_n^*](a_0) - h[g_n^*](a_0) = \int (h_n^*(b) - g_n^*(b))\, d\chi\,(b) \to 0.$$

So we have $\bigvee_n h[g_n^*] = h[f^*] = \bigwedge_n h[h_n^*]$. By Proposition 2.6, $\bigvee_n (h[g_n^*] \circ \varphi) = h[f^*] \circ \varphi = \bigwedge_n (h[h_n^*] \circ \varphi)$ and consequently

$$\bigvee_n h[g_n^* \circ \hat{\varphi}] = h[f^*] \circ \varphi = \bigwedge_n h[h_n^* \circ \hat{\varphi}].$$

On the other hand, it is clear that

$$\bigvee_n h[g_n^* \circ \hat{\varphi}] \leqslant h[f^* \circ \hat{\varphi}] \leqslant \bigwedge_n h[h_n^* \circ \hat{\varphi}].$$

Hence, we have $h[f^*] \circ \varphi = h[f^* \circ \hat{\varphi}]$.

So far, we have assumed that f^* is defined everywhere on Δ. Since $h[f^*]$ does not change by any change of f^* on a negligible subset of Δ, we infer that $f^* \circ \hat{\varphi}$ changes only on a negligible subset of ∂U by a mentioned change on f^*. Hence we conclude that the formula (13) is true of any class function $f^* \in L^\infty(d\chi)$, as was to be proved. Q.E.D.

COROLLARY 6.4. — *If a measurable subset* A *of* Δ *is negligible (resp. has positive measure), then* $\hat{\varphi}^{-1}(A)$ *is negligible (resp., has positive measure) in* ∂U.

This has essentially been proved in the last paragraph of the proof of Lemma 6.3. This shows us that $f^* \circ \hat{\varphi}$ is a well-defined class function on ∂U for any $f^* \in L^1(d\chi)$. We finally show the following.

PROPOSITION 6.5. — *If* $f^* \in L^1(d\chi)$, *then* $f^* \circ \hat{\varphi} \in L^1(d\sigma)$ *and* (13) *holds. If* $f^* \in L^p(d\chi)$ *with* $1 \leqslant p \leqslant \infty$, *then* $f^* \circ \hat{\varphi}$ *belongs to* $L^p(d\sigma)$ *and the correspondence* $f^* \to f^* \circ \hat{\varphi}$ *is an isometric isomorphism of* $L^p(d\chi)$ *onto* $L^p(d\sigma)_T$, *where* $L^p(d\sigma)_T$ *denotes the set of* T-*invariant functions in* $L^p(d\sigma)$.

Proof. — We may suppose that f^* is real and positive. We put $f_n^* = \inf\{f^*, n\}$, $n = 1, 2, \ldots$. By Lemma 6.3, we have $h[f_n^*] \circ \varphi = h[f_n^* \circ \hat{\varphi}]$, $n = 1, 2, \ldots$. Corollary 3.4 shows that $f_n^* \circ \hat{\varphi}$ can be regarded as the boundary function for the harmonic function $h[f_n^*] \circ \varphi$. Clearly, $h[f^*] \circ \varphi$ is a majorant of $h[f_n^*] \circ \varphi$ for each n. It follows that, if $\hat{h}$ is the boundary function for $h[f^*] \circ \varphi$, we have $\hat{h} \in L^1(d\sigma)$ and $f_n^* \circ \hat{\varphi} \leqslant \hat{h}$ a.e. for each n. So, $f^* \circ \hat{\varphi} \leqslant \hat{h}$ a.e. and consequently $f^* \circ \hat{\varphi} \in L^1(d\sigma)$. Moreover, we have

$$\begin{aligned} h[f^*] \circ \varphi &= h\,[\sup_n f_n^*] \circ \varphi = \Big(\bigvee_n h[f_n^*]\Big) \circ \varphi = \bigvee_n (h[f_n^*] \circ \varphi) \\ &= \bigvee_n h[f_n^* \circ \hat{\varphi}] = h\,[\sup_n (f_n^* \circ \hat{\varphi})] = h[f^* \circ \hat{\varphi}], \end{aligned}$$

as was to be proved.

Next we suppose $f^* \in L^p(d\chi)$ with $1 \leqslant p < \infty$. Then, $h[f^*]$ belongs to $h^p(R)$ and so $h[f^*] \circ \varphi \in h^p(U)$. Since $f^* \circ \hat{\varphi}$ is, by (13), the boundary function for $h[f^*] \circ \varphi$, it belongs to $L^p(d\sigma)$ by Theorem 3.5. Moreover, the same theorem shows that

$$\|f^*\|_p = \|h[f^*]\|_p = \|h[f^*] \circ \varphi\|_p = \|h[f^* \circ \hat{\varphi}]\|_p = \|f^* \circ \hat{\varphi}\|_p,$$

and therefore that the correspondence $f^* \to f^* \circ \hat{\varphi}$ is an isometric map of $L^p(d\chi)$ onto $L^p(d\sigma)_T$.

7. Invariant subspaces of $L^p(d\chi)$.

Let R be a hyperbolic Riemann surface which satisfies the conditions (A), (B) and (C). We know by Theorem 3.5 that the map $h \to \hat{h}$ gives an isometric linear injection of $H^p(R)$ into $L^p(d\chi)$ for each p with $1 \leqslant p \leqslant \infty$. By use of this map, we can identify, for each p, $H^p(R)$ with a subspace of $L^p(d\chi)$, which we shall denote by $H^p(d\chi)$. We define

$$H^p(d\chi)_0 = \left\{ u^* \in H^p(d\chi) : \int u^*(b)\, d\chi(b) = 0 \right\}.$$

We note that $H^\infty(d\chi)$ and $H^\infty(d\chi)_0$ are both subalgebras of $L^\infty(d\chi)$. In this section, we are going to determine closed (weakly*-closed, if $p = \infty$) subspaces of $L^p(d\chi)$ that are invariant under multiplication by functions in $H^\infty(d\chi)$. To do this, we first define the boundary values of multiplicative analytic functions. We say (cf. [5]) that a function $Q : \alpha \to Q(.\,;\alpha)$ of $H_1(R;\mathbf{Z})$ into the space of all measurable functions on Δ_1 modulo χ-null functions is an *m-function* of character $\theta \in \Pi$, if $Q(.\,;\alpha) = \theta(\alpha)\theta(\beta)^{-1}Q(.\,;\beta)$ a.e. for any α, β in $H_1(R;\mathbf{Z})$. Two m-functions Q_1 and Q_2 are called equivalent and denoted as $Q_1 \equiv Q_2$ if they have the same character θ and there is an $\alpha_0 \in H_1(R;\mathbf{Z})$ such that $Q_2(.\,;\alpha) = \theta(\alpha_0)Q_1(.\,;\alpha)$ a.e. for every $\alpha \in H_1(R;\mathbf{Z})$.

Now, we denote by $MH^p(R)$, $1 \leqslant p \leqslant \infty$, the set of all multiplicative analytic functions f on R such that $|f|^p$ has a harmonic majorant on R if $p < \infty$ and $|f|$ is bounded on R if $p = \infty$. Let f be a non-constant function in $MH^p(R)$ with character θ. Take any single-valued branch of f on the Green star region $\mathbf{G}'(R; a_0)$ (cf. Section 5) and denote it as $f(z;0)$, where 0 denotes the zero element of $H_1(R;\mathbf{Z})$. For any $\alpha \in H_1(R;\mathbf{Z})$, we denote by $f(z;\alpha)$ the single-valued branch of f on $\mathbf{G}'(R;a_0)$ which is obtained by an analytic continuation of $f(z;0)$ along the path α. We clearly have $f(z;\alpha) = \theta(\alpha)f(z;0)$ for each $\alpha \in H_1(R;\mathbf{Z})$ and $z \in \mathbf{G}'(R;a_0)$. By Lemma 5.5, $f(z;0)$ has a radial limit a.e. on $\mathbf{G}(R;a_0)$. We put $\hat{f}(b;0) = f(L;0)$ if $L \in \Lambda(a_0)$, $b = b_L$, and $f(L;0)$ exists in the sense explained in Section 5. In view of Lemma 5.5 and Proposition 5.8, $\hat{f}(b; 0)$ is well-defined as a class function on Δ_1 and belongs to $L^p(d\chi)$. For each $\alpha \in H_1(R; \mathbf{Z})$, $\hat{f}(b; \alpha)$ is defined similarly as the radial limit a.e. of the branch $f(z; \alpha)$. We have of course $\hat{f}(b; \alpha) = \theta(\alpha)\hat{f}(b; 0)$ a.e. for each $\alpha \in H_1(R; \mathbf{Z})$, so that $\alpha \to \hat{f}(\,.\,; \alpha)$ is an m-function. It is clear that a different choice of the initial branch gives rise to an equivalent m-function. Thus, each $f \in MH^p(R)$ defines a set of mutually equivalent m-functions, any one of them being denoted as $\hat{f}$.

Further, we say that an m-function Q is an *i-function* if $|Q(b; \alpha)| = 1$ a.e. on Δ_1 for each $\alpha \in H_1(R; \mathbf{Z})$. Now we are in the position to prove our main result.

THEOREM 7.1. — *Let* $1 \leqslant p \leqslant \infty$. *Let* $\mathfrak{M}$ *be a closed (weakly* closed, if* $p = \infty$) *subspace of* $L^p(d\chi)$ *such that* $H^\infty(d\chi)\,\mathfrak{M} \subseteq \mathfrak{M}$.

a) $\mathfrak{M}$ *is doubly invariant,* i.e., $H^\infty(d\chi)_0\,\mathfrak{M}$ *is dense (weakly* dense, if* $p = \infty$) *in* $\mathfrak{M}$, *if and only if there exists a measurable subset* Σ *of* Δ_1 *such that* $\mathfrak{M} = C_\Sigma L^p(d\chi)$, *where* C_Σ *denotes the characteristic function of* Σ. *The set* Σ *is determined by* $\mathfrak{M}$ *uniquely up to a null set.*

b) $\mathfrak{M}$ *is simply invariant,* i.e., $H^\infty(d\chi)_0\mathfrak{M}$ *is not dense (weakly* dense, if* $p = \infty$) *in* $\mathfrak{M}$, *if and only if there exists an i-function* Q *of some character* $\theta \in \Pi$ *such that*

$$(14)\quad \mathfrak{M} = \{f^* \in L^p(d\chi) : f^*/Q \equiv \hat{h} \text{ for some } h \in MH^p(R)\}.$$

The i-function Q *is determined uniquely by* $\mathfrak{M}$ *up to equivalence.*

Proof. — First we consider the case $1 \leqslant p < \infty$. Let $\mathfrak{M}$ be a closed subspace of $L^p(d\chi)$ invariant under the multiplication of functions in $H^\infty(d\chi)$. Let $\{\mathfrak{M}\}_p$ be the smallest closed subspace of $L^p(d\sigma)$ that contains all $f^* \circ \hat{\varphi}$ with $f^* \in \mathfrak{M}$ and is invariant under the multiplication by the coordinate function $e^{i\omega}$ on ∂U. Then, $\{\mathfrak{M}\}_p$ is either doubly invariant (i.e., $e^{i\omega}\{\mathfrak{M}\}_p = \{\mathfrak{M}\}_p$) or simply invariant (i.e., $e^{i\omega}\{\mathfrak{M}\}_p \subsetneq \{\mathfrak{M}\}_p$). We shall investigate these two cases separately.

(i) Suppose first that $\{\mathfrak{M}\}_p$ is doubly invariant. Then, by [13], $\{\mathfrak{M}\}_p = C_{S'}L^p(d\sigma)$, where S′ is a measurable subset of ∂U and $C_{S'}$ denotes its characteristic function. Since $\{\mathfrak{M}\}_p$ is invariant under T, we may assume that S′ is invariant under T, i.e., $\tau(S') = S'$ for any $\tau \in T$. So, $C_{S'} \in L^p(d\sigma)_T$. By Proposition 6.5, there exists an element Q in $L^\infty(d\chi)$ such that $Q \circ \hat{\varphi} = C_{S'}$ a.e. on ∂U. This shows that Q takes either 0 or 1 up to a null set. Namely, Q determines a measurable subset Σ of Δ_1 such that $Q = C_\Sigma$. We shall show that $\mathfrak{M} = C_\Sigma L^p(d\chi)$.

If $f^* \in \mathfrak{M}$, then $f^* \circ \hat{\varphi} \in \{\mathfrak{M}\}_p$ so that

$$(C_\Sigma f^*) \circ \hat{\varphi} = (C_\Sigma \circ \hat{\varphi})(f^* \circ \hat{\varphi}) = C_{S'}(f^* \circ \hat{\varphi}) = f^* \circ \hat{\varphi}.$$

Thus, $C_\Sigma f^* = f^*$ a.e. and so $f^* \in C_\Sigma L^p(d\chi)$. Hence,

$$\mathfrak{M} \subseteq C_\Sigma L^p(d\chi).$$

In order to show the reverse inclusion, we take any s^* in $L^{p'}(d\chi)$ with $p^{-1} + p'^{-1} = 1$. Then, $s^* \circ \hat{\varphi} \in L^{p'}(d\sigma)$. Now suppose that s^* is orthogonal to $\mathfrak{M}$, i.e.,

$$\int_{\Delta_1} s^*(b) f^*(b)\, d\chi(b) = 0$$

for every $f^* \in \mathfrak{M}$. Let g be the function defined by (2) with $a = a_0$ and define $B_0 \in H^\infty(R)$ by $|B_0| = g\delta(\theta(g)^{-1})$. Let u be any meromorphic function on R such that $g|u|$ is bounded on R. Then, $B_0 u \in H^\infty(R)$ and therefore

$$\hat{B}_0 \hat{u} f^* \in \mathfrak{M}$$

for any $f^* \in \mathfrak{M}$. So we have

$$\int_{\Delta_1} \hat{B}_0(b)\hat{u}(b) f^*(b) s^*(b)\, d\chi(b) = 0.$$

By Theorem 5.12, there exists an analytic function $M(f^*)$ in $H^1(R)$ such that $\hat{M}(f^*) = \hat{B}_0 f^* s^*$ a.e. on Δ_1. By considering the case $u = 1$, we have $M(f^*)(a_0) = 0$. Proposition 6.5 shows us that

$$h[(\hat{B}_0 f^* s^*) \circ \hat{\varphi}] = h[\hat{B}_0 f^* s^*] \circ \varphi = M(f^*) \circ \varphi.$$

So $(\hat{B}_0 f^* s^*) \circ \hat{\varphi}$ is the boundary function for

$$M(f^*) \circ \varphi \in H^1(U).$$

For any analytic function ν on U, continuous up to the boundary ∂U, we thus have

$$\begin{aligned}\int_{\partial U} \nu(e^{i\omega})((\hat{B}_0 f^* s^*) \circ \hat{\varphi})(e^{i\omega})\, d\sigma(\omega) &= \nu(0)(M(f^*) \circ \hat{\varphi})(0) \\ &= \nu(0)(M(f^*))(a_0) = 0.\end{aligned}$$

Taking L^p limits in $\nu(f^* \circ \hat{\varphi})$, we see that

$$\int_{\partial U} ((\hat{B}_0 s^*) \circ \hat{\varphi}) f_1 \, d\sigma = 0 \quad \text{for any } f_1 \in \{\mathfrak{M}\}_p.$$

Since $\{\mathfrak{M}\}_p = C_{S'} L^p(d\sigma)$, $(\hat{B}_0 s^*) \circ \hat{\varphi}$ must vanish a.e. on S' and consequently $\hat{B}_0 s^*$ must vanish a.e. on Σ. Since $\hat{B}_0$ can vanish only on a set of measure 0 in view of Lemma 5.1, s^* must vanish a.e. on Σ. This shows that $s^* \perp C_\Sigma L^p(d\chi)$ and therefore $C_\Sigma L^p(d\chi) \subseteq \mathfrak{M}$, as was to be proved.

(ii) Now suppose that $\{\mathfrak{M}\}_p$ is simply invariant. Then, by [14], there exists a function $q \in L^\infty(d\sigma)$ with $|q| = 1$ a.e. on ∂U such that $\{\mathfrak{M}\}_p = q H^p(d\sigma)$. Since $\{\mathfrak{M}\}_p$ is invariant under T, there exists a character $\tau \to c(\tau)$ of the group T such that $q \circ \tau = c(\tau) q$ a.e. on ∂U.

For any $\tau \in T$, we draw a curve Γ joining the origin 0 with $\tau(0)$ within U. Then $\varphi(\Gamma)$ is a 1-cycle starting from a_0. Clearly any two such curves define homologous cycles of R. Therefore, $\varphi(\Gamma)$ determines an element α in the group $H_1(R; \mathbf{Z})$. The correspondence $\tau \to \alpha$ preserves the group operations so that it gives a homomorphism of T onto $H_1(R; \mathbf{Z})$, which we call the canonical homomorphism of T onto $H_1(R; \mathbf{Z})$. Thus, the above character $\tau \to c(\tau)$ of T induces a character θ of $H_1(R; \mathbf{Z})$ such that $\theta(\alpha) = c(\tau)$, where $\tau \to \alpha$ is the canonical homomorphism of T onto $H_1(R; \mathbf{Z})$.

Now let $N \in MH^\infty(R)$ be such that $|N| = \delta(\theta)$ $(= u$, say) and let $N(z; \alpha)$ for $z \in \mathbf{G}'(R; a_0)$ and $\alpha \in H_1(R; \mathbf{Z})$ be defined as in the second paragraph of this section. Furthermore, let N_1 be the analytic function on U such that

$$|N_1| = |N| \circ \varphi$$

and $N_1(0) = N(a_0;\ 0)$. Then, $N_1 \circ \tau = c(\tau)N_1$ for any $\tau \in T$. Let $f^* \in \mathfrak{M}$. Then there exists a function $F \in H^p(U)$ such that $f^* \circ \hat{\varphi} = q\hat{F}$. Multiplying N_1 on both sides, we get $\hat{N}_1\bar{q}(f^* \circ \hat{\varphi}) = \hat{N}_1\hat{F}$, where N_1F is a T-invariant function in $H^1(U)$. So we can find an h in $H^1(R)$ with

$$h \circ \varphi = N_1F.$$

By (13), we have

$$h[\hat{h} \circ \hat{\varphi}] = h[\hat{h}] \circ \varphi = h \circ \varphi = N_1F = h[\hat{N}_1\hat{F}].$$

So, $\hat{h} \circ \hat{\varphi} = \hat{N}_1\hat{F}$ a.e. on ∂U and therefore

$$(\hat{h}/f^*) \circ \hat{\varphi} = (\hat{N}_1\hat{F})/(f^* \circ \hat{\varphi}) = \hat{N}_1\bar{q}$$

a.e. on ∂U. This shows that $\hat{h}/f^*$ is independent of the choice of f^* in $\mathfrak{M}$.

Since $u = \delta(\theta)$ is outer, $\log u$ is an outer function in HP(R) so that $\hat{u}$ exists a.e. on Δ_1 and $\log u = h\,[\log \hat{u}]$. Thus, we have

$$\begin{aligned} h\,[\log|\hat{N}_1|] &= \log|N_1| = \log(u \circ \varphi) \\ &= (\log u) \circ \varphi = h\,[\log \hat{u}] \circ \varphi = h\,[\log(\hat{u} \circ \hat{\varphi})]. \end{aligned}$$

Hence, $|\hat{N}_1| = \hat{u} \circ \hat{\varphi}$ a.e. on ∂U. Now, by Proposition 2.4,

$$\log u = \lim_{m\to\infty}\lim_{n\to\infty}\,[(-m) \vee (n \wedge (\log u))]. \tag{15}$$

Put $u_{m,n} = (-m) \vee (n \wedge (\log u))$ for $m,\ n = 1,\ 2,\ \ldots.$ Proposition 5.8 says that $u_{m,n}$ has a radial limit a.e. on $\mathbf{G}(R;\ a_0)$ and $u_{m,n}(L) = \hat{u}_{m,n}(b_L)$ a.e. on $\mathbf{G}(R;\ a_0)$. It follows from this and (15) that $\log u$ has a radial limit a.e. and $\log u(L) = \log \hat{u}(b_L)$ a.e., i.e., u has a radial limit a.e. and $u(L) = \hat{u}(b_L)$ a.e.

On the other hand, consider any branch $N(z;\ \alpha)$ of N on the Green star region $\mathbf{G}'(R;\ a_0)$. Then, Lemma 5.5 states that $N(z;\ \alpha)$ has a radial limit a.e. on $\mathbf{G}(R;\ a_0)$. Since $|N(z;\ \alpha)| = u(z)$ on R, we conclude that $|N(L;\ \alpha)| = \hat{u}(b_L)$ a.e. We define $\hat{N}(b;\ \alpha) = N(L;\ \alpha)$ with $b = b_L$, whenever $N(L;\ \alpha)$, $L \in \Lambda(a_0)$, exists. Finally we put

$$Q(b;\ \alpha) = f^*(b)\hat{N}(b;\ \alpha)/\hat{h}(b)$$

if the right-hand side is defined. For each $\alpha \in H_1(R; \mathbf{Z})$, $Q(b; \alpha)$ is well-defined up to a null set and

$$|Q(b; \alpha)| = |f^*(b)/\hat{h}(b)||\hat{N}(b; \alpha)| = |f^*(b)/\hat{h}(b)|\hat{u}(b) = 1$$

a.e. So $Q(b; \alpha)$ is an i-function. Thus, for each $f^* \in \mathfrak{M}$, we have $f^*(b)/Q(b; \alpha) = \hat{h}(b)/\hat{N}(b; \alpha)$ a.e. on Δ_1 for each $\alpha \in H_1(R; \mathbf{Z})$. Since $|h(z)/N(z; \alpha)| \circ \varphi = |N_1F|/|N_1| = |F|$, we see that h/N belongs to $MH^p(R)$. Hence,

$$f^* \in H^p(d\chi; Q),$$

where $H^p(d\chi; Q)$ denotes the right-hand side of (14). This shows that $\mathfrak{M}$ is included in $H^p(d\chi; Q)$.

Next we shall show the reverse inclusion. Let $s^* \in L^{p'}(d\chi)$ be orthogonal to $\mathfrak{M}$. Since $\{\mathfrak{M}\}_p = qH^p(d\sigma)$, we have, as in (i),

$$\int_{\partial U} q\hat{F}((\hat{B}_0 s^*) \circ \hat{\varphi})\, d\sigma = 0 \quad \text{for any} \quad F \in H^p(U).$$

So, $q((\hat{B}_0 s^*) \circ \hat{\varphi}) \in H^{p'}(d\sigma)$ and $\int_{\partial U} q((\hat{B}_0 s^*) \circ \hat{\varphi})\, d\sigma = 0$.

Let $\nu^* \in H^p(d\chi; Q)$ and $M \in MH^p(R)$ be such that $\nu^*(b)/Q(b; \alpha) = \hat{M}(b; \alpha)$ a.e. on Δ_1 for each $\alpha \in H_1(R; \mathbf{Z})$. We use the representation $Q(b; \alpha) = f^*(b)\hat{N}(b; \alpha)/\hat{h}(b)$ defined above. So we have $\nu^*(b) = f^*(b)\hat{N}(b; \alpha)\hat{M}(b; \alpha)/\hat{h}(b)$, where $N(z; \alpha)M(z; \alpha)$ is independent of α. Namely,

$$N(z; \alpha)M(z; \alpha)$$

defines a single-valued analytic function $K(z)$ in $H^p(R)$. Let M_1 be the analytic function on U such that

$$|M_1| = |M| \circ \varphi \quad \text{and} \quad M_1(0) = M(a_0; 0).$$

It follows that $K \circ \varphi = M_1N_1$ and therefore

$$(\nu^* \circ \hat{\varphi})((\hat{B}_0 s^*) \circ \hat{\varphi}) = \hat{M}_1 q((\hat{B}_0 s^*) \circ \hat{\varphi}) \in H^1(d\sigma).$$

So, $$h[\nu^*\hat{B}_0 s^*] \circ \varphi = h[(\nu^*\hat{B}_0 s^*) \circ \hat{\varphi}] = h[\hat{M}_1 q((\hat{B}_0 s^*) \circ \hat{\varphi})],$$

the last member being in $H^1(U)$ and vanishing at the origin 0. Hence, $h[\nu^*\hat{B}_0 s^*]$ belongs to $H^1(R)$ and vanishes at a_0.

We define $\nu = h[\nu^*\hat{B}_0 s^*]$, $\nu_+ = \exp\,((\log|\nu|) \wedge 0)$ and $\nu_- = \exp\,(-((\log|\nu|) \vee 0))$, where ν_- is an outer l.a.m.

by Proposition 2.7. Further, we define $k \in H^\infty(R)$ by $|k| = \delta(\theta(g))\delta(\theta(g)^{-1})$. Then, $\varphi k/B_0$ is meromorphic on R, vanishes at a_0 and satisfies

$$g|\varphi k/B_0| = \varphi\delta(\theta(g)).$$

Thus, $g|\varphi k/B_0|$ has a harmonic majorant on R. By Theorem 5.3, we have

$$\int_{\Delta_1} (\hat{\varphi}(b)\hat{k}(b)/\hat{B}_0(b))\, d\chi(b) = (\varphi k/B_0)(a_0) = 0.$$

Since $\hat{\varphi}(b) = \varphi^*(b)\hat{B}_0(b)s^*(b)$ a.e. on Δ_1, we get

$$(16) \qquad \int_{\Delta_1} \hat{k}(b)\varphi^*(b)s^*(b)\, d\chi(b) = 0$$

for any $\varphi^* \in H^p(d\chi;\ Q)$ and any $s^* \perp \mathfrak{M}$.

Since k is outer, it is β exterior. To show this, we put $k_n = (-\log|k|) \wedge n$ for $n = 1, 2, \ldots$ Then, each k_n is outer, $\exp(k_n)$ is bounded on R and $k_n \to -\log|k|$ pointwise on R. We have

$$|k| \exp(k_n) = \exp(-(-\log|k|) + k_n) \leqslant 1$$

and $|k|\exp(k_n) \to 1$ pointwise on R. We define t_n in $H^\infty(R)$ by the condition $|t_n| = \delta(\theta(\exp(k_n))^{-1})\exp(k_n)$. Then, $\{kt_n : n = 1, 2, \ldots\}$ is a norm-bounded sequence in $H^\infty(R)$, so that it has a β convergent subsequence $\{kt_{n(j)} : j = 1, 2, \ldots\}$. Let $t \in H^\infty(R)$ be the limit of this subsequence. We thus have

$$\begin{aligned} |t| &= \lim_{j\to\infty} |kt_{n(j)}| = \lim_{j\to\infty} \delta(\theta(\exp(k_{n(j)}))^{-1})|k| \exp(k_{n(j)}) \\ &= \lim_{j\to\infty} \delta(\theta(\exp(k_{n(j)}))^{-1}). \end{aligned}$$

By the condition (B), t is β exterior and consequently k is β exterior.

Thus, there exists a net $\{t_\lambda\}$ in $H^\infty(R)$ such that $t_\lambda k$ converge to 1 with respect to the β topology. Theorem 3.6 shows that $\hat{t}_\lambda\hat{k}$ converge to 1 with respect to the weak* topology $\sigma(L^\infty(d\chi), L^1(d\chi))$. Since $H^p(d\chi;\ Q)$ is invariant, (16) implies

$$\int_{\Delta_1} \hat{t}_\lambda(b)\hat{k}(b)\varphi^*(b)s^*(b)\, d\chi(b) = 0$$

for every λ. By taking limit in λ, we see finally that

$$\int_{\Delta_1} \varphi^*(b)s^*(b)\, d\chi(b) = 0.$$

As s^* is arbitrary, we have $\varphi^* \in \mathfrak{M}$. Hence,

$$H^p(d\chi; Q) \subseteq \mathfrak{M},$$

as desired.

(iii) We shall show that $C_\Sigma L^p(d\chi)$ is doubly invariant for any $\Sigma \subseteq \Delta_1$ and that $H^p(d\chi; Q)$ is simply invariant for any i-function Q.

We first consider the case $\mathfrak{M} = C_\Sigma L^p(d\chi)$. We put

$$u(z) = \exp(-G(a_0, z))$$

and define $B_1 \in H^\infty(R)$ by $|B_1| = u\delta(\theta(u)^{-1})$. Then,

$$B_1(a_0) = 0$$

so that $\hat{B}_1 \in H^\infty(d\chi)_0$. We know that $\hat{u}(b) = 1$ a.e. on Δ_1. Trivially, $\hat{u}(\operatorname{sgn} \hat{B}_1)\mathfrak{M} = \mathfrak{M}$ or, equivalently,

$$(\hat{u}(\operatorname{sgn} \hat{B}_1))^{-1}\mathfrak{M} = \mathfrak{M}.$$

Since $\mathfrak{M}$ is invariant, we have

$$\hat{\varphi}\mathfrak{M} = (\hat{u}(\operatorname{sgn} \hat{B}_1))^{-1}\hat{B}_1\mathfrak{M} = \hat{B}_1\mathfrak{M},$$

where $\varphi = \delta(\theta(u)^{-1})$. As we shall show below, $\hat{\varphi}\mathfrak{M}$ is dense in $\mathfrak{M}$ and therefore $\hat{B}_1\mathfrak{M}$ is dense in $\mathfrak{M}$, which implies that $\mathfrak{M}$ is doubly invariant.

In order to show that $\hat{\varphi}\mathfrak{M}$ is dense $\mathfrak{M}$, we note that $-\log \varphi$ is a positive outer harmonic function on R. Putting $\varphi_n = (-\log \varphi) \wedge n$ for $n = 1, 2, \ldots$, we see as before that φ_n are outer, $\exp \varphi_n$ as well as $\exp(-\varphi_n)$ are bounded on R, $\varphi \exp(\varphi_n) \leqslant 1$, and φ_n converge increasingly to $-\log \varphi$ pointwise on R. By Propositions 3.1 and 3.3, $\hat{\varphi}_n$ converge increasingly to $-\log \hat{\varphi}$ in $L^1(d\chi)$. So some subsequence $\{\hat{\varphi}_{n(j)}: j = 1, 2, \ldots\}$ of $\{\hat{\varphi}_n\}$ converges increasingly to $-\log \hat{\varphi}$ a.e. on Δ_1 and therefore $\exp(-\hat{\varphi}_{n(j)})$ converge decreasingly to $\hat{\varphi}$ a.e. on Δ_1. Now let $f^* \in \mathfrak{M}$. Then,

$$f^*\hat{\varphi}.\exp(\hat{\varphi}_{n(j)}) \in \hat{\varphi}\mathfrak{M}.$$

Since $\hat{\varphi}.\exp(\hat{\varphi}_{n(j)}) \leqslant 1$ a.e., we have

$$|f^*\hat{\varphi}.\exp(\hat{\varphi}_{n(j)})| \leqslant |f^*|$$

a.e. and $f^*\hat{\varphi}.\exp(\hat{\varphi}_{n(j)})$ converge a.e. to f^* on Δ_1. By Lebesgue's dominated converge theorem, we see that $f^*\hat{\varphi}.\exp(\hat{\varphi}_{n(j)})$ converge to f^* with respect to the weak topology $\sigma(L^p(d\chi), L^{p'}(d\chi))$. Thus, $\hat{\varphi}\mathfrak{M}$ is weakly dense in $\mathfrak{M}$. Since $\hat{\varphi}\mathfrak{M}$ is a convex subset of $L^p(d\chi)$, its weak closure is exactly equal to its norm-closure. Hence $\hat{\varphi}\mathfrak{M}$ is dense in $\mathfrak{M}$, as was to be proved.

Next we consider the case $\mathfrak{M} = H^p(d\chi; Q)$. We take any f^* in the closure of $H^\infty(d\chi)_0\mathfrak{M}$, i.e., there exists a sequence $\{u_n : u_n(a_0) = 0, n = 1, 2, \ldots\}$ in $H^\infty(R)$ and a sequence $\{f_n^* : n = 1, 2, \ldots\}$ in $\mathfrak{M}$ such that $\hat{u}_n f_n^*$ converge to f^* in $L^p(d\chi)$. Let h and h_n, $n = 1, 2, \ldots$, be in $MH^p(R)$ such that

$$f^*(b)/Q(b; \alpha) = \hat{h}(b; \alpha) \qquad \text{and} \qquad f_n^*(b)/Q(b; \alpha) = \hat{h}_n(b; \alpha)$$

a.e. on Δ_1 for each $\alpha \in H_1(R; \mathbf{Z})$. We also take $N \in MH^\infty(R)$ in such a way that $|N| = \delta(\theta)$, where θ denotes the character of Q. Then, $\hat{N}(b; \alpha)\hat{h}(b; \alpha) = \hat{N}(b; \alpha)f^*(b)/Q(b; \alpha)$ is the L^p limit of the sequence

$$\hat{N}(b; \alpha)\hat{u}_n(b)f_n^*(b)/Q(b; \alpha) = \hat{u}_n(b)\hat{N}(b; \alpha)\hat{h}_n(b; \alpha).$$

It follows easily that the sequence of single-valued analytic functions $u_n(z)N(z; \alpha)h_n(z; \alpha)$ on R converges to

$$N(z; \alpha)h(z; \alpha)$$

uniformly on compact subsets of R. Since $u_n(a_0) = 0$ for each n, we have $N(a_0; \alpha)h(a_0; \alpha) = 0$ and therefore

$$h(a_0; \alpha) = 0.$$

On the other hand, let $N' \in MH^\infty(R)$ be such that

$$|N'| = \delta(\theta^{-1}).$$

Then, the function $\hat{N}'(b; \alpha)Q(b; \alpha)$ is independent of α and so determines a function $f' \in L^\infty(d\chi)$. Since

$$f'(b)/Q(b; \alpha) = \hat{N}'(b; \alpha),$$

we have $f' \in H^p(d\chi; Q)$ for every $1 \leqslant p \leqslant \infty$ and in particular $f' \in \mathfrak{M}$. Since $N'(a_0; \alpha) \neq 0$, the above observation shows that f' is not in the closure of $H^\infty(d\chi)_0\mathfrak{M}$. Hence, $\mathfrak{M}$ is simply invariant.

The proof of the theorem in the case of $1 \leqslant p < \infty$ can now be obtained easily by combining (i), (ii) and (iii). The case $p = \infty$ can be shown in the same way as in the case $1 \leqslant p < \infty$ by using the weak* topology $\sigma(L^\infty(d\chi), L^1(d\chi))$ in place of the L^p norm topology. The statements concerning the uniqueness of Σ and Q can be shown easily. This completes the proof of Theorem 7.1.

Finally, we deduce Neville's main result in [8] from the preceding theorem.

Corollary 7.2 (Neville [8; Theorem 7.1.1]). — *Let* R *be a hyperbolic Riemann surface for which* (A), (B) *and* (C) *hold. Let* $1 \leqslant p \leqslant \infty$ *and let* $\mathfrak{M}$ *be a closed* (β *closed, if* $p = \infty$) *subspace of* $H^p(R)$. *Then,* $\mathfrak{M}$ *is an* $H^\infty(R)$*-submodule of* $H^p(R)$ *if and only if it is quasi-principal,* i.e., *there exists a bounded inner* l.a.m. I *such that, for* $1 \leqslant p < \infty$,

$$\mathfrak{M} = \{f \in H^p(R) : (|f|/I)^p \textit{ admits a harmonic majorant}\}$$

and, for $p = \infty$,

$$\mathfrak{M} = \{f \in H^\infty(R) : |f|/I \textit{ is bounded}\}.$$

Proof. — Let $\mathfrak{M}$ be a non-trivial closed (β closed, if $p = \infty$) $H^\infty(R)$-submodule of $H^p(R)$, $1 \leqslant p \leqslant \infty$. Put

$$\mathfrak{M}_\Delta = \{\hat{f} : f \in \mathfrak{M}\},$$

which is the set of the boundary functions of the elements in $\mathfrak{M}$. It follows from Theorems 3.5 and 3.6 that $\mathfrak{M}_\Delta$ is a closed (weakly* closed, if $p = \infty$) $H^\infty(d\chi)$-submodule of $H^p(d\chi)$. Every nonzero function in $\mathfrak{M}_\Delta$ cannot vanish identically on any subset of Δ_1 of positive measure. So, $\mathfrak{M}_\Delta$ cannot be doubly invariant in view of Theorem 7.1 *a*). $\mathfrak{M}_\Delta$ is thus simply invariant so that there exists an *i*-function Q of some character θ with $\mathfrak{M}_\Delta = H^p(d\chi; Q)$. If $f \in \mathfrak{M}$, then

$$\hat{f} \in H^p(d\chi; Q)$$

and so there exists an $h \in \mathrm{MH}^p(\mathrm{R})$ such that

$$\hat{f}(b)/\mathrm{Q}(b;\alpha) = \hat{h}(b;\alpha)$$

a.e. on Δ_1 for any $\alpha \in \mathrm{H}_1(\mathrm{R};\mathbf{Z})$. Namely, $\hat{f}(b)/\hat{h}(b;\alpha)$ is independent of the choice of f in $\mathfrak{M}$. Since $f(z)/h(z;\alpha)$ is a multiplicative meromorphic function of bounded characteristic on R whose boundary values are independent of f, we see that the function $f(z)/h(z;\alpha)$ itself is also independent of $f \in \mathfrak{M}$. We put $q(z;\alpha) = f(z)/h(z;\alpha)$ so that

$$\hat{q}(b;\alpha) = \mathrm{Q}(b;\alpha)$$

a.e. on Δ_1 for any $\alpha \in \mathrm{H}_1(\mathrm{R};\mathbf{Z})$. Thus a function $f \in \mathrm{H}^p(\mathrm{R})$ belongs to $\mathfrak{M}$ if and only if $f(z)/q(z;\alpha)$ is in $\mathrm{MH}^p(\mathrm{R})$.

On the other hand, Proposition 2.7 implies that

$$p_{\mathrm{I}}(\log|f|) \leqslant 0$$

for any $f \in \mathfrak{M}$. Since $\mathrm{I(R)}$ is order complete, we see that $\vee\{p_{\mathrm{I}}(\log|f|) : f \in \mathfrak{M}\}$ $(= u_{\mathrm{I}},$ say) exists in $\mathrm{I(R)}$. If we put $\mathrm{I} = \exp u_{\mathrm{I}}$, then I is an inner l.a.m. on R and $(|f|/\mathrm{I})^p$ admits a harmonic majorant on R ($|f|/\mathrm{I}$ is bounded on R, if $p = \infty$) for any $f \in \mathfrak{M}$. Let $\mathrm{J} \in \mathrm{MH}^\infty(\mathrm{R})$ be such that $|\mathrm{J}| = \mathrm{I}$ on R. We have shown that $f(z)/\mathrm{J}(z;\alpha) \in \mathrm{MH}^p(\mathrm{R})$ for any $f \in \mathfrak{M}$.

From these observation follows that $q(z;\alpha)/\mathrm{J}(z;\alpha)$ belongs to $\mathrm{MH}^\infty(\mathrm{R})$. In fact, $q(z;\alpha)/\mathrm{J}(z;\alpha)$ is evidently a multiplicative meromorphic function of bounded characteristic. Suppose that this has a pole at a point, a', in R. We then take any nonzero $f \in \mathfrak{M}$, so that $f(z)/q(z;\alpha) \in \mathrm{MH}^p(\mathrm{R})$. We suppose that f/q has a zero of order $c' \geqslant 1$ at the point a'. Let B' be a meromorphic function on R such that $|\mathrm{B}'| = \exp(c'\mathrm{G}_{a'}).\delta(\theta(\exp(-c'\mathrm{G}_{a'})))$. Then, $f\mathrm{B}'$ is a meromorphic function of bounded characteristic on R such that we have $f(z)\mathrm{B}'(z)/q(z;\alpha) \in \mathrm{MH}^p(\mathrm{R})$ and $\hat{f}\hat{\mathrm{B}}' \in \mathrm{L}^p(d\chi)$. Therefore, $\hat{f}\hat{\mathrm{B}}'$ also belongs to $\mathfrak{M}_\Delta$, i.e., the boundary function of an analytic function in $\mathrm{H}^p(\mathrm{R})$. Since $f\mathrm{B}'$ is of bounded characteristic, it is determined by its boundary values, so that $f\mathrm{B}'$ belongs to $\mathrm{H}^p(\mathrm{R})$, too. Hence, $f\mathrm{B}' \in \mathfrak{M}$ and therefore $f(z)\mathrm{B}'(z)/\mathrm{J}(z;\alpha)$ belongs to $\mathrm{MH}^p(\mathrm{R})$. But

$$f(z)\mathrm{B}'(z)/\mathrm{J}(z;\alpha) = (f(z)\mathrm{B}'(z)/q(z;\alpha))(q(z;\alpha)/\mathrm{J}(z;\alpha))$$

should have a pole at a' in view of our construction of B$'$. This contradiction shows that $q(z; \alpha)/J(z; \alpha)$ must be analytic. Since J is analytic, q is also analytic. Since $|\hat{q}| = 1$ a.e. on Δ_1, $|q|$ is an inner l.a.m. and $|q|/I \leqslant 1$. Since $(|f|/|q|)^p$ admits a harmonic majorant ($|f|/|q|$ is bounded, if $p = \infty$) and so $p_1 (\log |f|) \leqslant \log |q|$ for any $f \in \mathfrak{M}$, we see that $\log I \leqslant \log |q|$, or equivalently, $I/|q| \leqslant 1$. So, $|q| = I$ and therefore $\hat{q}$ and $\hat{J}$ are equivalent. Hence, the subspace $\mathfrak{M}$ has the desired form. The converse statement is obvious. Q.E.D.

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