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TOPOLOGICAL CONJUGACY OF LOCALLY FREE R^{n-1} ACTIONS ON n-MANIFOLDS

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This paper considers locally free differentiable actions of R^{n-1} on a compact orientable n-manifold which have no compact orbits. For such an action, it is known that all orbits are isomorphic to $T^k \times R^{n-k-1}$ for k fixed, $n - k \geqslant 2$, where T^k is the k-torus [1]. We associate to such an action a collection of rotation numbers which are elements of S^1. Theorem 1 states that when such an action is free $(k = 0)$, it is topologically conjugate (i.e. in a parameter preserving sense) to a linear action on a torus, provided one of the rotation numbers is sufficiently irrational (i.e. satisfies a Liouville inequality). Theorem 2 treats the case of a sufficiently irrational rotation number and $k \neq 0$. It gives again a topological conjugacy to a linear action on the n-torus when

$$n - k > 2.$$

When $n - k = 2$, it gives a topological conjugacy to a certain type of action on a principal T^k bundle over T^2.

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We will take the k-torus T^k to be the quotient of R^k by the integer lattice Z^k. An action $\bar{\alpha}: R^k \times T^n \to T^n$ of R^k on T^n is said to be linear if it is obtained as the projection of an action α of R^k on R^n defined by

$$\alpha(r, x) = x + Ar,$$

where A is an $n \times k$ matrix of real numbers. Two actions φ and φ' of a Lie group G on manifolds V and V' respectively are said to be topologically conjugate if there is a homeomorphism $f\colon V \to V'$ such that $f(\varphi(g, v)) = \varphi'(g, f(v))$ for all g in G, v in V. All actions will be differentiable and differentiable will mean of class C^∞. Further details of part I can be found in the thesis of R. Tischler [3].

1. Let φ be a locally free action of R^{n-1} on a compact connected orientable n-manifold V^n with no compact orbits. We begin by describing results in [1] about the foliation whose leaves are the orbits of φ. It is known that since no orbit of φ is compact, every orbit is dense in V^n and the holonomy group of each orbit has at most two elements ([1], theorems 8 and 9). Since V is orientable the foliation is orientable and hence there is no holonomy.

One can find a differentiably embedded circle S^1 in V^n which is transverse to the orbits of φ. Each orbit meets this circle since they are dense. There is induced by the foliation a pseudogroup Γ of local orientation preserving diffeomorphisms of S^1. These are defined by holonomy mappings from one interval to another interval on the embedded circle. Theorem 6 of [1] shows that there is a bundle-like metric on V^n for which the embedded circle is orthogonal to the leaves. This means that the orthogonal flow to the leaves parametrized by arc length leaves the foliation invariant, and in particular that the metric induced on the transversal circle is invariant under the action of Γ. In fact, we can assume S^1 to be parametrized as p_t, t in R/Z, such that for each element f in Γ there is a λ in S^1 such that $f(p_t) = p_{t+\lambda}$ for any p_t in the domain of f.

In order to have the bundle-like metric differentiable, it is necessary in general to choose a new atlas for V^n. The new atlas, ([1], p. 97), is constructed as follows: let

$$h\colon U \to R^1 \times R^{n-1}$$

be a chart such that the projection to R^1 defines the foliation locally. Then a chart for the new atlas is constructed from h by composition with a homeomorphism of $R^1 \times R^{n-1}$ which is the product of the identity on R^{n-1} and a particular homeomorphism of R^1. Because the change of atlas is of this form, the R^{n-1} action remains differentiable for the new atlas on V^n.

THEOREM 1. — *Let* φ *be a free differentiable action of* R^{n-1} *on a compact connected oriented n-manifold* V^n. *Suppose that there is a real number* λ *such that for some* f *in* Γ *described above, and some* p_t, $f(p_t) = p_{t+\lambda}$, *where* λ *satisfies the Liouville inequality*

$$(1.1) \qquad \left|\lambda \pm \frac{m}{n}\right| > \frac{C}{n^{\gamma}}$$

for arbitrary integers m, n, *and fixed real numbers* C *and* γ *with* $\gamma > 2$. *Then* φ *is topologically conjugate to a linear action* $\bar{\alpha}$ *of* R^{n-1} *on the n-torus* T^n.

It is known that all irrational numbers except a set of transcendental numbers of measure zero satisfy an inequality such as (1.1).

Remark. — Theorem I gives a topological conjugacy to a linear action. However, if one uses the atlas on V^n which makes the bundle-like metric differentiable, then the proof of theorem 1 actually gives a differentiable conjugacy with a linear action. This remark will be needed for theorem 2.

The proof of theorem 1 will involve a series of lemmas in which the hypotheses of theorem 1 will be assumed.

LEMMA 1.1. — *If for some* f *in* Γ, p_t *in the domain of* f, *and* λ *in* S^1, $f(p_t) = p_{t+\lambda}$, *then there exists an* $\bar{f}$ *in* Γ *such that* $\bar{f}$ *restricted to the domain of* f *is the same as* f, *and for all* p_t *in* S^1, $\bar{f}(p_t) = p_{t+\lambda}$.

Proof. — The orthogonal flow to the leaves parametrized by arclength leaves the foliation and embedded circle invariant. Hence it sends elements of Γ into elements of Γ by translating domain and range. Since the orbit of p_t under this flow is the entire embedded circle the lemma is proved.

We see therefore that Γ actually comes from a group of rotations of the circle. This group of rotations of the embedded circle is independent of the choice of bundle-like metric. We call λ as in lemma 1.1 a rotation number, and when φ is free we can define a unique « return function » $r_\lambda : S^1 \to R^{n-1}$ by the equation $\varphi(r_\lambda(t), p_t) = p_{t+\lambda}$. One can show that r_λ is differentiable. The next lemma shows that if one return function is constant, then all return functions are constant.

LEMMA 1.2. — *Let* r_λ *and* r_μ *be two return functions. Suppose that* λ *is irrational and* r_λ *constant. Then* r_μ *is constant.*

Proof. — Consider the closed curve q_t defined by

$$q_{t+\mu} = \varphi(r_\mu(0), p_t)$$

where t is in R/Z. Since $\varphi(r_\lambda, q_{t+\mu}) = q_{t+\mu+\lambda}$ the curve q_t is r_λ-invariant, and intersects the curve p_t at $p_\mu = q_\mu$. Since λ is irrational, the set $\{\mu + n\lambda | n \in Z\}$ is dense in R/Z. Thus $p_{\mu-n\lambda} = q_{\mu-n\lambda}$ for n in Z. We conclude that $q_{t+\mu} = p_{t+\mu}$ for all t in R/Z. Thus $\varphi(r_\mu(0), p_t) = p_{t+\mu}$ and hence $r_\mu(0) = r_\mu(t)$ for all t. This completes the proof of lemma 2.

We want to find an embedded circle with all return functions constant. To do this, we need the following lemma (see [5]).

LEMMA 1.3. — *Let* F *be a* C^∞ *mapping from* S^1 *to* R^{n-1} *such that* $\int_{S^1} F = 0$. *Define the map* T *of the set of continuous maps from* S^1 *to* R^{n-1} *to itself by* $Tg(t) = g(t + \lambda)$, *for* t *in* R/Z. *If* λ *satisfies the inequality* (1.1), *then there is a* C^∞ *solution* g *mapping* S^1 *to* R^{n-1} *of the equation*

$$g - Tg = F. \tag{1.2}$$

Proof. — The mapping F is C^∞ if and only if for each $k = 1, \ldots, n-1$ and for all integers $s \geqslant 0$,

$$\sum_{j=-\infty}^{\infty} j^s |a_j^k| < \infty \tag{1.3}$$

where a_j^k is the j^{th} Fourier coefficient of the mapping F^k from S^1 to R, where $F = (F^1, \ldots, F^{n-1})$ ([4], page 26.) Thus if the equation (1.2) has a solution $g = (g^1, \ldots, g^{n-1})$, where g^k has the Fourier expansion

$$g^k(t) \sim \sum_{j=-\infty}^{\infty} b_j^k e^{2\pi i j t},$$

then a simple calculation yields that

$$b_j^k = \left(\frac{1}{2\sin j\pi\lambda}\right) e^{\pi i\left(-j\lambda-\frac{1}{2}\right)} a_j^k$$

and so

$$|b_j^k| = \left(\frac{1}{2\,|\sin j\pi\lambda|}\right) |a_j^k|.$$

The inequality (1.3) and the fact that λ satisfies (1.1) allow one to show that if g is defined by $\{b_j^k\}$, then g is a C^∞ solution to (1.2) and lemma 1.3 is proved.

Main lemma 1.4. — *Suppose that there is a real number λ such that for some f in Γ, $f(p_t) = p_{t+\lambda}$, where λ satisfies (1.1). Then there is an embedding of the circle q_t in V^n such that all return functions are constant.*

Proof. — By lemma 1.2 it suffices to find an embedding q_t of the circle in V^n with just one return function r_λ constant, provided that λ is irrational. In some cases one can construct an embedding $\bar{p}_t$ of the circle with constant λ-return function from the original embedding p_t in the following way. If K is a mapping of S^1 to R^{n-1}, and we define $\bar{p}_t$ by $\bar{p}_t = \varphi(K(t), p_t)$, then the return functions r_λ and $\bar{r}_\lambda$ respectively corresponding to the rotation number λ are related as follows;

$$\bar{r}_\lambda(t) = r_\lambda(t) - K(t) + K(t+\lambda). \tag{1.4}$$

So if $\bar{r}_\lambda$ is to be a constant C, one must have

$$K(t) - K(t-\lambda) = r_\lambda(t) - C;$$

or, differentiating, $K'(t) - K'(t+\lambda) = r'_\lambda(t)$. Since λ satisfies (1.1), lemma 1.3 guarantees the existence of a C^∞ mapping K from S^1 to R^{n-1} such that equation (1.4) is true. $\bar{p}_t$ need not be an embedding; however if $\bar{p}_{t_1} = \bar{p}_{t_2}$, then for all integers n, $\bar{p}_{t_1+n\lambda} = \bar{p}_{t_2+n\lambda}$, so for all t, $\bar{p}_t = \bar{p}_{t+(t_2-t_1)}$. One can find an embedding as desired by considering

$$t_0 = \min \{t | p_t = p_0\}$$

and representing S^1 as $R/Z.t_0$. The minimum exists since q_t is an immersion, and we can assume that $t_0 = 1$. Thus the Main lemma is proved.

Proof of theorem 1. — We can assume that there is an embedding p_t of R/Z in V^n with all return functions constant. Such constants form a subgroup of R^{n-1}. We can now define an action Ψ of R^1 on V^n in the following way; every point v in V^n can be written as $\varphi(r, p_t)$ for some t in R/Z, r in R^{n-1}. Let $\Psi(s, \varphi(r, p_t)) = \varphi(r, p_{t+s})$. In particular, $\Psi(s, p_t) = p_{t+s}$. The action Ψ is easily seen to be well-defined, and clearly Ψ commutes with φ, and so we have an action Φ of R^n on V^n defined by

$$\Phi((r, s), v) = \Psi(s, \varphi(r, v)) = \varphi(r, \Psi(s, v))$$

for (r, s) in $R^{n-1} \times R$.

The orbit of any point v of V^n under Φ is the whole of V; thus the map $\bar{h}$ of R^n to V^n given by

$$\bar{h}(r) = \Phi(r, v)$$

induces a homeomorphism h of R^n/I_v to V^n, where I_v denotes the isotropy subgroup of v under Φ. The action Φ on V^n corresponds to the action of R^n on R^n/I_v induced by the natural action of R^n on itself. Note that

$$e = (0, \ldots, 0,1)$$

in R^n is an element of I_v; also if r_λ in R^{n-1} is a return constant for the action φ corresponding to a rotation number λ, then $(r_\lambda, -\lambda)$ is also in I_v.

In fact, it is known that the quotient of R^n by a discrete subgroup is a compact n-manifold if and only if the subgroup is isomorphic to Z^n. This implies that I_v is generated over the integers by n vectors in R^n; it is clear that we can choose a set of free generators of the following type;

$$\{e, \bar{r}_{\lambda_1} - \lambda_1 e, \ldots, \bar{r}_{\lambda_{n-1}} - \lambda_{n-1} e\}$$

where $\bar{r}_{\lambda_i} = (r_{\lambda_i}, 0)$ and r_{λ_i} is the return constant in R^{n-1}

corresponding to a rotation number λ_i. Note that we have used λ_i to denote both a real number and its class in R/Z. Since I_v is discrete, this set is also linearly independent over R. (Hence the set $\{r_{\lambda_1}, \ldots, r_{\lambda_{n-1}}\}$ in R^{n-1} is linearly independent and gives a set of free generators for the group of return constants.)

There is thus an isomorphism $\bar{l} : Z^n \to I_v$ which extends to a group automorphism $\bar{l}$ of R^n. This defines a homeomorphism $\bar{l} : R^n/Z^n \to R^n/I_v$ which respects the naturally induced R^n action on both spaces. If r and w are elements of R^n, $[w]$ the class of w in T^n, then one has that

$$l^{-1}h^{-1}(r, hl[w]) = [w] + [\bar{l}^{-1}r],$$

which means that Φ is conjugate by hl to a linear R^n action on T^n. Since $\varphi(r, v)$ is equal to $\Phi((r, 0), v)$, we have $(hl)^{-1}(r, hl[w]) = [w] + [A(r)]$ where we set

$$A(r) = \bar{l}^{-1}(r, 0).$$

This completes the proof of theorem 1.

One cannot expect to prove a result like theorem 1 without some condition on the degree of irrationality of a rotation number. That is, there do exist free actions with no compact orbits for which there are no circles with constant return functions. An example of such an action of R^1 on T^2 is given in [2].

2. In this section we will generalize the results of theorem 1 for free actions to locally free actions of R^{n-1} which are not free, but which have no compact orbits. For such actions, as mentioned above, all the orbits are dense, and it follows easily that the isotropy subgroups are equal at every point. Hence, every orbit is diffeomorphic to $R^{n-k-1} \times T^k$, $k \neq 0$, k constant, and we have that the action induces a free action φ of $R^{n-k-1} \times T^k$ on V^n. The restriction of φ to $\{0\} \times T^k$ defines a principal T^k bundle structure on V^n and φ projects to a free action $\bar{\varphi}$ of R^{n-k-1} on the quotient space V^n/T^k. For such a φ we have the lemma:

LEMMA 2.1. — *If $\bar{\varphi}$ is conjugate to a linear action of R^{n-k-1} on T^{n-k} by a diffeomorphism of T^{n-k} onto V^n/T^k, then* *either a) the action of* T^k *on* V^n *defines a trivial principal* T^k *bundle, or b) the dimension of* V^n/T^k *is* 2, i.e. $n - k = 2$.

This lemma, combined with theorem 1, will enable us to prove the following theorem:

THEOREM 2. — *If any of the rotation numbers of* $\bar{\varphi}$ *satisfy the Liouville inequality* (1.1) *then there are two possibilities:*

a) If $n - k > 2$, φ *is topologically conjugate to a free linear action of* $R^{n-k-1} \times T^k$ *on* T^n; *or*

b) If $n - k = 2$, V^n *is the quotient space of* R^n *by a group of covering transformations which are affine mappings of* R^n. *Furthermore,* φ *can be topologically conjugated to an action* Ψ *on* V^n *which is covered (relative to the above covering) by a linear action of* R^{n-1} *on* R^n.

Proof of lemma 2.1. — We will assume that $n - k > 2$ and show that *a*) is true. We are given that there is a diffeomorphism from T^{n-k} to V^n/T^k which conjugates $\bar{\varphi}$ to a linear action on T^{n-k}. This diffeomorphism pulls back V^n to a differentiable principal T^k bundle over T^{n-k}, and conjugates φ to a free $R^{n-k-1} \times T^k$ action on this bundle which projects to the linear action of R^{n-k-1} on T^{n-k} that is conjugate to $\bar{\varphi}$. Thus we lose no generality by assuming that $\bar{\varphi}$ is actually linear. The principal T^k bundle is a product of k principal S^1 bundles over T^{n-k}, each a quotient of the T^k bundle by a T^{k-1} subgroup of T^k. φ induces a free $R^{n-k-1} \times S^1$ action on each of these S^1 bundles. Hence, it suffices to assume that $k = 1$, for if each of these S^1 bundles is trivial, then so is the T^k bundle. Since $\bar{\varphi}$ is linear it induces a free linear action of R^2 on any linear 3-torus in T^{n-k}. Furthermore φ will restrict to a free action of $R^2 \times T^k$ on the portion of the T^k bundle sitting over the 3-torus. Since the Chern classes of the principal S^1 bundles are in $H^2(T^{n-k}; Z)$, and since we are assuming that

$$n - k > 2,$$

it is sufficient to prove the lemma in the case $n - k = 3$. Thus it suffices to prove lemma 2.1 in the special case where V^4 is a principal S^1 bundle over T^3 and where φ is a free action of $R^2 \times S^1$ on V^4 which projects to a free linear action of R^2 on T^3. Let (x_0, x_1, x_2) be the usual angular coordinates on T^3. Let T_{ij}, $0 \leqslant i < j \leqslant 2$, denote the 2-torus defined by $x_k = 0$, $0 \leqslant k \leqslant 2$, $k \neq i, j$. Then the R^2 action $\bar{\varphi}$ induces a linear vector field on each T_{ij}, unique up to a constant multiple. Let $\bar{X} = \frac{\partial}{\partial x_1} + \beta \frac{\partial}{\partial x_2}$, and

$$\bar{Y} = \frac{\partial}{\partial x_0} + \alpha \frac{\partial}{\partial x_1},$$

with α, β, real constants, be the induced vector fields on T_{12} and T_{01} respectively. We observe that the numbers 1, β, $\alpha\beta$ are independent if the reals are considered as a vector space over the rationals. Otherwise, it is easy to construct a periodic vector field on T^3 which is a linear combination of $\bar{X}$ and $\bar{Y}$ and this would contradict the fact that $\bar{\varphi}$ is a free action.

The action φ restricted to an R^1 subgroup of $R^2 \times S^1$ defines a vector field on V^4. Let X and Y be two such vector fields which project respectively to $\bar{X}$ and $\bar{Y}$. Y induces a flow a_t on V^4 and a_1 maps the portion of V^4 over T_{12} diffeomorphically onto itself. We denote V^4 restricted over T_{12} by N^3 and proceed to describe N^3 as a quotient space of R^3. Let (x_1, x_2, z) denote the usual coordinates on R^3. Then consider the group of diffeomorphisms of R^3 generated by

$$\begin{aligned}
\gamma_1(x_1, x_2, z) &= (x_1, x_2, z + 1)\\
\gamma_2(x_1, x_2, z) &= (x_1 + 1, x_2, z)\\
\gamma_3(x_1, x_2, z) &= (x_1, x_2 + 1, z + px_1),
\end{aligned}$$

where p, an integer, denotes (up to sign) the Chern class of the principal S^1 bundle N^3 evaluated on T_{12}. The quotient space of R^3 by this group is a principal S^1 bundle over T_{12} by projection to the first two coordinates, equivalent to N^3 since such a bundle is classified by its Chern class. Since Y commutes with the action of S^1 along the fibers of V^4, we see that a_1 is covered by a diffeomorphism $\tilde{a}_1$ of R^3 which can be written

$$\tilde{a}_1(x_1, x_2, z) = (x_1 + \alpha, x_2, z + g(x_1, x_2)),$$

where g is differentiable. From the form of the covering transformations γ_i, $1 \leqslant i \leqslant 3$, we see that g has the properties:

(2.1) $g(x_1 + 1, x_2) = g(x_1, x_2) + m,$ m an integer
(2.2) $g(x_1, x_2 + 1) = g(x_1, x_2) + p\alpha + q,$ q an integer

Since X is tangent to N^3 and commutes with the action of S^1 on N^3 we can lift X to a vector field $\tilde{X}$ on R^3 which has the form

$$\tilde{X} = \frac{\partial}{\partial x_1} + \beta \frac{\partial}{\partial x_2} + f(x_1, x_2)\frac{\partial}{\partial z}.$$

Since $\tilde{X}$ is invariant by γ_2 we have that

$$f(x_1 + 1, x_2) = f(x_1, x_2).$$

Since $[X, Y] = 0$ on V^4, $\tilde{a}_1$ must leave $\tilde{X}$ invariant. Hence, by applying the jacobian of $\tilde{a}_1$, we find that

(2.3) $f(x_1 + \alpha, x_2) - f(x_1, x_2) - g_1(x_1, x_2) = \beta g_2(x_1, x_2)$

where $g_i = \dfrac{\partial g}{\partial x_i}$.

If we integrate (2.3) with respect to x_1 from 0 to 1, we find, by (2.1), that

$$-m = \beta \int_0^1 g_2(s, x_2)\, ds.$$

If we then integrate with respect to x_2 from 0 to 1 we find from (2.2) that

$$-m = \beta \int_0^1 [g(s, 1) - g(s, 0)]\, ds = p\alpha\beta + q\beta$$

This contradicts the rational independence of 1, β, and $\alpha\beta$ if $p \neq 0$. Thus, $p = 0$, which means that the Chern class of V^4 evaluated on T_{12} is zero. A similar argument shows that the Chern class is zero on T_{01} and T_{02} as well. Hence V^4 is a trivial bundle, and lemma 2.1 is proved.

Proof of theorem 2. — We can apply theorem 1 to $\bar{\varphi}$. That is, there is a topological conjugacy of $\bar{\varphi}$ to a linear action $\bar{\Psi}$ of R^{n-k-1} on T^{n-k}. However, to apply lemma 2.1, we need a differentiable conjugacy. According to the remark after theorem 1, we can make the conjugation differentiable provided we use the right choice of atlas on V^n/T^k. If we change the atlas on V^n/T^k without changing the atlas on V^n, then the projection of V^n to V^n/T^k may not be differentiable. We give an indication of how to find atlases on V^n and V^n/T^k for which φ and $\bar{\varphi}$ are differentiable and for which the projection of V^n to V^n/T^k is differentiable. Recall that in order to define the rotation numbers, a bundle-like metric was constructed as well as a corresponding new atlas for which the bundle-like metric is C^∞. There is a bundle-like metric for φ which is invariant by the T^k action on the fibers of V^n. This is true because any metric which admits a closed loop orthogonal to the leaves can be modified to a bundle-like metric [1]. This metric and corresponding new atlas can be projected to V^n/T^k where it is a bundle-like metric for $\bar{\varphi}$. Hence, we can use the remark after theorem 1 to see that lemma 2.1 applies to $\bar{\varphi}$ and φ with the new atlases. The change of atlases causes no loss of generality since the conclusion of theorem 2 gives only a topological conjugation of φ.

First we consider the situation where case a) of lemma 2.1 applies. In this case, the diffeomorphism from T^{n-k} onto V^n/T^k conjugates $\bar{\varphi}$ to a linear action $\bar{\Psi}$ on T^{n-k} and conjugates φ to a differentiable action Ψ of $R^{n-k-1} \times T^k$ on the trivial T^k bundle, $T^{n-k} \times T^k$. The action Ψ restricted to $\{0\} \times T^k$ gives translation along the fibers of this T^k bundle.

We will be able to show that theorem 2 is true in case *a*) if we can find a section $\sigma : T^{n-k} \to T^{n-k} \times T^k$ with the property: σ is invariant under the action of Ψ restricted to a subgroup G of $R^{n-k-1} \times T^k$, where G is isomorphic to R^{n-k-1} and σ conjugates the action of Ψ restricted to G to the action $\bar{\Psi}$. For each x in T^{n-k}, let $\sigma_2(x)$ denote the projection of $\sigma(x)$ on the T^k factor. Then we can map $T^{n-k} \times T^k$ to itself by sending (x, t) to $(x, t + \sigma_2(x))$. This map is a principal T^k bundle isomorphism. Furthermore, this mapping conjugates Ψ to an action of $G \times T^k$ on $T^{n-k} \times T^k$, which is given by T^k translation τ in the second factor and where the action of G on T^{n-k} in the first factor is conjugate to the linear action $\bar{\Psi}$. Hence, we have that Ψ

is conjugate to the linear action $\overline{\Psi} \times \tau$ of $R^{n-k-1} \times T^k$ on $T^{n-k} \times T^k$. This conjugation involves an isomorphism of the group $G \times T^k$ to $R^{n-k-1} \times T^k$ as well as a homeomorphism of the space on which the group acts. However, this does not affect the conclusion that φ is topologically conjugate to a linear action since the isomorphism of groups is a linear isomorphism. Thus to prove case *a*) of theorem 2, it suffices to produce a section σ as above.

Since $\overline{\Psi}$ is linear, there is a linear circle $\overline{C}$ in T^{n-k} transverse to the orbits of $\overline{\Psi}$ which has a rotation number λ_1, satisfying the Liouville inequality (1.1). This follows from the proof of theorem 1, which is used to conjugate $\bar{\varphi}$ to a linear action $\overline{\Psi}$. Let $r_{\lambda_1}, \ldots, r_{\lambda_{n-k-1}}$ in R^{n-k-1} be a basis for the return constants for $\overline{C}$. The bundle over $\overline{C}$ is isomorphic to the trivial bundle $\overline{C} \times T^k$. Let the points of $\overline{C} \times T^k$ be denoted by (p_t, θ) for t in R/Z, and θ in T^k. Consider the automorphisms of $\overline{C} \times T^k$ given by

$$(p_t, \theta) \longmapsto \Psi((r_{\lambda_i}, 0), (p_t, \theta)), \ 1 \leqslant i \leqslant n-k-1.$$

Since $\overline{\Psi}(r_{\lambda_i}, p_t) = p_{t+\lambda_i}$, and since Ψ commutes with translation along the fibers, we have

$$\Psi((r_{\lambda_i}, 0), (p_t, \theta)) = (p_{t+\lambda_i}, \theta + \bar{\rho}(t)),$$

where $\bar{\rho}$ is a differentiable function from $\overline{C}$ to T^k. The mapping $\bar{\rho}$ is homotopic to a constant. This can be seen from the homotopy $\overline{C} \times T^k \times [0, 1] \to T^k$ given by

$$(p_t, \theta, s) \to \pi_2(\Psi((s . r_{\lambda_i}, 0), (p_t, \theta))),$$

where π_2 is the projection of $T^{n-k} \times T^k$ onto the T^k factor. Thus $\bar{\rho}$ comes from a map $\rho : S^1 \to R^k$, or in other words, $\int_{S^1} \bar{\rho}' = 0$. If $h : S^1 \to R^k$ is a differentiable function, we denote by $\bar{h}$ the corresponding function from S^1 to T^k. Given such a function h we can define a section C of $\overline{C} \times T^k$ by

$$C = \{(p_t, \bar{h}(t)) | t \in R/Z\}.$$

We would like to find such a section C so that, for each i,

$1 \leqslant i \leqslant n-k-1$, there is a θ_i in T^k, with

$$\Psi((r_{\lambda_i}, \theta_i), C) = C.$$

We have that $\Psi((r_{\lambda_i}, \theta_i), (p_t, \bar{h}(t))) = (p_{t+\lambda_i}, \theta_i + \bar{\rho}(t) + \bar{h}(t))$, and so we need an h such that $\bar{h}(t+\lambda_i) = \bar{h}(t) + \bar{\rho}(t) + \theta_i$. By lemma 1.3, the equation $h'(t+\lambda_1) = h'(t) + \rho'(t)$ can be solved for an R^k-valued function h since λ_1 satisfies (1.1) and $\int_{S^1} \rho' = 0$. Thus for $i = 1$, there is a θ_1 such that $\Psi((r_{\lambda_1}, \theta_1), C) = C$. Since any two points in the fiber differ by a translation of T^k, we can choose a θ_i for each i, $2 \leqslant i \leqslant n-k-1$, such that $\Psi((r_{\lambda_i}, \theta_i), C)$ intersects C. It then follows by an argument similar to the proof of lemma 1.2 that $\Psi((r_{\lambda_i}, \theta_i), C) = C$ for all i,

$$1 \leqslant i \leqslant n-k-1.$$

We take G to be the linear subgroup of $R^{n-k-1} \times T^k$ spanned by the elements $(r_{\lambda_i}, \theta_i)$, $1 \leqslant i \leqslant n-k-1$; i.e. G is covered by a linear subspace of R^n with a basis which projects to $\{(r_{\lambda_i}, \theta_i)\}$. The union of the orbits through all points of C by the action of Ψ restricted to G forms a submanifold which intersects each fiber once and only once. This submanifold defines a section $\sigma : T^{n-k} \to T^{n-k} \times T^k$ and σ clearly conjugates the action of G by Ψ to the action of R^{n-k-1} by $\bar{\Psi}$. Thus the proof of case $a)$ is complete.

In case $b)$ where $n-k=2$, we have that V^n is a principal T^k bundle over T^2, but not necessarily a trivial bundle, and that φ is topologically conjugate to Ψ which projects to a linear action $\bar{\Psi}$ of R^1 on T^2. We will simply indicate how to modify the proof of part $a)$ to take into account the fact that the bundle may not be trivial. Again let $\bar{C}$ be a linear circle in T^2 with associated rotation number λ and return constant r_λ in R^1. The bundle over $\bar{C}$ is again isomorphic to $\bar{C} \times T^k$. The automorphism of $\bar{C} \times T^k$ given by

$$(p_t, \theta) \longmapsto \Psi((r_\lambda, 0), (p_t, \theta))$$

is no longer homotopic to the identity. Hence, the n-tuple of integers $\eta = \int_{S^1} \bar{\rho}'$ is not necessarily zero as in case $a)$.

However, we can find an $h : S^1 \to R^k$ which solves the equation $h'(t+\lambda) - h'(t) = \rho'(t) - \eta$. As above we define a section over $\overline{C}$ by $C = \{(p_t, \overline{h}(t)) | t \in R/Z\}$. Then there is a θ_1 in T^k such that

$$\Psi((r_\lambda, \theta_1), (p_t, \overline{h}(t))) = (p_{t+\lambda}, \overline{h}(t+\lambda) + \eta . t),$$

where $\eta . t$ denotes scalar multiplication of η by t. We can again define a subgroup G of $R^1 \times T^k$ as in the proof of part *a*); i.e., in this case G is spanned by (r_λ, θ_1). The orbits through C by Ψ restricted to G do not form a section in general, so we must modify the results of part *a*). We lift Ψ to an action $\tilde{\Psi}$ on $R^1 \times T^k \times S^1$ which is a covering space of V^n by the covering projection

$$\pi : R^1 \times T^k \times S^1 \to V^n,$$

given by

$$\pi(r, \theta, t) = \Psi((r, \theta + \theta_1 . r/r_\lambda), (p_t, \overline{h}(t))).$$

The action $\tilde{\Psi}$ is simply translation by $R^1 \times T^k$ in the first two factors of $R^1 \times T^k \times S^1$. This is so because

$$\pi(\tilde{r} + r, \tilde{\theta} + \theta, t) = \Psi((\tilde{r}, \tilde{\theta} + \theta_1 . \tilde{r}/r_\lambda), \pi(r, \theta, t)),$$

where we can use $(\tilde{r}, \tilde{\theta} + \theta_1 . \tilde{r}/r_\lambda)$ in the right hand expression instead of $(\tilde{r}, \tilde{\theta})$ since we are able to make a linear isomorphism of $R^1 \times T^k$ to $G \times T^k$ without loss of generality. The cyclic group of covering transformations of

$$R^1 \times T^k \times S^1,$$

relative to π, is generated by the homeomorphism

$$(r, \theta, t) \longmapsto (r + r_\lambda, \theta - \eta . (t - \lambda), t - \lambda),$$

since this homeomorphism commutes with π and r_λ generates the return constants. (We are using λ as both a real number and as an element of R/Z as in the proof of theorem 1.) This homeomorphism is clearly covered by an affine mapping of R^n. Thus we see that V^n is the quotient space of R^n by affine transformations and $\tilde{\Psi}$ lifted to R^n is clearly a linear action. This completes the proof of case *b*) of theorem 2.

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