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SUFFICIENT CONDITIONS FOR THE CONTINUITY OF STATIONARY GAUSSIAN PROCESSES AND APPLICATIONS TO RANDOM SERIES OF FUNCTIONS

by Naresh C. JAIN [1, 2].
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1. Introduction.

Let $X(t)$, $t \in [0, 1]^n$, be a stochastically continuous separable Gaussian process with the property that

$$E(X(t + h) - X(t))^2 = \sigma^2(|h|).$$

In this case Dudley's sufficient condition, involving metric entropy, for the a.s. continuity of the sample paths of this process, can be given in terms of the familiar integral

$$I(\bar{\sigma}) = \int_0 \frac{\bar{\sigma}(u)}{u\,(\log 1/u)^{1/2}}\, du.$$

$I(\bar{\sigma}) < \infty$ is sufficient for $X(t)$ to have continuous sample paths a.s. The function $\bar{\sigma}$ is the non-decreasing rearrangement of σ for $h \in [0, 1]$. A useful corollary of this result is that $I(\sigma) < \infty$ is also a sufficient condition for the a.s. continuity of the sample paths of $X(t)$. These results are presented in section 2.

In Section 3 our results are generalized to processes with subgaussian increments. These results are used in Section 4

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(2) On sabbatical leave at the University of Illinois, Urbana, Illinois, when this research was carried out.

to obtain sufficient conditions for the continuity of a certain class of weakly stationary processes with discrete spectral distributions. We introduce the concept of a strongly symmetric sequence of random variables to describe this class. In section 5 we show that the results of Section 4, applied to random Fourier series, imply a theorem of Kahane on a sufficient condition for the a.s. uniform convergence of these series. Some other results on random Fourier series are also given in Section 5. Finally, in Section 6, we comment on the a.s. uniform convergence of some other random series of functions.

2. Sufficient conditions for the continuity of stationary Gaussian processes.

Let $\{X(t),\ t \in [0, 1]^n\}$ be a stochastically continuous separable stationary Gaussian process on the measure space $(\Omega, \mathscr{B}, P)$ with $EX(t) = 0$, $EX(t)^2 < \infty$. Let T be the map $T(t) = X(t)$ from $[0, 1]^n$ into $L^2(\Omega)$. Stochastic continuity of $X(t)$ implies that T is a continuous map; the topology on $L^2(\Omega)$ is given by the norm

$$\|X(t) - X(s)\| = \{E(X(t) - X(s))^2\}^{1/2}.$$

The process $\{X(t),\ t \in [0, 1]^n\}$ is said to have continuous sample paths almost surely if on a set of ω with P measure 1 the function $X(t, \omega)$ is continuous in t.

Let $C = \{X(t) : t \in [0, 1]^n\}$. Clearly C is a subset of $L^2(\Omega)$. Let $N(\varepsilon)$ be the minimal number of balls (determined by the norm) of radius $\leqslant \varepsilon$ necessary to cover C. The function $H(\varepsilon) = \log N(\varepsilon)$ is called the metric entropy of C. Dudley has shown [2] (see also [3]) that

$$J(H) = \int_0 H^{1/2}(\varepsilon)\, d\varepsilon < \infty$$

is a sufficient condition for a stochastically continuous separable Gaussian process $\{X(t),\ t \in [0, 1]^n\}$ to have continuous sample paths almost surely.

Let $X(t)$ have stationary increments and let

$$E(X(t + h) - X(t))^2 = \sigma^2(|h|)$$

for some non-negative continuous function σ. If σ is strictly increasing for $|h| \in [0, \delta]$ then for $\varepsilon < \sigma(\delta)$, $H(\varepsilon)$ is comparable to $\log|\sigma_1(\varepsilon)|$ where σ_1 is the inverse function of σ. This is not difficult to check; we shall show it for $t \in [0, 1]$ (i.e. $n = 1$). Suppose $\varepsilon < \sigma(\delta)$; let $h = \sigma_1(\varepsilon)$; then if $u \leqslant h$, $\|X(0) - X(u)\| \leqslant \varepsilon$. By stationarity and symmetry $[1/h] + 1$ balls of radius ε cover $X(t)$ but $[1/h]/4$ balls do not ([ ] denotes integral part). Therefore $N(\varepsilon)$ is comparable to $1/h = \dfrac{1}{\sigma_1(\varepsilon)}$ and the result follows by taking logarithms.

In the case of strictly increasing σ we see that $J(H) < \infty$ if and only if $I(\sigma) < \infty$ where

$$I(\sigma) = \int_0 \frac{\sigma(u)}{u\,(\log 1/u)^{1/2}}\, du$$

(see Lemma 2.2).

Of course the metric entropy is determined by σ whether or not σ is strictly increasing; however, since σ need not even be of bounded variation it is quite difficult to see the relationship between them. Nevertheless, we can compare the convergence or divergence of $J(H)$ with that of $I(\bar{\sigma})$, where $\bar{\sigma}$ is the non-decreasing rearrangement of σ. In other words when σ is not increasing $\bar{\sigma}$ substitutes as a smoothed version of σ. We proceed to develop this result.

Let $g : [0, a] \to [0, \delta]$ be a continuous non-negative function. Define

$$\mu(y) = \lambda\{h \in [0, a] : g(h) < y\}$$

where λ is Lebesgue measure; $\mu(y)$ is a left continuous strictly increasing function. Let $\bar{g}(h)$ be the generalized inverse of $\mu(y)$ given by

$$\bar{g}(h) = \sup\{y : \mu(y) < h\}. \tag{2.0}$$

The function $\bar{g}(h)$ is a continuous non-decreasing function called the *non-decreasing rearrangement* of $g(h)$ for $h \in [0, a]$. (Note that if $g(h)$ is strictly increasing, μ is the ordinary inverse of g and $\bar{g}(h) = g(h)$).

Our main result is a relationship between the metric entropy and the metric of a metric space with a very special property. Suppose that there exists a continuous map Q from $[0, 1]^n$

onto a subset S of a metric space $\mathscr{S}$ with metric $d(x, y)$, $x, y \in \mathscr{S}$. Let $x_t \in \mathrm{S}$ be the image of $t \in [0, 1]^n$ (x_0 the image of 0) under Q and assume that on S the metric has the property that $d(x_t, x_s) = f(|t - s|)$, ($|\ |$ denotes ordinary distance in R^n). Consider the balls in the metric space generated by the metric and having centers in S. Let $\mathrm{N}(\varepsilon)$ be the minimal number of such balls with radius less than or equal to ε that cover S. Such a number exists since S is compact. Define

$$(2.1) \qquad m(y) = \lambda\{h \in [0, 1]^n : d(x_0, x_h) < y\}.$$

The following lemma is obtained:

Lemma 2.1. — *Let* S *be the subspace of the metric space given above and let the metric* d, *the covering number* $\mathrm{N}(\varepsilon)$, *and the measure* m *also be as defined above. Then for all integers* $k \geqslant 1$ *we have*

$$\frac{2^{-n}}{m(2^{-k})} \leqslant \mathrm{N}(2^{-k}) \leqslant \frac{2^n + 1}{m(2^{-k-2})}.$$

Proof. — We have defined $\mathrm{S} = \{x_t : t \in [0, 1]^n\}$ and $\mathrm{N}(2^{-k})$ as the minimal number of balls of radius 2^{-k} that cover S. Denote these balls by S_i, $i = 1, \ldots, \mathrm{N}$, and let their centers be given by x_{t_i}, $i = 1, \ldots, \mathrm{N}$. Let

$$\mathrm{A}_i = \{u \in [0, 1]^n : d(x_u, x_{t_i}) < 2^{-k}\}.$$

By hypothesis d is radially symmetric; it follows that

$$\lambda(\mathrm{A}_i) \leqslant 2^n m(2^{-k}).$$

Note that $\bigcup_{i=1}^{\mathrm{N}} \mathrm{A}_i \supset [0, 1]^n$; therefore

$$1 \leqslant \lambda\left(\bigcup_{i=1}^{\mathrm{N}} \mathrm{A}_i\right) \leqslant \mathrm{N} 2^n m(2^{-k}).$$

Since $\mathrm{N} = \mathrm{N}(2^{-k})$, we have

$$\mathrm{N}(2^{-k}) \geqslant \frac{2^{-n}}{m(2^{-k})}.$$

To obtain the other inequality let $k \geqslant 1$ be fixed and let

$$\mathrm{S}_0' = \{x_u : d(x_0, x_u) < 2^{-k}\}.$$

Let A_0' be the pre-image of S_0' in $[0, 1]^n$. By definition

$$\lambda(A_0') = m(2^{-k}).$$

If there exists x_u such that $d(x_u, S_0') \geqslant 2^{-k}$, then pick x_{t_1} so that $d(x_{t_1}, S_0') = 2^{-k}$, equality is possible since $Q : [0, 1]^n \to S$ is continuous, and define

$$S_1' = \{x_u : d(x_u, x_{t_1}) < 2^{-k}\}.$$

Proceeding as before, pick x_{t_2} such that $d(x_{t_2}, S_0' \cup S_1') = 2^{-k}$, if such exists, and define

$$S_2' = \{x_u : d(x_u, x_{t_2}) < 2^{-k}\}.$$

Continue this process as long as it is possible and let N be the maximal number of sets obtained. Denote these sets by S_i', $i = 0, \ldots, N-1$, and their pre-images in $[0, 1]^n$ by A_i', $i = 0, \ldots, N-1$. These sets are disjoint; therefore

$$\sum_{i=0}^{N-1} \lambda(A_i') \leqslant 1.$$

Clearly S_i', $i = 0, \ldots, N-1$, need not be a cover of S but N sets having the same centers as S_i' and having radius 2^{-k+1} is a cover for S. Therefore

$$N(2^{-k+1}) \leqslant N. \tag{2.2}$$

We will show that

$$\lambda(A_i') \geqslant (2^n + 1)^{-1} m(2^{-k-1}); \tag{2.3}$$

then we will have

$$1 \geqslant \sum_{i=0}^{N-1} \lambda(A_i') \geqslant N(2^n + 1)^{-1} m(2^{-k-1}).$$

Using (2.2) we see that

$$N(2^{-k+1}) \leqslant \frac{2^n + 1}{m(2^{-k-1})}.$$

We now obtain (2.3). It follows from the radial symmetry of d that

$$\lambda(A_i') \geqslant \lambda[u \in [0, 1/2]^n : d(x_0, x_u) < 2^{-k}]. \tag{2.4}$$

Observe that

$$(2.5)\quad \begin{aligned} m(2^{-k-1}) &= \lambda\{u \in [0, 1]^n : d(x_0, x_u) < 2^{-k-1}\} \\ &= \lambda\{u \in [0,1/2]^n : d(x_0, x_u) < 2^{-k-1}\} \\ &\quad + \lambda[u \in ([0, 1/2]^n)^c : d(x_0, x_u) < 2^{-k-1}\} \end{aligned}$$

where $([0, 1/2]^n)^c$ denotes the set $[0, 1]^n - [0, 1/2]^n$.

If the set in the last term of (2.5) is empty then (2.3) holds, otherwise there exists a $u_0 \in ([0,1/2]^n)^c$ for which

$$d(x_0, x_{u_0}) < 2^{-k-1}.$$

Since

$$d(x_u, x_{u_0}) \leqslant d(x_u, x_0) + d(x_{u_0}, x_0),$$

it follows that

$$\begin{aligned} &\lambda\{u \in ([0, 1/2]^n)^c : d(x_u, x_0) < 2^{-k-1}\} \\ &\qquad \leqslant \lambda\{u \in ([0, 1/2]^n)^c : d(x_u, x_{u_0}) < 2^{-k}\} \\ &\qquad \leqslant 2^n\lambda\{u \in [0, 1/2]^n : d(x_u, x_0) < 2^{-k}\}. \end{aligned}$$

By (2.5)

$$m(2^{-k-1}) \leqslant (2^n + 1)\lambda\{u \in [0, 1/2]^n : d(x_u, x_0) < 2^{-k}\}.$$

Combining this with (2.4) we obtain (2.3).

In our applications of Lemma 2.1 we will be concerned with a generalized inverse of the function $m(y)$ defined in (2.1). The following technical lemma says that for our purposes we can treat a generalized inverse as though it is an ordinary inverse. See [1] for further details.

LEMMA 2.2. — *Let f be a non-increasing function and f^{-1} a generalized inverse of f* (i.e. *a function f^{-1} satisfying* $\sup\{x : f(x) > y\} \leqslant f^{-1}(y) \leqslant \inf\{x : f(x) < y\}$). *Then*

$$\int_0 f(u)\,du < \infty \Longleftrightarrow \int^\infty f^{-1}(u)\,du < \infty.$$

We now return to the stationary separable stochastically continuous Gaussian process $X(t)$, $t \in [0, 1]^n$, with

$$E(X(t + h) - X(t))^2 = \sigma^2(|h|).$$

The hypotheses of Lemma 2.1 are satisfied by this process; the metric is the norm on $L^2(\Omega)$ and $d(x_{t+h}, x_t) = \sigma(|h|)$.

As in (2.0) define

$$\bar{\sigma}(h) = \sup \{y : m(y) < h\}, \tag{2.6}$$

so that $\bar{\sigma}$ is the non-decreasing rearrangement of σ. We prove

THEOREM 2.3. — *Let* $X(t)$, $t \in [0,1]^n$, *be a separable stochastically continuous Gaussian process with stationary increments such that* $EX(t) = 0$ *and* $E(X(t+h) - X(t))^2 = \sigma^2(|h|)$. *Let* $\bar{\sigma}$ *be the non-decreasing rearrangement of* σ *given in* (2.6). *Then a sufficient condition for* $X(t)$ *to have continuous sample paths almost surely is that*

$$I(\bar{\sigma}) = \int_0 \frac{\bar{\sigma}(u)}{u\,(\log 1/u)^{1/2}}\, du < \infty. \tag{2.7}$$

Proof. — We show that under these hypotheses

$$J(H) < \infty \quad \text{if and only if} \quad I(\bar{\sigma}) < \infty. \tag{2.8}$$

To see this first note that the convergence or divergence of $J(H)$ is not affected if ε is replaced by $k\varepsilon$. Therefore by Lemma (2.1) applied to $f(\varepsilon) = (\log 1/m(\varepsilon))^{1/2}$, $J(H) < \infty$ if and only if

$$\int_0 \left(\log \frac{1}{m(\varepsilon)}\right)^{1/2} d\varepsilon < \infty.$$

Since $\bar{\sigma}$ is a generalized inverse of m it follows from Lemma 2.2 that $J(H) < \infty$ if and only if

$$\int^{\infty} \bar{\sigma}(e^{-y^2})\, dy < \infty.$$

Statement (2.8) follows by a change of variables.

For applications of Theorem 2.3 we will use a stronger condition than (2.7). To obtain this we need another lemma.

LEMMA 2.4. — *Let* $h(u)$ *be a non-negative, non increasing function on* $(0, 1]$ *and* $\alpha_i(x)$, $i = 1, 2$, *be distribution functions with* $\alpha_i(0) = 0$, $i = 1, 2$, *and* $\alpha_1(x) \geqslant \alpha_2(x)$. *Then* $\int_0^x h(u)\, d\alpha_1(u) < \infty$ *implies*

$$\int_0^x h(u)\, d\alpha_2(u) \leqslant \int_0^x h(u)\, d\alpha_1(u).$$

Proof. — By integration by parts

$$(2.9)\qquad \int_0^x h(u)\, d\alpha_1(u) = h(x)\alpha_1(x) - \int_0^x \alpha_1(u)\, dh(u),$$

where we use the fact that $\int_0^x h(u)\, d\alpha_1(u) < \infty$ and h and α monotone imply that $\lim_{u \to 0} h(u)\alpha_1(u) = 0$. The proof follows immediately from (2.9) taking into account the fact that h is non-increasing.

We can now show that $I(\sigma) < \infty$ is also a sufficient condition for sample path continuity.

Corollary 2.5. — *Let* $\sigma, \bar\sigma$ *be defined as in Theorem* 2.3; *then for* $0 \leqslant x \leqslant 1$,

$$\int_0^x \frac{\bar\sigma(u)}{u\,(\log 1/u)^{1/2}}\, du \leqslant \int_0^x \frac{\sigma(u)}{u\,(\log 1/u)^{1/2}}\, du.$$

In particular $I(\sigma) < \infty$ *is a sufficient condition for the continuity of the sample paths of* $X(t)$ *considered in Theorem* 2.3.

Proof. — Write $I(\bar\sigma)$ and $I(\sigma)$ as

$$\int_0^x h(u)\, d\left(\int_0^u \bar\sigma(s)\, ds\right) \qquad \text{and} \qquad \int_0^x h(u)\, d\left(\int_0^u \sigma(s)\, ds\right)$$

respectively, where $h(u) = u^{-1}\left(\log \frac{1}{u}\right)^{-\frac{1}{2}}$.

It follows, as in Hardy, Littlewood and Polya [5], page 277, that for $0 \leqslant x \leqslant 1$

$$\int_0^x \bar\sigma(u)\, du \leqslant \int_0^x \sigma(u)\, du.$$

The proof now follows from Lemma 2.4.

It was shown in [12] that there are continuous stationary Gaussian processes for which $I(\sigma)$ is infinite. In [10] it was shown that for some of these processes $J(H)$ and consequently $I(\bar\sigma)$ is finite. Also, as discussed in [10], there are no known examples to contradict the claim that $I(\bar\sigma) < \infty$ is a necessary and sufficient condition for the continuity of stationary Gaussian processes satisfying $E(X(t+h) - X(t))^2 = \sigma^2(|h|)$. So far we have not been able to use the new results of this section to obtain further information on the problem of finding necessary and sufficient conditions for the continuity of stationary Gaussian processes (*).

The monotone rearrangement $\bar{\sigma}$ of σ is a kind of smoothing of σ. In [11] one of us commented on a different attempt to smooth σ. Let $X(t)$, $t \in [0, 1]$, be a stationary Gaussian process with $E(X(t+h) - X(t))^2 = \sigma^2(h)$. Consider the function

$$\psi^2(h) = \frac{1}{h}\int_0^h \sigma^2(u)\,du. \tag{2.10}$$

There is a stationary Gaussian process $Y(t)$ such that $E(Y(t+h) - Y(t))^2 = \psi^2(h)$. In [11] it was shown that it is possible to have a continuous process $X(t)$ for which the associated process $Y(t)$, corresponding to the increments variance $\psi^2(h)$ given by (2.10), is discontinuous. We add to this the following:

THEOREM 2.6. — *Let* $X(t)$, $Y(t)$, $t \in [0, 1]$, *be stationary Gaussian processes with* $E(X(t+h) - X(t))^2 = \sigma^2(h)$,

$$E(Y(t+h) - Y(t))^2 = \psi^2(h)$$

with σ *and* ψ *related as in* (2.10). *Then if* $Y(t)$ *has continuous sample paths, so does* $X(t)$. *The converse is false.*

Proof. — If a stationary Gaussian process $Y(t)$ has its increments variance $\psi^2(h)$ given by (2.10) then it follows from ([9], Theorem 4.5.1) that the spectral distribution F of $Y(t)$ is concave. It was shown in ([11], Theorem 1) that for stationary Gaussian processes with concave spectral distribution F

$$\tilde{I}(F) = \int^{\infty} \frac{(1 - F(x))^{1/2}}{x\,(\log x)^{1/2}}\,dx < \infty \tag{2.11}$$

is a necessary and sufficient condition for a.s. continuity of the sample paths.

In [10], Theorem 1, as well as in [6], it was seen that $\tilde{I}(F) < \infty$ implies $I(\psi) < \infty$ (this result holds for all F, not just for concave F). On the other hand $I(\psi) < \infty$

(*) *Added in proof :* Professor Fernique has now shown that $I(\bar{\sigma}) < \infty$ is a necessary and sufficient condition fur the sample continuity of a stationary Gaussian process.

is a sufficient condition for continuity by Corollary 2.5. Therefore for stationary Gaussian processes with increments variance $\psi^2(h)$ given by (2.10)

$$(2.12) \qquad \mathrm{I}(\psi) < \infty$$

is also a necessary and sufficient condition for the a.s. continuity of the sample paths. Therefore if $Y(t)$ is continuous $\mathrm{I}(\psi) < \infty$. We complete the proof by showing that $\mathrm{I}(\psi) < \infty$ implies $\mathrm{I}(\sigma) < \infty$. This follows from the following inequalities:

$$(2.13) \qquad \mathrm{I}(\sigma) < \infty \Longleftrightarrow \mathrm{I}\left(\frac{1}{u}\int_0^u \sigma(s)\, ds\right) < \infty,$$

and

$$(2.14) \quad \mathrm{I}\left(\frac{1}{u}\int_0^u \sigma(s)\, ds\right) \leqslant \mathrm{I}\left(\left(\frac{1}{u}\int_0^u \sigma^2(s)\, ds\right)^{1/2}\right) = \mathrm{I}(\psi).$$

We get (2.14) by Schwarz's inequality; (2.13) is obtained by observing that

$$\int_0 \frac{1}{u\,(\log 1/u)^{1/2}}\, d\left(\int_0^u \sigma(s)\, ds\right) < \infty$$
$$\Longleftrightarrow \int_0 \left(\int_0^u \sigma(s)\, ds\right) d\left(\frac{1}{u\,(\log 1/u)^{1/2}}\right) < \infty.$$

This proves the first assertion of the theorem. The second assertion is proved in [11] by a counter-example.

3. Processes with subgaussian increments.

The results of Section 2 are not restricted to Gaussian processes. They apply to stochastic processes with increments which have a probability distribution with tail estimate similar to that of a normal random variable. We proceed to make this precise.

A random variable is called subgaussian if for any real λ

$$(3.1) \qquad \mathrm{E}(e^{\lambda X}) \leqslant e^{\frac{\sigma^2\lambda^2}{2}},$$

where $\sigma^2 = \mathrm{E}X^2$. A subgaussian random variable must have zero mean. The random variable X taking values ± 1, each with probability $1/2$, is called a Rademacher random variable and is subgaussian. For later use we define a sequence of independent Rademacher random variables as a *Rademacher sequence.*

It follows from the exponential Chebychev inequality that if X is subgaussian (which implies that $-X$ is subgaussian) then

$$\text{(3.2)} \qquad P\{|X| > \lambda\} \leqslant 2e^{-\frac{\lambda^2}{2\sigma^2}}.$$

We will now display stochastic processes $X(t)$, $t \in [0, 1]$, such that $EX(t) = 0$ and $X(t) - X(s)$ are subgaussian for all s, $t \in [0, 1]$. Let $\{\varphi_n(t)\}$ be a sequence of continuous functions, $t \in [0, 1]$, such that $\sum\limits_{n=0}^{\infty} \varphi_n^2(t)$ converges uniformly. Let $\{\varepsilon_n\}$ be a sequence of independent subgaussian random variables, $E[\varepsilon_n^2] = 1$. Consider

$$\text{(3.3)} \qquad X(t) = \sum_{n=0}^{\infty} \varphi_n(t)\varepsilon_n$$

(the sequence converges a.s. for fixed t). Since

$$\begin{aligned} E[e^{\lambda(X(t)-X(s))}] &= \prod_{n=0}^{\infty} E[e^{\lambda(\varphi_n(t)-\varphi_n(s))\varepsilon_n}] \\ &\leqslant e^{\frac{\lambda^2}{2}\sum\limits_{n=0}^{\infty}(\varphi_n(t)-\varphi_n(s))^2}, \end{aligned}$$

$X(t)$ has the required properties. The following theorem provides a sufficient condition for the continuity of (3.3).

THEOREM 3.1. — *Let* $X(t)$, $t \in [0, 1]$, *be given by* (3.3) *and suppose*

$$E(X(t) - X(s))^2 \leqslant \psi^2(|t - s|),$$

where ψ *is non-decreasing and* $I(\psi) < \infty$. *Then* $X(t)$ *has continuous sample paths* a.s.

Proof. — If $\{\varepsilon_n\}$ were normal random variables this would be a consequence of Fernique's theorem [4]. Dudley's theorem implies Fernique's theorem (see [2], Theorem 7.1 and [10]) and the proof of Dudley's theorem (as well as the proof of Fernique's theorem) depends only on the increments of $X(t)$ satisfying (3.2) rather than on the somewhat finer estimate available for Gaussian random variables. Therefore the theorem is proved.

Suppose that $X(t)$ also satisfies

$$(3.4)\quad E(X(t) - X(s))^2 = \sum_{n=0}^{\infty} (\varphi_n(t) - \varphi_n(s))^2 = \sigma^2(|t-s|)$$

for some continuous function σ. Applying the results of Section 2 and by the same reasoning used in the proof of Theorem 3.1 we obtain.

THEOREM 3.2. — *Let* $X(t)$, $t \in [0, 1]$, *be given by* (3.3) *and suppose that it also satisfies* (3.4); *then* $I(\bar{\sigma}) < \infty$ *is sufficient for* $X(t)$ *to have continuous sample paths a.s.*

Remark. — In this case using Corollary 2.5 we see that $I(\sigma) < \infty$ is also sufficient for $X(t)$ to have continuous sample paths a.s.

4. Weakly stationary processes with discrete spectral distribution.

We will show that $I(\sigma) < \infty$ is a sufficient condition for the a.s. continuity of the sample paths of a large class of weakly stationary processes with discrete spectral distribution.

Let $R(.,.)$ be a real valued continuous stationary covariance on $[0, 1] \times [0, 1]$ with discrete spectral distribution F. (Since R is real we take it as a cosine transform of F concentrated on the right half line). Let F have weights a_n^2 at λ_n, $\lambda_n \geqslant 0$, $\sum_{n=0}^{\infty} a_n^2 = 1$. Then

$$(4.1)\qquad R(s, t) = \sum_{n=0}^{\infty} a_n^2 \cos \lambda_n(s-t) = p(|s-t|).$$

Let

$$(4.2)\qquad \sigma^2(h) = 2(1 - p(h)) = 4 \sum_{n=0}^{\infty} a_n^2 \sin^2 \frac{\lambda_n h}{2}.$$

Every second order, zero mean, weakly stationary process with discrete spectral distribution and covariance given by (4.1) has a representation (in the mean-square sense) in the form

$$(4.3) \qquad X(t) = \sum_{n=0}^{\infty} a_n\{\xi_n \cos \lambda_n t + \xi'_n \sin \lambda_n t\}$$

where $\{\xi_n\}$, $\{\xi'_n\}$ are two mutually orthogonal sequences of orthogonal random variables with $E(\xi_n^2) = E(\xi_n'^2) = 1$ and $E(\xi_n) = E(\xi'_n) = 0$ for all n and with $\sum_{n=0}^{\infty} a_n^2 = 1$. (See Doob [13] Chapter XI, Theorem 4.1.).

We impose one important condition on the process (4.3) that $\{\xi_n\}$ and $\{\xi'_n\}$ be strongly symmetric. We will call a sequence of real or complex valued random variables $\{\xi_n\}$ *strongly symmetric* if the finite dimensional distributions of $\{\xi_n\}$ are the same as the finite dimensional distributions of $\{\varepsilon_n\xi_n\}$, where $\{\varepsilon_n\}$ is a Rademacher sequence independent of $\{\xi_n\}$. (Note that $\{\xi_n\}$ is symmetric if the finite dimensional distributions of $\{\xi_n\}$ are the same as the finite dimensional distributions of $\{\varepsilon\xi_n\}$ where ε is a Rademacher random variable independent of $\{\xi_n\}$.)

If $\{\xi_n\}$ is any arbitrary sequence of random variables and $\{\varepsilon_n\}$ is a Rademacher sequence independent of $\{\xi_n\}$, then clearly $\{\varepsilon_n\xi_n\}$ is strongly symmetric. It is also clear that if $\{\xi_n\}$ is strongly symmetric then $E(\xi_n\xi_m) = 0$ for $n \neq m$, i.e. they are orthogonal.

If $\{\xi_n\}$ is a sequence of symmetric independent random variables then it is strongly symmetric. However, the class of strongly symmetric random variables is larger than this. We were led to define strongly symmetric random variables by recognizing that in some previous studies of random Fourier series (i.e. $\lambda_n = n$), $\{\xi_n\}$ was taken to be a sequence of symmetric and independent random variables only to insure that they were strongly symmetric.

We need the following lemmas:

LEMMA 4.1. — *Let* $\{X_n(t)\}$, $\{X'_n(t)\}$, $n \geqslant 0$, $t \in [0, 1]$, *be independent sequences of continuous symmetric stochastic processes on some probability space* $(\Omega, \mathscr{F}, P)$. *If*

$$(4.4) \qquad \sum_{n=0}^{\infty} [X_n(t) + X'_n(t)]$$

converges uniformly a.s., then each of the series $\sum_{n=0}^{\infty} X_n(t)$ *and* $\sum_{n=0}^{\infty} X_n'(t)$ *converges uniformly a.s. (Note that a stochastic process is symmetric if all of its finite dimensional distributions are symmetric.)*

Proof. — Since the sequences are symmetric and independent, the a.s. uniform convergence of (4.4) implies the a.s. uniform convergence of $\sum_{n=0}^{\infty} [X_n(t) - X_n'(t)]$. Adding these two series gives the a.s. uniform convergence of $\sum_{n=0}^{\infty} X_n(t)$. The same method gives the result for $\sum_{n=0}^{\infty} X_n'(t)$.

LEMMA 4.2. — *Let* $\{e_j\}$ *and* $\{e_j'\}$ *be two sequences of real-valued continuous functions on a compact metric space* T *such that* $\sum_j [e_j^2(t) + e_j'^2(t)]$ *converges uniformly on* T. *Let* $\{\eta_j\}$, $\{\eta_j'\}$ *be mutually independent sequences of independent symmetric random variables with* $E(\eta_j^2) = E(\eta_j'^2) = 1$ *for all* j. *If the stochastic process represented by*

$$X(t) = \sum_j [e_j(t)\eta_j + e_j'(t)\eta_j'] \tag{4.5}$$

(in the sense of a.s. convergence for each t *by Kolmogorov's theorem) has a version with almost all paths continuous, then the series in* (4.5) *converges uniformly a.s.*

Proof. — The proof is a slight modification of arguments given in [7].

In the next theorem a sufficient condition for continuity obtained originally for subgaussian processes is extended to a large class of random trigonometric functions.

THEOREM 4.3. — *Let* $\{X_n, n \geqslant 1\}$ *and* $\{\Phi_n, n \geqslant 1\}$ *be two sequences of real-valued random variables on some probability space* $(\Omega, \mathcal{F}, P)$ *such that the complex sequence* $\{X_n e^{i\Phi_n}\}$ *is strongly symmetric. Let*

$$E(X_n^2) = a_n^2, \qquad \sum_{n=0}^{\infty} a_n^2 = 1 \tag{4.6}$$

and

$$\sigma^2(h) = 4 \sum_{n=0}^{\infty} a_n^2 \sin^2 \frac{\lambda_n h}{2},$$

where $\lambda_n \geqslant 0$. *Then* $I(\sigma) < \infty$ *implies the a.s. uniform convergence of the series*

$$(4.7) \qquad \sum_{n=0}^{\infty} X_n \cos(\lambda_n t + \Phi_n),$$

$t \in [0, 1]$.

Proof. — We construct a product probability space $(\Omega, \mathscr{F}, P)$ such that $\Omega = \Omega_1 \times \Omega_2$, $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$ and $P = P_1 \times P_2$ where $\{X_n\}$ and $\{\Phi_n\}$ are defined on $(\Omega_1, \mathscr{F}_1, P_1)$ and two independent Rademacher sequences $\{\varepsilon_n\}$ and $\{\varepsilon_n'\}$ are defined on $(\Omega_2, \mathscr{F}_2, P_2)$. E_i will denote the expectation operator relative to P_i, $i = 1, 2$, and E relative to P. Let

$$(4.8) \quad Z(t) = \sum_{n=0}^{\infty} X_n\{\varepsilon_n \cos(\lambda_n t + \Phi_n) + \varepsilon_n' \sin(\lambda_n t + \Phi_n)\}.$$

From (4.6) we see that

$$(4.9) \qquad \sum_{n=0}^{\infty} X_n^2 < \infty \quad \text{a.s. } (P).$$

For a fixed $\omega_1 \in \Omega_1$, we denote the process in (4.8) by $Z_{\omega_1}(t)$, i.e.

$$(4.10) \quad Z_{\omega_1}(t) = \sum_{n=0}^{\infty} X_n(\omega_1)\{\varepsilon_n \cos(\lambda_n t + \Phi_n(\omega_1)) + \varepsilon_n' \sin(\lambda_n t + \Phi_n(\omega_1))\}.$$

Note that $Z_{\omega_1}(t)$ is a stochastic process on Ω_2. Define

$$\sigma_{\omega_1}^2(h) = E_2(Z_{\omega_1}(t+h) - Z_{\omega_1}(t))^2 = 4 \sum_{n=0}^{\infty} X_n^2(\omega_1) \sin^2 \frac{\lambda_n h}{2}.$$

Note that $I(\sigma) < \infty$ implies $I(\sigma_{\omega_1}) < \infty$ a.s. (P_1) since

$$E_1\{I(\sigma_{\omega_1})\} = I\{E_1(\sigma_{\omega_1})\} \leqslant I(\sigma).$$

The last step follows by Schwarz's inequality since

$$E_1(\sigma_{\omega_1}) \leqslant \{E_1(\sigma_{\omega_1}^2)\}^{1/2} = \sigma.$$

The process $Z_{\omega_1}(t)$ is a process of the type given in (3.3);

therefore by Theorem 3.2 there is a set $\overline{\Omega}_1 \subset \Omega_1$ such that $P(\overline{\Omega}_1) = 1$, and $\omega_1 \in \overline{\Omega}_1$ implies that $Z_{\omega_1}(t)$ has continuous sample paths a.s. (P_2). It follows from Lemma 4.2 that the series for $Z_{\omega_1}(t)$ converges uniformly a.s. (P_2) for $\omega_1 \in \overline{\Omega}_1$. We now apply Lemma 4.1 to the two sequences of independent (*w.r.t.* P_2) symmetric stochastic processes

$\{X_n(\omega_1)\varepsilon_n \cos(\lambda_n t + \Phi_n(\omega_1))\}$ and $\{X_n(\omega_1)\varepsilon_n' \sin(\lambda_n t + \Phi_n(\omega_1))\}$

and obtain that

$$\sum_{n=0}^{\infty} X_n(\omega_1)\varepsilon_n \cos(\lambda_n t + \Phi_n(\omega_1))$$

and

$$\sum_{n=0}^{\infty} X_n(\omega_1)\varepsilon_n' \sin(\lambda_n t + \Phi_n(\omega_1))$$

both converge uniformly a.s. (P_2). It follows by Fubini's theorem that

$$\sum_{n=0}^{\infty} X_n\varepsilon_n \cos(\lambda_n t + \Phi_n) \tag{4.11}$$

converges uniformly a.s. (P). The hypothesis that $\{X_n e^{i\Phi_n}\}$ is strongly symmetric is equivalent to both $\{X_n \cos \Phi_n\}$ and $\{X_n \sin \Phi_n\}$ being strongly symmetric. Consequently, the finite dimensional joint distributions of (4.7) and (4.11) are the same. Therefore (4.7) converges uniformly a.s. (P).

Note that the condition (4.6) that $\sum_{n=0}^{\infty} a_n^2 = 1$ is only a convenient normalization. Any finite value suffices.

We can now state our result for weakly stationary processes with discrete spectral distribution.

THEOREM 4.4. — *Let* $X(t)$ *be given by* (4.3). *Suppose* $\{\xi_n\}$ *and* $\{\xi_n'\}$ *are strongly symmetric sequences. Then* $I(\sigma) < \infty$ *is a sufficient condition for the a.s. uniform convergence of the series in* (4.3).

Proof. — By taking $X_n = a_n\xi_n$ and $\Phi_n = 0$ in Theorem 4.3 we conclude that $\sum_{n=0}^{\infty} a_n\xi_n \cos \lambda_n t$ converges uniformly a.s. With $X_n = a_n\xi_n'$ and $\Phi_n = \pi/2$ we get the uniform convergence a.s. of $\sum_{n=0}^{\infty} a_n \xi_n' \sin \lambda_n t$. The theorem follows.

It was convenient for us to prove Theorem 4.3 before Theorem 4.4 but Theorem 4.3 can be obtained as a corollary of Theorem 4.4, even though the uniform convergence of (4.7) seems like a more general result than the uniform convergence of (4.10). In other words our results apply essentially to weakly stationary processes. The apparently more general Theorem 4.3 is a consequence of Theorem 4.4 and Lemma 4.1.

Theorem 4.4 can be restated in an interesting manner. We do this as a corollary.

Corollary 4.5. — *Let* $X(t)$ *be given by* (4.3) *except that* $\{\xi_n\}$ *and* $\{\xi_n'\}$ *are only required to have uniformly bounded second moments. Let* $\{\theta_n\} = \{\varepsilon_n \xi_n\}$ *and* $\{\theta_n'\} = \{\varepsilon_n' \xi_n'\}$ *where* $\{\varepsilon_n\}$ *and* $\{\varepsilon_n'\}$ *are independent Rademacher sequences independent of* $\{\xi_n\}$, $\{\xi_n'\}$. *Then* $I(\sigma) < \infty$ *implies the a.s. uniform convergence of*

$$\sum_{n=0}^{\infty} a_n \{\theta_n \cos \lambda_n t + \theta_n' \sin \lambda_n t\}$$

where σ *is given by* 4.2.

Finally we take up the question of how essential is the property of strong symmetry. It turns out that some such restriction is essential. Let $\lambda_n = 2\pi n$ in (4.3), i.e.

$$X(t) = \sum_{n=0}^{\infty} a_n \{\xi_n \cos 2\pi n t + \xi_n' \sin 2\pi n t\}. \tag{4.12}$$

We give an example of bounded orthogonal sequences $\{\xi_n\}$ and $\{\xi_n'\}$, with $E\xi_n = E\xi_n' = 0$ (in fact each ξ_n, ξ_n' is symmetric) and $E\xi_n^2 = E\xi_n'^2 = \text{Const.}$, for which $Z(t)$ in (4.12) has all of its sample paths discontinuous whenever $a_n \geqslant 0$ and $\{a_n\} \notin l^1$. Simply let $\xi_n = \cos 2\pi n\omega$ and $\xi_n' = -\sin 2\pi n\omega$, $\omega \in [0, 1]$. Then

$$X(t) = \sum_{n=0}^{\infty} a_n \cos 2\pi n(t + \omega)$$

and this is unbounded at $t = (1 - \omega)$ whenever $\{a_n\} \notin l^1$. One can check that $\{\xi_n\}$ and $\{\xi_n'\}$ are not strongly symmetric.

Note that in (4.12) the process $X(t)$ is continuous if $\{a_n\} \in l^1$. Therefore as soon as this condition is violated (and $a_n \geqslant 0$) we can find weakly stationary processes with discrete spectral distributions that are discontinuous. Hence restricting $\{\xi_n\}$, $\{\xi'_n\}$ to the strongly symmetric class seems a reasonable way to get any further classification of these weakly stationary processes with respect to sample path continuity. (Note that if $a_n = \left(\dfrac{1}{n\,(\log n)^\alpha}\right)^{1/2}$ for $\alpha > 2$, then $I(\sigma) < \infty$; see [10] Theorem 1.)

5. Random Fourier series.

For $\lambda_n = 2\pi n$ the series in (4.7) are random Fourier series. Applying the results of Section 4, Kahane's result for the a.s. uniform convergence of these series is obtained ([8], Theorem 1, page 64). We also add to a different result of Kahane ([8], page 77) on necessary conditions for the a.s. uniform convergence of random Fourier series. Finally, we apply some of our results to obtain a property of Sidon sets.

The following theorem is simply a restatement of theorem 4.3 in the case $\lambda_n = 2\pi n$.

THEOREM 5.1. — *Consider the series*

$$\sum_{n=0}^{\infty} X_n \cos(2\pi nt + \Phi_n) \tag{5.1}$$

where $\{X_n\}$ *and* $\{\Phi_n\}$ *are sequences of real valued random variables such that* $\{X_n e^{i\Phi_n}\}$ *is strongly symmetric,* $t \in [0, 1]$. *Let*

$$E(X_n^2) = a_n^2, \qquad \sum_{n=0}^{\infty} a_n^2 < \infty, \tag{5.2}$$

$$\sigma^2(h) = 4 \sum_{n=0}^{\infty} a_n^2 \sin^2 n\pi h. \tag{5.3}$$

Then $I(\sigma) < \infty$ *is sufficient for the a.s. uniform convergence of* (5.1).

Define

$$s_j^2 = \sum_{2^j \leqslant n < 2^{j+1}} a_n^2. \tag{5.4}$$

Kahane has shown that if $\{X_n e^{i\Phi_n}\}$ are independent and symmetric then

$$(5.5) \qquad \sum_{k=1}^{\infty} 2^{k/2} \left(\sum_{j=2^k}^{2^{k+1}-1} s_j^2 \right)^{1/2} < \infty$$

implies the a.s. uniform convergence of (5.1). It has been shown in [11], and independently in [6], that (5.5) implies $I(\sigma) < \infty$. Therefore Theorem 5.1 implies this result of Kahane.

For future use we note that if $s_j \searrow$ and $\Sigma s_j < \infty$ then (5.5) holds.

We have a few remarks to make concerning the relationship of our work to Kahane's. Although he requires $\{X_n e^{i\Phi_n}\}$ to be independent and symmetric all he uses in Theorem 1, page 64 [8], is that they are strongly symmetric. The technique of replacing $\{X_n e^{i\Phi_n}\}$ by $\{\varepsilon_n X_n e^{i\Phi_n}\}$ where $\{\varepsilon_n\}$ is a Rademacher sequence independent of $\{X_n e^{i\Phi_n}\}$ is due to Kahane; it is of major importance in our Section 4. A condition such as (5.5), which involves the coefficients of the random series (5.1), is all right when $\lambda_n = 2\pi n$, but is meaningless when the λ_n are arbitrary. A condition involving σ clearly depends upon λ_n as well as the coefficients of the series. Finally, Kahane's method in proving his Theorem 1, page 64 [8], which incorporates earlier work of Paley, Zygmund and Salem, uses properties of trigonometric polynomials and does not extend to the case where $2\pi n$ is replaced by λ_n (as far as we can see).

The following theorem is a necessary condition for the a.s. uniform convergence of random Fourier series.

THEOREM 5.2. — *Consider the series* (5.1) *where* $\{X_n\}$ *and* $\{\Phi_n\}$ *are sequences of real valued random variables such that* $X_n e^{i\Phi_n}$ *are symmetric and independent. Let* a_n^2 *be given by* (5.2) *and* s_j^2 *by* (5.4). *Suppose one of the following conditions holds:*

a) There is a positive constant c *such that*

$$(5.7) \qquad E(X_n^4) \leqslant cE^2(X_n^2) = ca_n^4;$$

b) $|a_n|$ *and there is a constant* c_1 *such that*

$$(5.8) \qquad a_n^2 = E(X_n^2) \leqslant c_1 E^2|X_n|.$$

Then $\Sigma s_j = \infty$ *implies that* (5.1) *represents an unbounded function on* $[0, 1]$ *a.s.*

Proof. — Under hypothesis *a*) this is Kahane's Theorem 1, page 77, [8]. We use it to prove the theorem under hypothesis *b*). Let $\Omega = \Omega_1 \times \Omega_2$, $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$, $P = P_1 \times P_2$ be a product probability space such that $\{X_n\}$ and $\{\Phi_n\}$ are defined on Ω_1 and a Rademacher sequence $\{\varepsilon_n\}$ is defined on Ω_2. Consider the series $\sum_{n=0}^{\infty} \varepsilon_n X_n \cos(2\pi nt + \Phi_n)$. We will show that for each fixed $\omega_1 \in \overline{\Omega}_1 \subset \Omega_1$, $P_1(\overline{\Omega}_1) = 1$, the series

$$\sum_{n=0}^{\infty} \varepsilon_n X_n(\omega_1) \cos(2\pi nt + \Phi_n(\omega_1)) \tag{5.9}$$

represents an unbounded function on $[0, 1]$ a.s. (P_2). Using this theorem under hypothesis *a*) with $\varepsilon_n X_n(\omega_1)$ replacing X_n and $\Phi_n(\omega_1)$ replacing Φ_n, (5.9) represents an unbounded function on $[0, 1]$ a.s. (P_2) if $\Sigma s_j(\omega_1) = \infty$, where

$$s_j^2(\omega_1) = \sum_{n=2^j}^{2^{j+1}-1} X_n^2(\omega_1).$$

Therefore we need to show that

$$\sum_{j=1}^{\infty} \left(\sum_{n=2^j}^{2^{j+1}-1} X_n^2 \right)^{\frac{1}{2}} = \infty \quad \text{a.s.} \quad (P_1). \tag{5.10}$$

By Schwarz's inequality,

$$\sum_{j=1}^{\infty} \left(\sum_{n=2^j}^{2^{j+1}-1} X_n^2 \right)^{1/2} \geqslant \sum_{j=1}^{\infty} 2^{-j/2} \sum_{n=2^j}^{2^{j+1}-1} |X_n|. \tag{5.11}$$

Since the $|X_n|$ are independent, by Chebychev's inequality,

$$P\left\{ \sum_{n=2^j}^{2^{j+1}-1} (E|X_n| - |X_n|) \geqslant \varepsilon 2^{j/2} s_j \right\} \leqslant \frac{2^{-j}}{\varepsilon^2}, \tag{5.12}$$

where s_j^2 is given in (5.4). By the Borel-Cantelli lemma only finitely many of the events in (5.12) occur and we conclude that a.s. (P_1) for all j sufficiently large

$$(5.13)\quad \sum_{n=2^j}^{2^{j+1}-1} |X_n| \geqslant \sum_{n=2^j}^{2^{j+1}-1} E|X_n| - \varepsilon 2^{j/2} s_j \geqslant c_1^{-\frac{1}{2}} \sum_{n=2^j}^{2^{j+1}-1} a_n - \varepsilon 2^{j/2} s_j,$$

by (5.8). Choose $0 < \varepsilon < (8c_1)^{-1/2}$; since $a_n \searrow$ we have

$$\sum_{n=2^j}^{2^{j+1}-1} a_n \geqslant 2^j a_{2^{j+1}} = 2^{j/2}[2^j a_{2^{j+1}}^2]^{\frac{1}{2}} \geqslant 2^{j/2} 2^{-\frac{1}{2}} s_{j+1}.$$

Hence, substituting in (5.13), we get

$$(5.14)\quad \sum_{n=2^j}^{2^{j+1}-1} |X_n| \geqslant 2^{j/2}\left\{(2c_1)^{-\frac{1}{2}} \cdot \frac{1}{2} - \varepsilon\right\} s_j \quad \text{for} \quad j \in I,$$

where $I = \{j : s_{j+1} \geqslant 2^{-1} s_j\}$. Since $\Sigma s_j = \infty$ we have $\sum_{j \in I} s_j = \infty$. Hence by using (5.14) in (5.11) we get (5.10). The remainder of the proof follows by Fubini's theorem.

Properties of random Fourier series can be used to obtain results in harmonic analysis. For example let $\Lambda = \{\lambda_n\}$, $n = 0, 1, \ldots$, where λ_n are integers, be a Sidon set. We will not define a Sidon set here (see [8], page 57), but observe that if $\{\lambda_n\}$ is a lacunary sequence, i.e. $\inf_n (\lambda_{n+1}/\lambda_n) > q > 1$, then Λ is a Sidon set. For Λ a Sidon set, $\{\xi_n\}$, $\{\xi'_n\}$ two independent sequences of independent random variables with $E\xi_n = E\xi'_n = 0$, $E\xi_n^2 = E\xi_n'^2 = 1$, the series

$$(5.15)\quad \sum_{n=0}^{\infty} a_n[\xi_n \cos 2\pi\lambda_n t + \xi'_n \sin 2\pi\lambda_n t],$$

converges uniformly a.s. if and only if $\{a_n\} \in l^1$. (This is equivalent to the statement that $\Sigma|a_n||\xi_n| < \infty$ if and only if $\{a_n\} \in l^1$). We obtain the following property of Sidon sets.

THEOREM 5.3. — *Let* $\Lambda = \{\lambda_n\}$, $n = 0, 1, \ldots$, *be a Sidon set and consider*

$$\sigma^2(h) = \sum_{n=0}^{\infty} a_n^2 \sin^2 \pi\lambda_n h.$$

Then $I(\bar{\sigma}) < \infty$ *if and only if* $\{a_n\} \in l^1$.

Proof. — Let the $\{\xi_n\}.\{\xi'_n\}$ in (5.15) be subgaussian. If $I(\bar{\sigma}) < \infty$ then by Theorem 3.2 and Lemma 4.2, (5.15) converges uniformly a.s.; hence $\{a_n\} \in l^1$. If $\{a_n\} \in l^1$, then (5.15) is a subset of a block in a Hilbert space as defined in ([10], (1.12)). By Theorem 2 of [10] $J(H) < \infty$ where H is the metric entropy of the block. Since (5.15) is weakly stationary it follows from (2.8) that $I(\bar{\sigma}) < \infty$.

6. Other random series of functions.

Let $[\varphi_n(t)]$, $t \in [0, 1]$, be a sequence of continuous functions such that $\sum_{n=0}^{\infty} \varphi_n^2(t)$ converges uniformly and

$$\sum_{n=0}^{\infty} (\varphi_n(t) - \varphi_n(s))^2 = \sigma^2(|t - s|)$$

for some continuous function σ. If $\{\varepsilon_n\}$ is a sequence of independent subgaussian random variables, then $I(\sigma) < \infty$ is a sufficient condition for the a.s. uniform convergence of $\sum_{n=0}^{\infty} \varphi_n(t)\varepsilon_n$. This result was the starting point in Section 4 in which we specialized to the case where the $\varphi_n(t)$ were sine and cosine functions. The methods of Section 4 do not carry over for more general $\varphi_n(t)$. Recall that when we wished to study $\sum_{n=0}^{\infty} \varphi_n(t)\xi_n$ for some strongly symmetric sequence $\{\xi_n\}$ with $E\xi_n^2 = 1$, defined on a probability space $(\Omega_1, \mathscr{F}_1, P_1)$, we introduced a Rademacher sequence $\{\varepsilon_n\}$ on $(\Omega_2, \mathscr{F}_2, P_2)$ and considered the series $\sum_{n=0}^{\infty} \varphi_n(t)\varepsilon_n\xi_n$ on $(\Omega, \mathscr{F}, P)$ where $\Omega = \Omega_1 \times \Omega_2$, $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$ and $P = P_1 \times P_2$. Then for fixed $\omega_1 \in \Omega_1$ we studied

$$Z_{\omega_1}(t) = \sum_{n=0}^{\infty} \xi_n(\omega_1)\varphi_n(t)\varepsilon_n \tag{6.1}$$

which is a subgaussian sequence if $E_2(Z_{\omega_1}(t) - z_{\omega_1}(s))^2$ exists. Notice, however, that if the $\varphi_n(t)$ are not uniformly bounded we don't even know if

$$(6.2) \qquad E_2 Z^2_{\omega_1}(t) = \sum_{n=0}^{\infty} \xi_n^2(\omega_1)\varphi_n^2(t)$$

converges uniformly. Even if the $\varphi_n(t)$ are taken to be uniformly bounded it is not clear whether

$$(6.3) \quad E_2(Z_{\omega_1}(t) - Z_{\omega_1}(s))^2 = \sum_{n=0}^{\infty} \xi_n^2(\omega_1)(\varphi_n(t) - \varphi_n(s))^2$$

is a continuous function of $|t - s|$.

We can apply our methods only if we take $\{\xi_n\}$ to be uniformly bounded and then we obtain a weaker result analagous to Theorem 3.1.

THEOREM 6.1. — *Let* $\{\varphi_n(t)\}$, $t \in [0, 1]$, *be a sequence of continuous functions such that* $\sum_{n=0}^{\infty} \varphi_n^2(t)$ *converges uniformly; let* $\sum_{n=0}^{\infty} (\varphi_n(t) - \varphi_n(s))^2 \leqslant \psi^2(|t - s|)$ *for a non-decreasing function* ψ *and* $\{\xi_n\}$ *be a uniformly bounded strongly symmetric sequence. Then* $I(\psi) < \infty$ *is a sufficient condition for*

$$X(t) = \sum_{n=0}^{\infty} \varphi_n(t)\xi_n$$

to converge uniformly a.s.

Proof. — Consider $Z_{\omega_1}(t)$ in (6.1). This function satisfies the hypothesis of Theorem 3.1 since

$$E_2(Z_{\omega_1}(t) - Z_{\omega_1}(s))^2 = \sum_{n=0}^{\infty} \xi_n^2(\omega_1)(\varphi_n(t) - \varphi_n(s))^2 \leqslant M^2\psi^2(|t - s|)$$

where M is the uniform bound for $\{\xi_n\}$. Therefore $Z_{\omega_1}(t)$ has continuous sample paths a.s. (P_2) on Ω_2. By Lemma 4.2 the series for $Z_{\omega_1}(t)$ converges a.s. The rest of the proof follows the proof of Theorem 4.3.

Example. — Let $X(t)$, $t \in [0, 1]$, be a real valued, stochastically continuous, second order weakly stationary process, i.e. $EX^2(t) < \infty$ (assume $EX(t) = 0$). All processes of this kind can be given as

$$X(t) = \sum_{n=0}^{\infty} \sqrt{\lambda_n}\theta_n(t)\xi_n, \tag{6.4}$$

where λ_n and $\theta_n(t)$ are the eigenvalues and eigenfunctions of $R(s, t) = EX(t)X(s)$. It is known that

$$E\,(X(t) - X(s))^2 = \sum_{n=0}^{\infty} \lambda_n(\theta_n(t) - \theta_n(s))^2$$

converges uniformly to $R(s, s) + R(t, t) - 2R(s, t)$. If $\{\xi_n\}$ is uniformly bounded and strongly symmetric and if $E(X(t) - X(s))^2 \leqslant \psi^2(|t - s|)$ where ψ is non-decreasing and $I(\psi) < \infty$ then the series (6.4) converges uniformly a.s.

We have seen that it is necessary to impose some property like strong symmetry on the $\{\xi_n\}$. The major open problem is to investigate the continuity of the series (6.4) when the $\{\varphi_n(t)\}$ are not sine or cosine functions and the $\{\xi_n\}$ are not uniformly bounded.

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