

## COMPLETENESS AND EXISTENCE OF BOUNDED BIHARMONIC FUNCTIONS ON A RIEMANNIAN MANIFOLD

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A.S. Galbraith has communicated to us the following intriguing problem : Does the completeness of a manifold imply, or is it implied by, the emptiness of the class  $H^2B$  of bounded nonharmonic biharmonic functions ? Among all manifolds considered thus far in biharmonic classification theory (cf. Bibliography), those that are complete fail to carry  $H^2B$ -functions, and one might suspect that this is always the case. We shall show, however, that there do exist complete manifolds of any dimension that carry  $H^2B$ -functions. Moreover, there exist both complete and incomplete manifolds not permitting these functions, and, trivially, incomplete manifolds possessing them.

We attach a Bibliography of recent work in the field.

1. Let  $C$  be the totality of complete Riemannian manifolds  $M$ , characterized by an infinite distance of any point of  $M$  to the ideal boundary. Denote by  $\mathcal{O}_{H^2B}^N$  and  $\tilde{\mathcal{O}}_{H^2B}^N$  the classes of  $N$ -manifolds,  $N \geq 2$ , for which  $H^2B = \emptyset$  or  $H^2B \neq \emptyset$ , respectively.

**THEOREM 1.** —  $C \cap \tilde{\mathcal{O}}_{H^2B}^N \neq \emptyset$  for every  $N$ .

*Proof.* — Take the  $N$ -cylinder

$$|x| < \infty, \quad |y_i| \leq 1, \quad i = 1, 2, \dots, N - 1,$$

with each face  $y_i = 1$  identified with  $y_i = -1$ , so as to obtain a covering space of the  $N$ -torus in the same manner as a conventional cylinder is a covering surface of the torus. Let  $T$  be this  $N$ -cylinder with the Riemannian metric

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$$ds^2 = \mu^{-2}(x) dx^2 + \mu^{4/(N-1)}(x) \sum_{i=1}^{N-1} dy_i^2$$

where

$$\mu(x) = (2 + x^2)^{\frac{1}{2}} \log(2 + x^2).$$

To see that  $T \in C$ , it suffices to show, in view of the symmetry, that  $\int_0^\infty \mu^{-1}(x) dx = \infty$ . The verification is immediate :

$$\begin{aligned} \int_0^\infty (2 + x^2)^{-\frac{1}{2}} \log^{-1}(2 + x^2) dx &> \frac{1}{2} \int_0^\infty (2 + x)^{-1} \log^{-1}(2 + x) dx \\ &= \frac{1}{2} \int_0^\infty \log \log(2 + x) = \infty. \end{aligned}$$

We introduce the function

$$u(x) = \int_0^x \mu^{-3}(t) \int_0^t \mu(s) \int_0^s \mu^{-3}(r) dr ds dt.$$

The Laplace-Beltrami operator  $\Delta = d\delta + \delta d$  gives

$$\Delta u = -g^{-\frac{1}{2}} (g^{\frac{1}{2}} g^{xx} u')' = -\mu^{-1} (\mu \mu^2 u')' = -\int_0^x \mu^{-3}(r) dr$$

and

$$\Delta^2 u = -\mu^{-1} (\mu \mu^2 (-\mu^{-3}))' = 0.$$

Thus  $u$  is nonharmonic biharmonic.

To see that  $u$  is bounded it suffices to show that it is so for  $x > 0$ . For all  $s > 0$ ,

$$\int_0^s \mu^{-3}(r) dr = \int_0^s (2 + r^2)^{-3/2} \log^{-3}(2 + r^2) dr = o(1),$$

and for all  $t > 0$ ,

$$\begin{aligned} \int_0^t \mu(s) \int_0^s \mu^{-3}(r) dr ds &< c \int_0^t (2 + s^2)^{\frac{1}{2}} \log(2 + s^2) ds \\ &< 2c \int_0^t (2 + s) \log(2 + s) ds \\ &= c \left[ (2 + t)^2 \log(2 + t) \right. \\ &\quad \left. - \frac{1}{2} (2 + t)^2 + \text{const.} \right]. \end{aligned}$$

Here and later  $c$  is a constant, not always the same. We let  $[ \ ]$  stand for the expression in brackets and obtain

$$u(x) < c \int_0^x (2 + t^2)^{-3/2} \log^{-3}(2 + t^2) [ \ ] dt.$$

The dominating term in the integrand is majorized by

$$\frac{1}{2} t^{-3} \log^{-3} t \times (2 + t)^2 \log(2 + t).$$

The integral from 1 to  $x > 1$  is bounded, and consequently so is  $u$  for all  $x$ .

This completes the proof of Theorem 1.

2. The following simple example, valid for  $N \geq 3$ , is perhaps also of interest. Let

$$T : \quad |x| < \infty, \quad |y| \leq \pi, \quad |z_i| \leq 1, \quad i = 1, \dots, N - 2,$$

with the metric

$$ds^2 = dx^2 + e^{-x} dy^2 + e^{(2e^x - x)/(N-2)} \sum_{i=1}^{N-2} dz_i^2,$$

the opposite faces again identified by pairs. Clearly  $T \in C$ .

The function

$$u = \cos y$$

belongs to  $H^2B$ . In fact,

$$\begin{aligned} \Delta u &= -e^{-e^x+x} (e^{e^x-x} e^x) (-\cos y) \\ &= e^x \cos y, \end{aligned}$$

and

$$\Delta^2 u = -e^{-e^x+x} [(e^{e^x-x} e^x)' \cos y + e^{e^x-x} e^x e^x (-\cos y)] = 0.$$

Thus  $T \in C \cap \tilde{\mathcal{O}}_{H^2B}$ .

3. The reason that we are only interested in nonharmonic biharmonic functions is, of course, that completeness is known not

to exclude bounded harmonic functions (Nakai-Sario [6]). For  $N \geq 3$ , we insert here a simple proof of this fact.

Take the  $N$ -cylinder

$$T: |x| < \infty, \quad |y| \leq 1, \quad |z_i| \leq 1, \quad i = 1, \dots, N - 2,$$

with the metric

$$ds^2 = dx^2 + e^{2x^2} dy^2 + \sum_{i=1}^{N-2} dz_i^2.$$

Trivially  $T \in C$ . The function

$$h(x) = \int_0^x e^{-t^2} dt$$

is harmonic,

$$\Delta h = -e^{-x^2} (e^{x^2} e^{-x^2})' = 0.$$

It also is bounded and, in fact, even Dirichlet finite :

$$D(h) = c \int_{-\infty}^{\infty} e^{-2x^2} e^{x^2} dx < \infty.$$

4. We return to nonharmonic biharmonic functions.

THEOREM 2. —  $C \cap \mathcal{O}_{H^2B}^N \neq \emptyset$  for every  $N$ .

*Proof.* — The Euclidean  $N$ -space  $E^N \in C$ . Every biharmonic function  $u$  has an expansion in spherical harmonics  $S_{nm}$

$$u = \sum_{n=0}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{n+2} + b_{nm} r^n) S_{nm}.$$

If  $u \in H^2B$ , then

$$\int_{|x|=r} u S_{nm} d\omega = c(a_{nm} r^{n+2} + b_{nm} r^n)$$

is bounded in  $r$ , hence  $a_{nm} = b_{nm} = 0$  for all  $n$ , except for  $b_{01}$ . Therefore  $u$  is constant.

5. In view of  $u = r^2 \in H^2B$  on the Euclidean  $N$ -ball, we have trivially  $\tilde{C} \cap \tilde{\mathcal{O}}_{H^2B}^N \neq \emptyset$  for every  $N$ , with  $\tilde{C}$  the totality of incomplete Riemannian manifolds. It remains to show :

THEOREM 3. —  $C \cap \mathcal{O}_{H^2B}^N \neq \emptyset$  for every  $N$ .

*Proof.* — Let  $E_\alpha^N$  be the  $N$ -space  $0 < r < \infty$  with the metric

$$ds = r^\alpha |dx|,$$

$\alpha$  a constant. It is known (Sario-Wang [19, 21]) that if  $N \geq 4$ ,  $E_\alpha^N \in \mathcal{O}_{H^2B}$  for every  $\alpha$ ;  $E_\alpha^2 \in \mathcal{O}_{H^2B}$  if and only if  $\alpha \neq -1 \mp n/2$ ,  $n = 1, 2, \dots$ ;  $E_\alpha^3 \in \mathcal{O}_{H^2B}$  if and only if  $\alpha \neq -1 \mp \left[ \frac{1}{2} n(n+1) \right]^{1/2}$ . On the other hand,  $E_\alpha^N \in \tilde{C}$  for every  $\alpha$ , hence the theorem.

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