

ON THE LOWER ORDER (R) OF AN ENTIRE DIRICHLET SERIES

by P.K. JAIN ⁽¹⁾ and D.R. JAIN

1. Introduction.

For an entire Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \quad (s = \sigma + it, \lambda_1 \geq 0, \lambda_n \rightarrow \infty \text{ with } n) \quad (1.1)$$

the lower order (R) λ is defined as :

$$\lim_{\sigma \rightarrow \infty} \inf \frac{\log \log M(\sigma)}{\sigma} = \lambda, \quad (0 \leq \lambda \leq \infty), \quad (1.2)$$

where $M(\sigma) = \sup \{|f(\sigma + it)| : -\infty < t < \infty\}$.

Improving upon a result of Rahman [6], Juneja and Singh [4] have, very recently, proved the following theorem :

THEOREM A. — *Let $f(s)$ be an entire Dirichlet series given by (1.1) of lower order (R) λ ($0 \leq \lambda \leq \infty$). Then*

$$\lim_{n \rightarrow \infty} \inf \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}} \leq \lambda. \quad (1.3)$$

Further, if

$$\lim_{n \rightarrow \infty} \sup \frac{\log n}{\lambda_n} < \infty, \quad (1.4)$$

and

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$$\varphi_n \equiv \frac{\log |a_{n-1}/a_n|}{\lambda_n - \lambda_{n-1}} \quad (1.5)$$

forms a non-decreasing function of n for $n > n_0$, then

$$\lim_{n \rightarrow \infty} \inf \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}} = \lambda. \quad (1.6)$$

In this paper, we obtain the estimations for the lower order (R) λ in terms of the sequences $\{\lambda_n\}$ and $\{a_n\}$ which hold for every entire Dirichlet series, and one of our estimations includes (1.6) and the result of Rahman [6] as the special cases. In fact, we prove :

THEOREM. — Let $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ be an entire Dirichlet series given by (1.1) of lower order (R) λ ($0 \leq \lambda \leq \infty$) such that (1.4) is satisfied. Then

$$\lambda = \max_{\{\lambda_{n_p}\}} \lim_{p \rightarrow \infty} \inf \frac{\lambda_{n_p} \log \lambda_{n_p-1}}{\log |a_{n_p}|^{-1}} \quad (1.7)$$

$$\lambda = \max_{\{\lambda_{n_p}\}} \lim_{p \rightarrow \infty} \inf \frac{(\lambda_{n_p} - \lambda_{n_p-1}) \log \lambda_{n_p-1}}{\log |a_{n_p-1}/a_{n_p}|} \quad (1.8)$$

2. Preliminary Discussions.

Let $\mu(\sigma)$ denote the maximum term of $f(s)$ for $\text{Re}(s) = \sigma$ and $\lambda_{\nu(\sigma)} = \max\{\lambda_n : \mu(\sigma) = |a_n| e^{\sigma\lambda_n}\}$. Let $\{\rho_n\}$ be the sequence of jump points of $\lambda_{\nu(\sigma)}$ (points of discontinuity of $\lambda_{\nu(\sigma)}$), every jump point is listed with multiplicity equal to size of the jump, such that $\rho_1 \leq \rho_2 \leq \rho_3 \leq \dots \rho_n \leq \dots$. Since $\lambda_{\nu(\sigma)} \rightarrow \infty$ as $\sigma \rightarrow \infty$, $\rho_n \rightarrow \infty$ as $n \rightarrow \infty$. We denote by $\{\lambda_{n_k}\}$ the range of $\lambda_{\nu(\sigma)}$, so that $\lambda_{\nu(\sigma)} = \lambda_{n_k}$ for $\sigma = \rho_{n_k}$. Then

i) $0 < \rho_{n_k} < \rho_{n_k+1} = \rho_{n_k+2} = \dots = \rho_{n_k+1}$, $k = 1, 2, 3, \dots$

ii) $\lambda_{\nu(\sigma)} = \lambda_{n_k}$, when $\rho_{n_k} \leq \sigma < \rho_{n_k+1}$, $k = 1, 2, 3, \dots$

These arguments are analogue to those for entire power series, given and used by Gray and Shah in their works [1,2,3].

Since $|a_{n_{k-1}}| e^{\sigma \lambda_{n_{k-1}}}$ and $|a_{n_k}| e^{\sigma \lambda_{n_k}}$ are the two consecutive maximum terms, we have

$$\rho_{n_k} = \log |a_{n_{k-1}}/a_{n_k}| / (\lambda_{n_k} - \lambda_{n_{k-1}}) . \tag{2.1}$$

Also, we need the following :

LEMMA

$$\liminf_{k \rightarrow \infty} \frac{\log \lambda_{n_k}}{\rho_{n_{k+1}}} = \lambda .$$

Proof. – Since ([5], Theorem B)

$$\liminf_{\sigma \rightarrow \infty} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \lambda, \quad (0 \leq \lambda \leq \infty) ,$$

there exists a sequence $\{x_i\}$, $x_i \rightarrow \infty$ with i such that

$$\lim_{i \rightarrow \infty} \frac{\log \lambda_{\nu(x_i)}}{x_i} = \lambda .$$

It is always possible to find a subsequence $\{\rho_{n_{k_i}}\}$ of $\{\rho_{n_k}\}$ which satisfies the inequalities :

$$\rho_{n_{k_i}} \leq x_i < \rho_{n_{k_{i+1}}}, \quad i = 1, 2, 3, \dots$$

In either of the cases, for $i \geq i_0 = i_0(\epsilon)$, $\epsilon > 0$, we have

$$\frac{\log \lambda_{n_{k_i}}}{\rho_{n_{k_{i+1}}}} \leq \frac{\log \lambda_{\nu(x_i)}}{x_i} \leq \lambda + \epsilon ,$$

which implies

$$\liminf_{k \rightarrow \infty} \frac{\log \lambda_{n_k}}{\rho_{n_{k+1}}} \leq \lambda .$$

The reverse inequality is obvious. Hence the lemma is proved.

3. Proof of the Theorem.

Using (2.1) in the lemma, we get

$$\lambda = \lim_{k \rightarrow \infty} \inf \frac{(\lambda_{n_k} - \lambda_{n_{k-1}}) \log \lambda_{n_{k-1}}}{\log |a_{n_{k-1}}/a_{n_k}|} . \quad (3.1)$$

We have proved (3.1) for a particular subsequence $\{\lambda_{n_k}\}$ which is the range of the rank function $\lambda_{\nu(\sigma)}$. Thus the theorem will be proved completely if we establish, for any arbitrary subsequence (say) $\{\lambda_{n_p}\}$ of $\{\lambda_n\}$, the following inequalities :

$$\lambda \geq \lim_{p \rightarrow \infty} \inf \frac{\lambda_{n_p} \log \lambda_{n_{p-1}}}{\log |a_{n_p}|^{-1}} \geq \lim_{p \rightarrow \infty} \inf \times \quad (3.2)$$

$$\frac{(\lambda_{n_p} - \lambda_{n_{p-1}}) \log \lambda_{n_{p-1}}}{\log |a_{n_{p-1}}/a_{n_p}|} .$$

Proof of the first inequality in (3.2) : Let

$$\lim_{p \rightarrow \infty} \inf \frac{\lambda_{n_p} \log \lambda_{n_{p-1}}}{\log |a_{n_p}|^{-1}} = \alpha .$$

Assume that $\alpha > 0$, for otherwise the result is trivially true. Therefore, for any $\epsilon > 0$, $\exists a N = N(\epsilon)$ such that

$$|a_{n_p}| > \frac{-\lambda_{n_p}}{\lambda_{n_{p-1}}^{\alpha - \epsilon}}, \quad (p \geq N) .$$

Let $e^{\sigma_p} = 2 \lambda_{n_{p-1}}^{\frac{1}{\alpha - \epsilon}}$, $p = 1, 2, 3, \dots$. So if

$\sigma_p \leq \sigma \leq \sigma_{p+1}$, we have

$$\begin{aligned} \log M(\sigma) &\geq \log |a_{n_p}| + \sigma_p \lambda_{n_p} \\ &\geq \log |a_{n_p}| + \sigma_p \lambda_{n_p} \\ &\geq \lambda_{n_p} \log 2 \\ &= e^{(\alpha - \epsilon)\sigma_{p+1}} \log 2 / 2^{\alpha - \epsilon} , \end{aligned}$$

i.e.

$$\log \log M(\sigma) \geq (\alpha - \epsilon) \sigma_{p+1} + \log \log 2 - (\alpha - \epsilon) \log 2 .$$

which gives

$$\lambda = \lim_{\sigma \rightarrow \infty} \inf. \frac{\log \log M(\sigma)}{\sigma} \geq \alpha .$$

Proof of the Second Inequality in (3.2) : Let

$$\lim_{p \rightarrow \infty} \inf \frac{(\lambda_{n_p} - \lambda_{n_{p-1}}) \log \lambda_{n_{p-1}}}{\log |a_{n_{p-1}}/a_{n_p}|} = \beta .$$

Again, without any loss of generality, assume $\beta > 0$, so that

$$|a_{n_{p-1}}/a_{n_p}| < \frac{\lambda_{n_p} - \lambda_{n_{p-1}}}{\lambda_{n_{p-1}}^{\beta - \epsilon}} ,$$

for $p \geq p_0 = p_0(\epsilon)$, $\epsilon > 0$. This implies

$$\left| \frac{a_{n_{p_0}}}{a_{n_p}} \right| = \left| \frac{a_{n_{p_0}}}{a_{n_{p_0+1}}} \right| \cdot \left| \frac{a_{n_{p_0+1}}}{a_{n_{p_0+2}}} \right| \cdots \left| \frac{a_{n_{p-1}}}{a_{n_p}} \right|$$

$$< \prod_{m=p_0+1}^p \frac{\lambda_{n_m} - \lambda_{n_{m-1}}}{\lambda_{n_{m-1}}^{\beta - \epsilon}}$$

$$\Rightarrow \log |a_{n_p}|^{-1} < O(1) + \frac{1}{\beta - \epsilon} \sum_{m=p_0+1}^p (\lambda_{n_m} - \lambda_{n_{m-1}}) \log \lambda_{n_{m-1}}$$

$$\Rightarrow \frac{\log |a_{n_p}|^{-1}}{\lambda_{n_p} \log \lambda_{n_{p-1}}} < o(1) + \frac{1}{\beta - \epsilon} - \frac{1}{\beta - \epsilon} \frac{\lambda_{n_{p_0}} \log \lambda_{n_{p_0}}}{\lambda_{n_p} \log \lambda_{n_{p-1}}}$$

$$\frac{(\beta - \epsilon)^{-1} \sum_{m=p_0+1}^{p-1} \lambda_{n_m} (\log \lambda_{n_m} - \log \lambda_{n_{m-1}})}{\lambda_{n_p} \log \lambda_{n_{p-1}}}$$

$$\Rightarrow \lim_{p \rightarrow \infty} \sup \frac{\log |a_{n_p}|^{-1}}{\lambda_{n_p} \log \lambda_{n_{p-1}}} \leq \frac{1}{\beta} .$$

Hence

$$\lim_{p \rightarrow \infty} \inf. \frac{\lambda_{n_p} \log \lambda_{n_p-1}}{\log |a_{n_p}|^{-1}} \geq \lim_{p \rightarrow \infty} \inf. \frac{(\lambda_{n_p} - \lambda_{n_p-1}) \log \lambda_{n_p-1}}{\log |a_{n_p-1}/a_{n_p}|},$$

Remark. — If, in addition to the hypothesis of our theorem, (1.5) is satisfied, then our result (1.7) reduces to (1.6). Further, if $\log \lambda_{n+1} \sim \log \lambda_n$, as $n \rightarrow \infty$, the result of Rahman [6] is also obtained.

Justification. — Since (1.5) is satisfied, each term of $f(s)$ is a maximum term and so $\lambda_{n_k} = \lambda_k$, for $k = 1, 2, \dots$. Therefore, the result (3.1) reduces to

$$\lambda = \lim_{k \rightarrow \infty} \inf. \frac{(\lambda_k - \lambda_{k-1}) \log \lambda_{k-1}}{\log |a_{k-1}/a_k|}. \quad (3.3)$$

Further, as the result (3.2) is true for every subsequence $\{\lambda_{n_p}\}$ of $\{\lambda_n\}$, it is also true for the sequence $\{\lambda_n\}$, since $\{\lambda_n\}$ may be regarded as a subsequence of $\{\lambda_n\}$. Hence

$$\lambda \geq \lim_{n \rightarrow \infty} \inf. \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}} \geq \lim_{n \rightarrow \infty} \inf. \frac{(\lambda_n - \lambda_{n-1}) \log \lambda_{n-1}}{\log |a_{n-1}/a_n|} \quad (3.4)$$

Thus, the result (1.6) follows from (3.3) and (3.4).

Furthermore, if $\log \lambda_{n+1} \sim \log \lambda_n$, as $n \rightarrow \infty$, then (1.6) implies that

$$\lambda = \lim_{n \rightarrow \infty} \inf. \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}} = \lim_{n \rightarrow \infty} \inf. \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}$$

which is a result of Rahman [6].

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D.R. JAIN, Department of Mathematics, A.R.S.D. College, Delhi University, NEW-DELHI-23, India	P.K. JAIN , Department of Mathematics, Hans Raj College, Delhi University, DELHI-7, India.
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