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Parametrized *h*-cobordism theory

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PARAMETRIZED h-COBORDISM THEORY

by A. E. HATCHER

1. Introduction.

The h-cobordism theorem is concerned with the following question: Let (W, M, M') be a smooth, compact, connected h-cobordism (i.e. ∂W is the disjoint union of the closed manifolds M and M' which are each deformation retracts of W). Is W diffeomorphic to $M \times I$? This is equivalent to asking if there exists a smooth map $(W, M, M') \to (I, 0, 1)$ having no critical points, or in fact whether an arbitrary smooth map $(W, M, M') \to (I, 0, 1)$ can be deformed so as to eliminate all its critical points. If the answer is affirmative, one can then ask the parametrized question: Let $f_t : M \times (I, 0, 1) \to (I, 0, 1)$, $t \in I^k$, be a k-parameter family of smooth maps such that f_t for $t \in \partial I^k$ has no critical points. Can the family f_t be deformed, staying fixed over ∂I^k, to a k-parameter family of functions without critical points? In other words one seeks to compute $\pi_k(\mathcal{F}, \mathcal{E})$, where $\mathcal{F}$ is the space of smooth maps $M \times (I, 0, 1) \to (I, 0, 1)$ and $\mathcal{E}$ is the subspace of maps without critical points (C^∞ topologies throughout). With no extra trouble we can allow M to have a boundary, provided that functions in $\mathcal{F}$ are required to restrict on $\partial M \times I$ to the projection $\partial M \times I \to I$.

The beauty of this problem, it seems to me, is that it links in a highly non-trivial way certain rather deep questions in topology, analysis, and algebra:

(1) Computing $\pi_k(\mathcal{F}, \mathcal{E})$ gives information about the global homotopy properties of various spaces of diffeomorphisms, via the isomorphism $\pi_k(\mathcal{F}, \mathcal{E}) \approx \pi_{k-1}\mathcal{P}(M, \partial M)$, where $\mathcal{P}(M, \partial M) = \{\text{diffeomorphisms } F : M \times I \to M \times I \text{ such that } F|M \times \{0\} \cup \partial M \times I = id.\}$ is the space of (relative) pseudo-isotopies of $id : M \to M$. (In fact

$\pi_k(\mathscr{F}, \mathscr{E}) \approx \pi_{k-1}\mathscr{E}$ since $\mathscr{F}$ is convex; and one has a fibration $\mathscr{S} \to \mathscr{P} \to \mathscr{E}$ whose fiber, the space of isotopies of $\mathrm{id} : \mathrm{M} \to \mathrm{M}$, is contractible so that $\pi_{k-1}\mathscr{E} \approx \pi_{k-1}\mathscr{P}(\mathrm{M}, \partial\mathrm{M})$.) Computing $\pi_{k-1}\mathscr{P}(\mathrm{M}, \partial\mathrm{M})$ can be thought of as answering uniqueness questions about product structures on an h-cobordism.

In two special cases $\mathscr{P}(\mathrm{M}, \partial\mathrm{M})$ has an immediate interest:

(a) Diff (D^n) splits homotopically as $\mathrm{O}(n) \times \mathscr{P}(\mathrm{S}^{n-1})$. Is $\mathscr{P}(\mathrm{S}^{n-1})$ contractible? If $n \leqslant 2$ the answer is yes, but for n large the answer is presumably no. According to a lemma of [17], $\mathscr{P}(\mathrm{S}^{n-1})$ is homotopy equivalent to $\mathscr{P}(\mathrm{D}^{n-1}, \partial\mathrm{D}^{n-1})$. Thus the remarks in ($b$) show indirectly that for some n, $\mathscr{P}(\mathrm{S}^{n-1})$ is not contractible. We shall describe below some candidates for non-zero elements of $\pi_{k-1}\mathscr{P}(\mathrm{S}^{n-1})$, $k \ll n$, for example when $k = 2$.

(b) Diff (S^n) $\approx 0(n+1) \times$ Diff (D^n, $\partial\mathrm{D}^n$), and one has a fibration

$$\mathrm{Diff}\,(\mathrm{D}^n, \partial\mathrm{D}^n) \to \mathscr{P}(\mathrm{D}^{n-1}, \partial\mathrm{D}^{n-1}) \to \mathrm{Diff}\,(\mathrm{D}^{n-1}, \partial\mathrm{D}^{n-1}).$$

The boundary homomorphism

$$\pi_j\,\mathrm{Diff}\,(\mathrm{D}^{n-1}, \partial\mathrm{D}^{n-1}) \xrightarrow{\partial} \pi_{j-1}\,\mathrm{Diff}\,(\mathrm{D}^n, \partial\mathrm{D}^n)$$

is the Gromoll homomorphism. All known exotic elements of π_j Diff (S^n) have been detected by their images in

$$\pi_0\,\mathrm{Diff}\,(\mathrm{D}^{n+j}, \partial\mathrm{D}^{n+j}) \approx \Gamma_{n+j+1}$$

under iteration of ∂ (see [1]). Clearly, not all the ∂'s can be isomorphisms, else one would have

$$\Gamma_{n+j+1} \approx \pi_0\,\mathrm{Diff}\,(\mathrm{D}^{n+j}, \partial\mathrm{D}^{n+j}) \approx \pi_{n+j}\,\mathrm{Diff}\,(\mathrm{D}^0) \approx 0.$$

As in (a) above we shall give candidates for non-trivial elements of $\pi_j\mathscr{P}(\mathrm{D}^{n-1}, \partial\mathrm{D}^{n-1})$, i.e., for Gromoll homomorphisms which are not isomorphisms, $j \ll n$.

For more general manifolds M one can study Diff (M) by means of a certain space of block diffeomorphisms $\widetilde{\mathrm{Diff}}$ (M) which is more accessible, using surgery theory to compare with the corresponding space of block homotopy equivalences for example [2, 12, 16]. In computing $\widetilde{\mathrm{Diff}}\,(\mathrm{M})/\mathrm{Diff}\,(\mathrm{M})$ one has first the local problem of « k-concordance modulo k-isotopy », i.e., deciding whether a diffeomorphism of $\mathrm{M} \times \mathrm{I}^k$

which on $M \times \partial I^k$ preserves projection onto ∂I^k can be deformed to preserve projection onto I^k throughout. Proceeding in this local problem one I factor at a time one encounters successive obstructions in $\pi_l \mathscr{P}(M \times I^{k-l-1}, \partial)$, $0 \leqslant l \leqslant k-1$.

(2) Generically a k-parameter family $f_t : M \times I \to I$ meets arbitrary singularities of « codimension » $\leqslant k$, together with their « universal unfoldings ». For example when $k \leqslant 4$ one encounters all the elementary catastrophes of Thom [13]. Computing $\pi_k(\mathscr{F}, \mathscr{E})$ thus involves a global study of such singularities of smooth real-valued functions, a study which already at the local level is quite involved.

(3) Analyzing k-parameter families leads one inexorably to algebraic K-theory, and in particular to a definition of higher K_n's which, although *a priori* unrelated to the K-theories of Karoubi, Gersten-Swan, and Quillen, has good reason to be the « one true » algebraic K-theory.

Computation of the groups $\pi_k(\mathscr{F}, \mathscr{E})$ is still in the early stages. In addition to the h-cobordism theorem itself, which may be paraphrased as saying « $\pi_0(\mathscr{F}, \mathscr{E}) \approx Wh_1(\pi_1 M)$ » if $\dim M \geqslant 5$, one has only the following:

1.1. PSEUDO-ISOTOPY THEOREM. — *Provided* $\dim M \geqslant 7$,

$$\pi_0 \mathscr{P}(M, \partial M) \approx \pi_1(\mathscr{F}, \mathscr{E}) \approx Wh_2(\pi_1 M) \oplus Wh_1(\pi_1 M; \mathbf{Z}_2 \times \pi_2 M).$$

In the simply-connected case this theorem is due to Cerf [3], who proved then that $\pi_1(\mathscr{F}, \mathscr{E}) = 0$ when $\dim M \geqslant 5$. For $\pi_1 M \neq 0$, the Wh_2 obstruction was discovered independently by J. B. Wagoner [14] and myself [6]. More recently, I have completed the determination of the second obstruction, the description of which is the main point of this paper.

The group $Wh_2(\pi_1 M)$ is defined as a certain quotient of $K_2 \mathbf{Z}[\pi_1 M]$, where K_2 is Milnor's K_2 (see [10]). The computation of these groups K_2 and Wh_2 seems to be quite difficult. In fact, until very recently only when $\pi_1 M = 0$ was the computation fully known: $Wh_2(0) = 0$, in line with Cerf's theorem.

Now one hears that Quillen has proved results which imply that $Wh_2(\pi_1 M) = 0$ when $\pi_1 M$ is free abelian. For certain groups π_1 it is also known that $Wh_2 \pi_1 \neq 0$, e.g., $\pi_1 = \mathbf{Z}_{20}$ ([11]) or $\pi_1 = \mathbf{Z} \oplus \mathbf{Z}_p$ for primes $p \geqslant 5$ ([15]).

In contrast with this, the groups $Wh_1(\pi_1 M; \mathbf{Z}_2 \times \pi_2 M)$ are readily computable. To state the result, let $(\mathbf{Z}_2 \times \pi_2 M)[\pi_1 M]$ denote the additive group of finite formal sums $\Sigma \alpha_i \sigma_i$ where $\alpha_i \in \mathbf{Z}_2 \times \pi_2 M$ and $\sigma_i \in \pi_1 M$. Thus $(\mathbf{Z}_2 \times \pi_2 M)[\pi_1 M]$ is the direct sum of $|\pi_1 M|$ copies of $\mathbf{Z}_2 \times \pi_2 M$.

1.2. Proposition.

$$Wh_1(\pi_1 M; \mathbf{Z}_2 \times \pi_2 M) \approx \frac{(\mathbf{Z}_2 \times \pi_2 M)[\pi_1 M]}{(\beta.1, \alpha\sigma - \alpha^\tau \tau\sigma\tau^{-1})}.$$

Here $(\beta.1, \alpha\sigma - \alpha^\tau\tau\sigma\tau^{-1})$ *denotes the additive subgroup of* $(\mathbf{Z}_2 \times \pi_2 M)[\pi_1 M]$ *generated by the elements* $\beta.1$ *and* $\alpha\sigma - \alpha^\tau\tau\sigma\tau^{-1}$ *where* $\alpha, \beta \in \mathbf{Z}_2 \times \pi_2 M$, $\sigma, \tau \in \pi_1 M$, 1 *is the identity of* $\pi_1 M$, *and* α^τ *denotes* τ *acting trivially on the* $\mathbf{Z}_2$ *component of* α *and in the usual way on the* π_2 *factor.*

For example, if $\pi_2 M = 0$ then $Wh_1(\pi_1 M; \mathbf{Z}_2)$ is the direct sum of $\mathbf{Z}_2$ with itself as many times as there are conjugacy classes in $\pi_1 M$ other than the trivial class $\{1\}$. Thus $Wh_1(\pi_1 M; \mathbf{Z}_2)$ vanishes if and only if $\pi_1 M = 0$. Since in general $Wh_1(\pi_1 M; \mathbf{Z}_2 \times \pi_2 M)$ splits naturally as

$$Wh_1(\pi_1 M; \mathbf{Z}_2) \oplus Wh_1(\pi_1 M; \pi_2 M),$$

this implies the following:

1.3. Corollary. — *When* $\dim M \geqslant 7$,

$$\pi_0 \mathscr{P}(M, \partial M) \approx \pi_1(\mathscr{F}, \mathscr{E})$$

is zero if and only if $\pi_1 M = 0$.

As an amusing application of the $Wh_1(\pi_1 M; \mathbf{Z}_2 \times \pi_2 M)$ obstruction we have:

1.4. Proposition. — *The group*

$$\pi_0 \operatorname{Diff}(S^1 \times D^n, \partial(S^1 \times D^n)), \quad n \geqslant 7,$$

is not finitely generated.

For by writing $D^n = D^{n-1} \times I$ one obtains a map

$$\pi_0 \operatorname{Diff}(S^1 \times D^n, \partial(S^1 \times D^n)) \xrightarrow{\Psi} \pi_0 \mathscr{P}(S^1 \times D^{n-1}, \partial(S^1 \times D^{n-1})).$$

The latter group contains (and is in fact equal to, using the result of Quillen), $Wh_1(\mathbf{Z}; \mathbf{Z}_2) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \ldots$ Writing elements of $Wh_1(\mathbf{Z};\ \mathbf{Z}_2)$ as finite sums $\sum_{i\neq 0} \alpha_i t^i$ where t generates $\mathbf{Z}$ and $\alpha_i \in \mathbf{Z}_2$, one finds that the image of Ψ in $Wh_1(\mathbf{Z};\ \mathbf{Z}_2)$ consists exactly of sums $\sum_{i\neq 0} \alpha_i t^i$ for which $\alpha_i = \alpha_{-i}$ (This involves a duality formula for the $Wh_1(\pi_1 M;\ \mathbf{Z}_2 \times \pi_2 M)$ obstruction). In particular, the image of Ψ, hence also $\pi_0 \operatorname{Diff}(S^1 \times D^n,\ \partial(S^1 \times D^n))$, is not finitely generated.

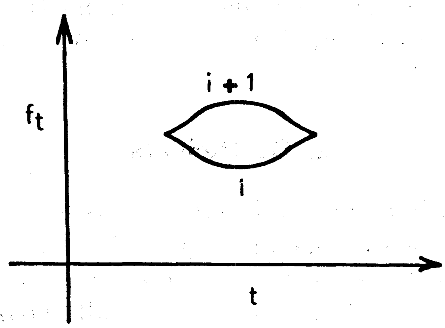
2. Coefficients.

In the following we shall have occasion to use only the singularities of codimension 0 and 1. The codimension 0 singularities are the well-known *non-degenerate* critical points. Critical points of codimension 1 we call *birth-death points*, since in a transversal one-parameter family they occur when a pair of non-degenerate critical points (of index i and $i+1$ for some $0 \leqslant i \leqslant n = \dim M$) are introduced or cancelled.

We shall also need to provide the f_t with *gradient-like vector-fields*. These are smooth vector fields η_t which near the critical point of f_t are the actual gradient of f_t with respect to some Riemannian metric μ_t, and which give f_t a positive directional derivative away from the critical points. Such η_t always exist and are unique up to isotopy. For non-degenerate critical points, specifying a gradient-like vector field for f_t is tantamount to giving a handlebody decomposition of $M \times I$. In effect, the stable and unstable manifolds of a non-degenerate critical point of index i are the core disc D^i and transverse disc D^{n+1-i} of an i-handle $D^i \times D^{n+1-i}$, where $n = \dim M$. In the language of handlebody theory birth-death points correspond to cancelling or introducing a complementary pair of handles of dimension i and $i+1$.

The simplest possibly non-trivial k-parameter family f_t: $M \times I \to I$, $t \in I^k$, representing an element of $\pi_k(\mathcal{F},\ \mathcal{E})$ has its critical points arranged as follows: For t belonging to a disc D^k in the interior of I^k, f_t has exactly two critical points, which are non-degenerate of index i and $i+1$, and these two critical points coalesce into a birth-death point of f_t for $t \in \partial D^k$, so that f_t has no critical points for $t \in I^k - D^k$.

For example when $k = 1$, the path f_t consists of the introduction and subsequent cancellation of a pair of non-degenerate critical points of index i and $i + 1$. This situation can be visualized via the *graphic* of f_t, which is the set $\{(t, f_t) | f_t \text{ is a critical value}\} \subset I^k \times I$. In the case $k = 1$:



(Birth-death points appear as cusps).

One can now define the local obstruction to eliminating the critical points of the k-parameter family f_t. Let V_t^n be a level surface between the two critical points, $t \in D^k$. The intersection of the stable manifold of the critical point of f_t of index $i + 1$ (respectively, the unstable manifold of the critical point of index i) with V_t^n is a sphere S_t^i (respectively S_t^{n-i}). Taking the union over $t \in D^k$, one has $S^i \times D^k \subset V^n \times D^k$, $S^{n-i} \times D^k \subset V^n \times D^k$. In general position the intersection $N^k = S^i \times D^k \cap S^{n-i} \times D^k \subset V^n \times D^k$ will be a submanifold of dimension k.

2.1. Parametrized Smale lemma. — *The following are equivalent:*

(i) *The critical points of the family f_t can be cancelled directly, i.e. by a deformation which simply shrinks the graphic of f_t to zero.*

(ii) *The gradient-like vector field can be deformed so that N^k becomes a k-disc intersecting each slice V_t^n in one point transversely.*

(iii) *There exists a parameter-preserving isotopy of $S^i \times D^k \to V^n \times D^k$ fixed over ∂D^k which changes N^k to a k-disc intersecting each slice V_t^n in one point transversely.*

We now define a bordism invariant which in a stable range of dimensions gives the obstruction to finding an isotopy as in (iii). If $k+2 \leqslant i \leqslant n-k-2$, N^k has contractions in $S^i \times D^k$ and $S^{n-i} \times D^k$ which are unique up to homotopy. These two contractions give a map of the suspension $\Sigma N^k \to V^n \times D^k \to M \times I \times D^k \to M$. Adjoint to this is a map $N^k \to \Omega M$ into the loopspace of M. Moreover, the contractions of $N^k \to S^i \times D^k$ and $N^k \to S^{n-i} \times D^k$ induce a canonical framing of the stable normal bundle of N,

$$\nu_N \approx \nu(S^i \times D^k, V^n \times D^k)_{|N} \oplus \nu(S^{n-i} \times D^k, V^n \times D^k)_{|N} \oplus \nu_{V^n \times D^k|N}.$$

(for the framing of the last summand one must choose one of the two contractions, say that in $S^i \times D^k$).

If N^k were a closed manifold then the map $N^k \to \Omega M$ together with the framing of ν_N would define an element of the framed bordism group $\Omega_k^{fr}(\Omega M)$. However, since the two critical points of f_t cancel over ∂D^k one knows that $\partial N^k \subset V^n \times \partial D^k$ consists of one point in each t slice, i.e. $\partial N^k \approx \partial D^k = S^{k-1}$. Thus one is lead to consider a relative bordism group $(\Omega, \pi)_k^{fr}(\Omega M)$ whose elements are framed bordism classes of maps $N^k \to \Omega M$, where the bordisms are required to be trivial over $\partial N^k \approx S^{k-1}$. Strictly speaking, maps $N \to \Omega M$ should take a basepoint $* \in \partial N$ to the constant path $* \in \Omega M$. Addition in $(\Omega, \pi)_k^{fr}(\Omega M)$ is via boundary connected sum at $* \in \partial N$. (This is not quite well defined when $k=1$). The zero element is represented by a map $D^k \to \Omega M$.

By transversality the bordism class of $N \to \Omega M$ in $(\Omega, \pi)_k^{fr}(\Omega M)$ is an invariant of the isotopies described in (iii) above. The converse is a parametrized Whitney procedure (see [8]):

2.2. LEMMA. — *If the class* $[N \to \Omega M]$ *vanishes in* $(\Omega, \pi)_k^{fr}(\Omega M)$ *and* $k+2 < i < n-k-2$, *then condition* (iii) *above is satisfied. Moreover, if* $k > 0$ *and*

$$k+2 \leqslant i \leqslant n-k-2$$

then all elements of $(\Omega, \pi)_k^{fr}(\Omega M)$ *are realizable as such invariants for some k-parameter family* f_t.

When $k=1$ this lemma was proved in [5] in a slightly different form.

The notation $(\Omega, \pi)_k^{fr}(\Omega M)$ is chosen because one has a natural long exact sequence

$$(*) \quad \cdots \to \pi_k^{fr}(\Omega M) \to \Omega_k^{fr}(\Omega M) \to (\Omega, \pi)_k^{fr}(\Omega M) \xrightarrow{\partial} \pi_{k-1}^{fr}(\Omega M) \to \cdots$$

where the « framed homotopy group » $\pi_k^{fr}(\Omega M)$ is just $\pi_k(\Omega M) \oplus \pi_k 0$.

In the classical case $k = 0$ it is easy to see that $(\Omega, \pi)_0^{fr}(\Omega M) = \Omega_0^{fr}(\Omega M)$ is isomorphic to the integral group ring $\mathbf{Z}[\pi_1 M]$, whose role in geometric topology is well known. When $k = 1$ the sequence $(*)$ becomes:

$$\begin{array}{ccccccccc} 0 \to & \pi_1^{fr}(\Omega M) & \to & \Omega_1^{fr}(\Omega M) & \to & (\Omega, \pi)_1^{fr}(\Omega M) & \to 0 \\ & \wr\wr & & \wr\wr & & \wr\wr & \\ 0 \to & (\mathbf{Z}_2 \times \pi_2 M)[1] & \to & (\mathbf{Z}_2 \times \pi_2 M)[\pi_1 M] & \to & \dfrac{(\mathbf{Z}_2 \times \pi_2 M)[\pi_1 M]}{(\mathbf{Z}_2 \times \pi_2 M)[1]} & \to 0 \end{array}$$

One has in fact

$$\Omega_k^{fr}(\Omega M) \approx \Omega_k^{fr}(\Omega \tilde{M})[\pi_1 M] \approx (\Omega_k^{fr}(*) \times \tilde{\Omega}_k^{fr}(\Omega \tilde{M}))[\pi_1 M]$$

for any k, where $\tilde{M}$ is the universal cover of M (so that $\Omega\tilde{M}$ can be identified with the identity component of ΩM) and $\tilde{\Omega}_k^{fr}$ is the reduced bordism group. For $k = 1$, $\Omega_1^{fr}(*) \approx \mathbf{Z}_2$ and $\tilde{\Omega}_1^{fr}(\Omega M) \approx \pi_1(\Omega M) \approx \pi_2 M$.

The sequence $(*)$ contains as a direct summand the sequence

$$\begin{array}{ccccccccc} \cdots \to & \pi_k^{fr}(*) & \xrightarrow{J_k} & \Omega_k^{fr}(*) & \to & (\Omega, \pi)_k^{fr}(*) & \xrightarrow{\partial} & \pi_{k-1}^{fr}(*) & \to \cdots \\ & \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & \\ \cdots \to & \pi_k 0 & \xrightarrow{J_k} & \pi_k^s & \to & \operatorname{cok}(J_k) \oplus \ker(J_{k-1}) & \to & \pi_{k-1} 0 & \to \cdots \end{array}$$

Here J_k is the classical J homomorphism to the k^{th} stable homotopy group of spheres π_k^s. For most k, J_k is not an isomorphism (e.g., $k = 2$), so lemma 2.2 gives many candidates for non-zero elements of $\pi_k(\mathscr{F}, \mathscr{E})$ on an arbitrary n-dimensional manifold M, $n \geqslant 2k + 4$, for example on D^n or S^n. These local obstructions were first identified in [4].

To prove that the k-parameter families with non-vanishing local invariant in $(\Omega, \pi)_k^{fr}(\Omega M)$ really give non-zero elements of $\pi_k(\mathscr{F}, \mathscr{E})$ one must somehow show that quite general deformations of the family f_t (fixed over ∂I^k) do not destroy this invariant completely. This is a much harder problem, and in fact I only know how to do this when $k = 1$. The procedure will be outlined in the following two sections.

3. Algebra.

To define $K_1(\mathbf{Z}_2 \times \pi_2)[\pi_1]$, of which $Wh_1(\pi_1; \mathbf{Z}_2 \times \pi_2)$ is a quotient, we need $(\mathbf{Z}_2 \times \pi_2)[\pi_1] \approx \Omega_1^{fr}(\Omega M)$ to be a ring with identity, or at least an ideal in such a ring. A natural way of doing this is via the sequence

$$0 \to \Omega_1^{fr}(\Omega M) \to \Omega_1^{fr}(\Omega M) \times \Omega_0^{fr}(\Omega M) \to \Omega_0^{fr}(\Omega M) \to 0$$

where the multiplication on the middle group is given by the graded ring structure of $\Omega_*^{fr}(\Omega M)$ (induced by the natural H-space structure on ΩM), truncated above dimension 1. This amounts to giving $\Omega_1^{fr}(\Omega M)$ trivial multiplication (all products zero) and letting $\sigma \in \pi_1 M \subset \mathbf{Z}[\pi_1 M] \approx \Omega_0^{fr}(\Omega M)$ act on $\alpha\tau \in (\mathbf{Z}_2 \times \pi_2 M)[\pi_1 M] \approx \Omega_1^{fr}(\Omega M)$ via $\sigma(\alpha\tau) = \alpha^\sigma \sigma\tau$ and $(\alpha\tau)\sigma = \alpha\tau\sigma$.

Recall the definition of K_1 of an ideal $\mathcal{A}$ in a ring $\mathcal{R}$ with identity [10] : Let

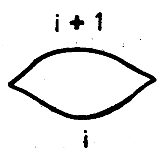
$$GL(\mathcal{A}) = \ker (GL(\mathcal{R}) \to GL(\mathcal{R}/\mathcal{A})) = \{I + A \in GL(\mathcal{R}) | A$$

has entries in $\mathcal{A}\}$ and let $E(\mathcal{A})$ be the mixed commutator subgroup $[GL(\mathcal{A}), GL(\mathcal{R})]$. Then $K_1\mathcal{A} = GL(\mathcal{A})/E(\mathcal{A})$.

3.1. Proposition. — *If* $\mathcal{A}^2 = 0$ *then* $K_1\mathcal{A} \approx \mathcal{A}/(\text{ra-ar})$ *via* $[I + A] \longmapsto \text{trace}\,(A)$.

Here (ra-ar) denotes the additive subgroup of $\mathcal{A}$ generated by elements ra-ar for $a \in \mathcal{A}$, $r \in \mathcal{R}$.

Now define $Wh_1(\pi_1; \mathbf{Z}_2 \times \pi_2)$ as $K_1(\mathbf{Z}_2 \times \pi_2)[\pi_1]$ modulo matrices of the form $I + (\beta . 1)$ for $\beta \in \mathbf{Z}_2 \times \pi_2$. Passing to this quotient corresponds to passing from $\Omega_1^{fr}(\Omega M)$ to $(\Omega, \pi)_1^{fr}(\Omega M)$, which is where our local obstruction for graphics



lies.

With this definition Proposition 1.2 is then a corollary of 3.1.

4. Geometry.

To define a homomorphism θ :

$$\pi_1(\mathscr{F}, \mathscr{E}) \to Wh_1(\pi_1 M; \mathbf{Z}_2 \times \pi_2 M)$$

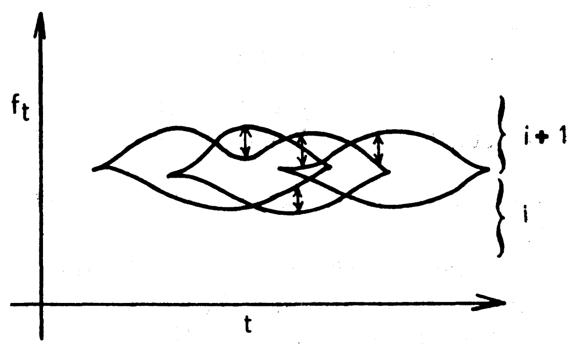
we use the fact that when $\dim M$ is large enough (e.g., $\geqslant 4$) any element of $\pi_1(\mathscr{F}, \mathscr{E})$ can be represented by a path f_t: $M \times I \to I$ with the following properties:

(i) As a one-parameter family f_t is generic, i.e., its singular set consists of arcs of non-degenerate critical points and isolated birth-death points.

(ii) For some fixed i, the non-degenerate critical points of each f_t are either of index i or $i + 1$.

(iii) The critical points of index i have f_t values which are strictly less than the f_t values of critical points of index $i + 1$.

Moreover f_t has a gradient-like vector field which we can assume to be in general position, i.e., stable and unstable manifolds intersect transversely (as one-parameter families). Thus one might have a graphic like



The vertical arrows denote the isolated trajectories connecting critical points of the same index. In handlebody theory these are usually called handle additions.

For technical reasons we need also the following condition to be satisfied:

(+) Birth-death points are independent in the sense that there are no trajectories of the vector field connecting these critical points with other critical points.

In other words complementary pairs of handles are introduced or cancelled disjointly from other handles.

The condition $(+)$ is easily achieved by a deformation of the vector field, at the expense of introducing more handle additions. One consequence of $(+)$ is that, as far as intersections of stable and unstable manifolds are concerned, the situation is as if all birth points occured near $t = 0$ all death points near $t = 1$.

If one supposes that there are no handle additions, so that all stable manifolds of index $i + 1$ critical points and all unstable manifolds of index i critical points run uninterrupted to a single intermediate leval surface V_t^n, then the intersections of these stable and unstable manifolds for the various arcs of non-degenerate critical points determine as in § 2 a whole matrix A with entries in $\Omega_1^{fr}(\Omega M)$. In this case one sets $\theta[f_t] = (-1)^i[I + A] \in Wh_1(\pi_1 M; \mathbf{Z}_2 \times \pi_2 M)$.

The existence of such families having no handle additions is intimately related to the Wh_2 invariant, according to one of the main results of [9]:

4.1. Proposition. — *When $2 \leqslant i \leqslant n - 2$ the family f_t can be deformed, preserving $(+)$, so as to eliminate all its handle additions if and only if an obstruction in $Wh_2(\pi_1 M)$ vanishes.*

Thus the natural domain of definition of the $Wh_1(\pi_1 M; \mathbf{Z}_2 \times \pi_2 M)$ invariant is the kernel of the Wh_2 invariant. In this sense the $Wh_1(\pi_1 M; \mathbf{Z}_2 \times \pi_2 M)$ obstruction is a secondary obstruction. It is also secondary with respect to indeterminacy. For the natural way to deform a family f_t having no handle additions is to pass through a two-parameter family consisting of a circle of handle additions, or more generally, of a number of concentric circles of handle additions. This corresponds exactly to multiplying $I + A$ by an element of the kernel of $GL((\mathbf{Z}_2 \times \pi_2 M)[\pi, M]) \to K_1(\mathbf{Z}_2 \times \pi_2)[\pi_1 M]$. *A priori* the indeterminacy of $I + A$ might be much greater than this since one can alter f_t by deformations much more violent than just traversing circles of handle additions.

The difficulty then is to show that the $Wh_1(\pi_1 M; \mathbf{Z}_2 \times \pi_2 M)$ obstruction is actually a primary obstruction, i.e., that it is universally defined and has no further indeterminacy. This is done as follows: Subdivide the t interval $I = [0, 1]$ into a finite number of subintervals, with subdivisions occuring at least in the t slices containing birth-death points or handle additions. Then in a t slice in the j^{th} t interval one has, after a suitable choice of bases, an $i + 1/i$ intersection matrix $M_j \in GL(\mathbf{Z}[\pi_1 M])$, just as in the zero-parameter h-cobordism theorem (where the image of this matrix in $Wh_1(\pi_1 M)$ is the torsion of the h-cobordism). Furthermore, by a rather complicated normalization procedure at handle additions, which involves a number of choices, one can define a matrix A_j over $(\mathbf{Z}_2 \times \pi_2 M)[\pi_1 M]$ which measures the one-dimensional intersections of stable and unstable manifolds in a level surface, even though these intersections do not form a manifold but only a one-dimensional complex, due to singularities at the handle addition points. The general definition of θ is then $\theta = (-1)^i\left[I + \sum_j M_j^{-1} A_j\right] \in Wh_1(\pi_1 M;\ \mathbf{Z}_2 \times \pi_2 M)$.

To show that θ is well defined one must show that it does not depend on the various choices which were made in its definition. In particular one must show that a deformation of f_t through a second parameter, fixing f_0 and f_1, to another such f_t involving only the indices i and $i+1$ does not change θ. Since we only define θ in the two-index case, we must use the rather difficult geometric result that a two-parameter family can also be deformed to push all its critical points into two indices, at least « stably ». Then one examines a small list of « catastrophes » which happen to f_t during this deformation and one sees that they all preserve θ. The details of this argument will appear in [7].

5. Questions.

The cases $k = 0$ and $k = 1$ lead one to ask whether in general $\pi_k(\mathscr{F}, \mathscr{E}) \approx \sum_{j=0}^{k}$ « $Wh_{k-j+1}(\Omega_j^{fr}(\Omega M))$ », at least for $k \ll \dim M$. There seem to be several obstacles to proving this. The first and most formidable is the problem of describing the global behavior of singularities of finite codimension. Ideally, one would like to prove, at least in a stable range, that k-parameter families can always be deformed into two indices i

and $i+1$. More precisely, let $\mathcal{G}_i \subset \mathcal{F}$ be the interior of the closure of the set of morse functions having critical points only of index i and $i+1$.

Conjecture. — For some function $\varphi : \mathbf{N} \to \mathbf{N}$, $\pi_k(\mathcal{G}_i) = 0$ whenever $\varphi(k) \leqslant i \leqslant \dim M - \varphi(k)$.

The conjecture is true for $k \leqslant 1$ [9].

This conjecture would imply that one need only consider the « nicest » singularities of codimension k, namely those with a local representation of the form

$$\sum_{j=1}^{n} \pm x_j^2 \pm x_{n+1}^{k+2}$$

Then presumably it would be just a matter of book-keeping to define groups $Wh_{k-j+1}(\Omega_j^{fr}(\Omega M))$ in which the $(j+1)^{st}$ order obstruction would lie. There would remain only the question of deciding if these higher order obstructions were in fact primary, i.e. of computing differentials in a certain spectral sequence.

One would also like to know the relationship between these hypothetical higher Whitehead groups and higher algebraic K groups. I expect there to be only a map $K_n \to Wh_n$, corresponding geometrically to passing from families of morse functions to general families of functions, which will not necessarily be surjective as it is when $n \leqslant 2$.

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