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ON SIGNATURES ASSOCIATED WITH RAMIFIED COVERINGS AND EMBEDDING PROBLEMS (1)

by John WOOD and Emery THOMAS

This note describes a way of using signatures associated with a ramified covering to study codimension two embeddings. Further results and more details will appear in [12]. The idea was first used by W. Massey to prove Whitney's conjecture about the possible normal bundles of a nonorientable surface smoothly embedded in $\mathcal{R}^4$, [9]. Beginning with $K^2 \subset S^4$, Massey constructed a ramified double cover $N^4 \to S^4$ branched along K and with covering transformation g. The signature of g provides a lower bound for the Betti number $\beta_2(N)$ in terms of the Euler class of the normal bundle of K in S^4. On the other hand $\beta_2(N) = 1 + \beta_2(K)$, so for fixed K a bound is obtained on the possible normal Euler classes.

A similar idea was applied by Hsiang and Szczarba [4] and by Rokhlin [10] to study smooth embeddings $i : K^2 \hookrightarrow M^4$ of oriented manifolds. In general, given a submanifold $i : K^{4n-2} \hookrightarrow M^{4n}$, K and M oriented, with $i_*[K] = y \cap [M]$ for some class $y \in H^2(M; Z)$, we will say that K is dual to y. By a fundamental result of Thom, given $y \in H^2(M; Z)$ there always exists a submanifold dual to y.

Suppose now that K is dual to dx, for some class $x \in H^2(M; Z)$ and integer $d \geqslant 2$. Then, there is a d-fold ramified cover $p : N \to M$ branched along K, see [3, § 6].

(1) Lecture given by J. Wood.

More precisely there is a $\mathbf{Z}/d\mathbf{Z}$ action on N, with quotient M and fixed set K, generated by $g\colon N \to N$, and such that g acts on the normal bundle of K in N by rotation in the fibre by $2\pi/d$.

The map g induces an action on $H = H^{2n}(N; Q) \otimes \mathbf{C}$ by isometries of the hermitian inner product defined by $\langle x \otimes \alpha, y \otimes \beta\rangle = \alpha\bar{\beta}x \cup y[N]$. Thus, if $H(l)$ denotes the eigenspace of g^* with eigenvalue $e^{2\pi i l/d}$, then H is an orthogonal direct sum $\bigoplus_{l=0}^{d-1} H(l)$. The signatures of the iterates of g can be computed using the G-signature theorem of Atiyah and Singer [1; 6.12] and these yield formulas for the signatures of the spaces $H(l)$. Rokhlin computes Sign $H(l) = \text{Sign } M - 2l(d-l)x^2[M]$ in the 4-dimensional case.

In higher dimensions one has (see [12])

Theorem 1. — *For any* $x \in H^2(M^{4n}; Z)$ *and integers* $0 \leqslant l < d$,

$$\text{Sign } H(l) = \{\cosh (d-2l)x \operatorname{sech} dx\, L(M)\}[M].$$

Here $L(M) = 1 + L_1(M) + \cdots \in H^{4*}(M; \mathbf{Q})$ is the Hirzebruch L-genus.

This result can be regarded as an integrality theorem and it has the following

Corollary. — *If* $L(M) = 1$, *for example if* M *is a* π*-manifold, then* $(2x)^{2n}[M] = 0 \bmod (2n)\,!$

Several important simplifications occur in the proof of Theorem 1 for the case $d = 2$, of a ramified double cover. We prove Theorem 1 in this case in the second half of this note.

To apply Theorem 1 to embedding problems we need to relate $\dim H(l)$ with K and M. The subspace $H(0)$ is the set of vectors fixed by g^* which, by a fundamental result on finite group actions, is the injective image

$$p^* H^{2n}(M; \mathbf{C}).$$

For $0 < l < d$, Rokhlin showed that $\dim H(l)$ is independent of l and hence

$$\dim H(l) = \frac{1}{d-1}\,(\beta_{2n}(N) - \beta_{2n}(M)).$$

(For d prime this follows from the fact that the eigenvalues are conjugate over $\mathbf{Q}$ and the map g^* is defined over $\mathbf{Q}$. In general one considers a family of coverings corresponding to divisors of d.)

In Rohlin's case, with the hypothesis $H_1(M) = 0$ and with d a prime power or $\pi_1(M - K)$ abelian,

$$\dim H(l) = \beta_1(K) + \beta_2(M).$$

Thus his result is

$$\beta_1(K) \geqslant |2l(d-l)x^2[M] - \operatorname{Sign} M| - \beta_2(M).$$

The strongest inequality is obtained for $l = [d/2]$, the greatest integer in $d/2$. Applied to $M = \mathbf{CP}_2$ this shows that the only classes representable by the 2-sphere are 0, $\pm x$, $\pm 2x$. (The $\pm$ sign comes from choice of orientation of S^2.)

Nonsingular algebraic hypersurfaces in $\mathbf{CP}_m$ provide natural examples of codimension 2 embeddings. If $K \subset \mathbf{CP}_m$ is given by a polynomial of degree d then K is dual to dx where $x \in H^2(\mathbf{CP}_m)$ is the canonical generator. In the case of $\mathbf{CP}_2$ Thom has conjectured that the algebraic surface should have minimal genus among all smoothly embedded, orientable surfaces dual to dx. (Recall that Kervaire and Milnor showed that any dx is dual to a combinatorially embedded S^2, [8].) For an algebraic $K^2 \subset \mathbf{CP}_2$,

$$\beta_1(K) = (d-1)(d-2).$$

Rokhlin's result confirms the conjecture for $d \leqslant 4$ but in general the best lower bound for $\beta_1(K)$ obtained this way grows as $d^2/2$. Can these results be improved by a finer study of the representation of $\mathbf{Z}/d\mathbf{Z}$ on $H^2(N)$?

For higher dimensions we need a natural class of embeddings in which $\beta_{2n}(N)$ can be computed from M^{4n} and K^{4n-2}. We say K is *taut* in M^{2m} if $(C, \partial C)$ is $(m-1)$-connected where C is the closed complement of a tubular neighbourhood of K in M. A strengthened version of the Lefschetz theorem on hyperplane sections shows that the above algebraic examples are taut in $\mathbf{CP}_m$. Kato and Matsumoto [7] have recently shown that any class in $H^2(M; \mathbf{Z})$ is dual to a taut submanifold $K \subset M$ and further that the homology of K is completely determined by $H_*(M; \mathbf{Z})$ and the middle Betti number $\beta_{m-1}(K)$. Thus the notion of tautness carries much of the topological character of the algebraic embeddings.

We remark that two algebraic hypersurfaces dual to the same class are diffeomorphic. Are two taut submanifolds dual to the same class and with the same middle Betti number homeomorphic, at least if $\pi_1(M) = 0$? This is of course true when $\dim K = 2$, and also, by results of Wall [10] and Jupp [6], for $\dim K = 6$ when $\pi_1(M) = 0$ and $H_2(M; \mathbf{Z})$ is torsion free.

It is always possible to add homologically trivial handles in the middle dimension to a taut submanifold preserving tautness. Thus, if $\mu(\xi)$ denotes the minimal $\beta_{m-1}(K)$ among taut K dual to ξ, with $\xi \in H^2(M^{4n}; \mathbf{Z})$, our basic question is: *how does* $\mu(\xi)$ *depend on* ξ?

If $N \to M$ is ramified over a taut K, then for $0 < l < d$, $\dim H(l) = \beta_{m-1}(K) - 2\beta_{m-1}(M) + \beta_m(M)$. Thus a lower bound for μ is given by

THEOREM 2. — *If* $K^{4n-2} \subset M^{4n}$ *is a taut embedding dual to* dx *and* $0 < l < d$, *then*

$$\beta_{2n-1}(K) \geqslant |\operatorname{Sign} H(l)| + 2\beta_{2n-1}(M) - \beta_{2n}(M).$$

This result, for d even and $l = d/2$, can be obtained from the special case of a ramified double cover treated below. In fact, by Theorem 1,

$$\operatorname{Sign} H(d/2) = \{\operatorname{sech} dx\, L(M)\}[M] = \{\operatorname{sech} 2x' L(M)\}[M]$$
when $x' = (d/2)x$.

For an example, the inequality of Theorem 2 can be compared with the nonsingular algebraic hypersurfaces in the case $M = \mathbf{C}P_{2n}$. This gives the inequalities:

$$\beta_{2n-1}(K_{\text{algebraic}}) = d^{-1}(d-1)[(d-1)^{2n} - 1] \geqslant \mu(dx)$$
$$\geqslant \frac{E_n}{(2n)!} d^{2n} + (\text{terms of lower order in } d).$$

Here E_n is the n-th Euler number which can be defined by the series $\operatorname{sech} x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} E_n x^{2n}$. Thus as a function of d, $\mu(dx)$ must grow as a polynomial of the same degree, $2n$, as the polynomial giving $\beta_{2n-1}(K)$ for K an algebraic hypersurface.

We conclude by deriving formulas for the signatures associated with a ramified double cover. In that case we have a manifold N^{4n} with an orientation preserving involution g with quotient space M and fixed set a codimension 2 submanifold K dual to an even class $2x \in H^2(M; \mathbf{Z})$. The map g induces an isometry g^* of $H = H^{2n}(N; \mathbf{Q})$ with respect to the inner product $(x, y) = x \cup y[N]$ which is symmetric because $2n$ is even and nondegenerate by Poincaré duality. The eigenspaces, $H(j) = \{v : g^*v = (-1)^j v\}$ for $j = 0, 1$, of g^* give an orthogonal direct sum decomposition of

$$H = H(0) \otimes H(1).$$

Also, the projection $p : N \to M$ induces an isometry

$$p^* : H^{2n}(M; \mathbf{Q}) \approxeq H(0).$$

Hence Sign $H(0)$ = Sign M and

$$(*) \qquad \begin{aligned} \text{Sign } g &= \text{Sign } (g|H(0)) + \text{Sign } (g|H(1)) \\ &= \text{Sign M} - \text{Sign H}(1) \\ \text{Sign N} &= \text{Sign M} + \text{Sign H}(1). \end{aligned}$$

We wish to express these signatures in terms of M and the class x.

THEOREM 3. — *For a ramified double cover as above we have*:

$$\begin{aligned} \text{Sign } g &= \{\tanh x \tanh 2x \, L(M)\}[M], \\ \text{Sign H}(1) &= \{\operatorname{sech} 2x \, L(M)\}[M], \end{aligned}$$

and
$$\text{Sign N} = \{(1 + \operatorname{sech} 2x)L(M)\}[M].$$

These expressions are easily seen to be equivalent using the equations $(*)$, the Hirzebruch signature formula,

$$\text{Sign M} = L(M)[M],$$

and the identity $1 - \tanh x \tanh 2x = \operatorname{sech} 2x$.

Let $K \circ K$ denote the self-intersection of K in N which can be defined by approximating the embedding $K \subset N$ by a map $s : K \hookrightarrow N$ transverse regular to K and setting

$K \circ K = s^{-1}(K)$ [1, *p.* 583]. The oriented bordism class represented by the inclusion $j: K \circ K \hookrightarrow K$ is well-defined and the formula $\text{Sign } g = \text{Sign } K \circ K$ is a consequence of the Atiyah-Singer G-signature theorem [1, 6.15] which has also been given an independent, elementary proof by Jänich and Ossa [5].

The method of [2, § 9] can be used to express $\text{Sign } K \circ K$ in terms of M an the class x. The map s above can be thought of as a section of the normal bundle $\nu_N(K)$ of K in N whose Euler class is i^*x. Hence $j_*[K \circ K] = i^*x \cap [K]$ and $\nu_K(K \circ K) = j^*\nu_N(K)$. The bundle equation

$$\tau(K \circ K) \oplus j^*\nu_N(K) = j^*\tau(K)$$

together with the multiplicative property of the L-genus and its naturality with respect to bundle maps imply

$$L(K \circ K) = j^*(L(\nu_N(K))^{-1} . L(K)).$$

Since $\nu_N(K)$ is an SO(2)-bundle with Euler class i^*x, $L(\nu_N(K))^{-1} = i^*(\tanh x/x)$ and

$$\begin{aligned} \text{Sign } K \circ K &= j^*\{i^*(\tanh x/x)L(K)\}[K \circ K] \\ &= \{i^*(\tanh x)L(K)\}[K], \quad [2;\ \S\ 9(3)]. \end{aligned}$$

Similarly $\tau K \oplus \nu_M(K) = i^*\tau M$ where the Euler class of $\nu_M(K)$ is $2i^*x$. This implies $L(K) = i^*((\tanh 2x/2x)L(M))$ and hence $\text{Sign } K \circ K = \{\tanh x \tanh 2x L(M)\}[M]$, completing the proof of Theorem 3.

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