SOME EXAMPLES ON QUASI-BARRELLED SPACES (1)

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J. Dieudonné has proved in [2] the following theorem:

a) Let $E$ be a bornological space. If $F$ is a subspace of $E$, of finite codimension, then $F$ is bornological.

We have given in [6] and [7], respectively, the following results:

b) Let $E$ be a quasi-barrelled space. If $F$ is a subspace of $E$, of finite codimension, then $F$ is quasi-barrelled.

c) Let $E$ be an ultrabornological space. If $F$ is a subspace of $E$, of infinite countable codimension, then $F$ is bornological.

The results a), b) and c) lead to the question if the results a) and b) will be true in the case of being $F$ a subspace of infinite countable codimension. In this paper we give an example of a bornological space $E$, which has a subspace $F$, of infinite countable codimension, such that $F$ is not quasi-barrelled.

In [8] we have proved the two following theorems:

d) Let $E$ be a $\mathcal{DF}$-space. If $G$ is a subspace of $E$, of finite codimension, then $G$ is a $\mathcal{DF}$-space.

e) Let $E$ be a sequentially complete $\mathcal{DF}$-space. If $G$ is a subspace of $E$, of infinite countable codimension, then $G$ is a $\mathcal{DF}$-space.

Another question is if the result d) is also true for subspaces of infinite countable codimension. Here we give an example of a quasi-barrelled $\mathcal{DF}$-space, which has a subspace $G$, of infinite countable codimension, which is not a $\mathcal{DF}$-space.

(1) Supported in part by the « Patronato para el Fomento de la Investigación en la Universidad ». 
N. Bourbaki, [1, p. 35], notices that it is not known if every bornological barrelled space is ultrabornological. In [9] we have proved that if $E$ is the topological product of an infinite family of bornological barrelled space, of non-zero dimension, there exists an infinite number of bornological barrelled subspaces of $E$, which are not ultrabornological. In this paper we give an example of a bornological barrelled space, which is not inductive limit of Baire spaces.

We use here vector spaces on the field $K$ of real or complex numbers. The topologies on these spaces are separated.

In [10] we have proved the following result:

\[ f) \text{Let } E \text{ be a barrelled space. If } \{E_n\}_{n=1}^\infty \text{ is an increasing sequence of subspaces of } E, \text{ such that } \bigcup_{n=1}^\infty E_n = E, \text{ then } E \text{ is the inductive limit of } \{E_n\}_{n=1}^\infty. \]

**Theorem 1.** — Let $E$ be the strict inductive limit of an increasing sequence $\{E_n\}_{n=1}^\infty$ of metrizable locally convex spaces. Let $F$ be a sequentially dense subspace of $E$. If $E$ is barrelled, then $F$ is bornological.

**Proof.** — Let $E_n$, $n = 1, 2, \ldots$, be the closure of $E_n$ in $E$. Obviously $E$ is the strict inductive limit of the sequence $\{E_n\}_{n=1}^\infty$. Let $F_n$ be the closure in $E$ of $F \cap E_n$, $n = 1, 2, \ldots$. If $x \in E$ there exists a sequence $\{x_n\}_{n=1}^\infty$ of points of $F$, which converges to $x$. Since the set of points of this sequence is bounded, there exists a positive integer $n_0$ such that $x_n \in E_{n_0}$, $n = 1, 2, \ldots$, and, therefore, $x \in F_{n_0}$. Hence $E = \bigcup_{n=1}^\infty F_n$. Since $E$ is barrelled, applying the result $f)$, we obtain that $E$ is the strict inductive limit of the sequence $\{F_n\}_{n=1}^\infty$.

Given any Banach space $L$ and a linear and locally bounded mapping $u$ from $F$ into $L$, we must to prove that $u$ is continuous. Let $u_n$ be the restriction of $u$ to $F \cap E_n$. Since $F \cap E_n$ is a metrizable space and $u_n$ is locally bounded, $u_n$ is continuous. Let $v_n$ be the continuous extension of $u_n$ to $F_n$. Let $v$ be the linear mapping from $E$ into $L$, which coincides with $v_n$ in $F_n$, $n = 1, 2, \ldots$. Since $v_n$ is
equal to $v_{n+1}$ on $F \cap \overline{E}_n$, then they are equal on $F_n$ and, therefore, $v$ is well defined. Since $E$ is the inductive limit of $\{F_n\}_{n=1}^{\infty}$ and since the restriction of $v$ to $F_n$ is continuous, $n = 1, 2, \ldots$, then $v$ is continuous. On other hand $u$ is the restriction of $v$ to $F$ and, therefore, $u$ is continuous. Q.E.D.

**Example 1.** — A. Grothendieck, [3], has given an example of a space $E$, which is strict inductive limit of an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of separable Frechet spaces, so that there exists in $E$ a non-closed subspace $G$, such that $G \cap E_n$ is closed, $n = 1, 2, \ldots$. In this example let $A_n$ be a countable set of $E_n$, dense in $E_n$. Let $P$ be the linear space generated by $\bigcup_{n=1}^{\infty} A_n$. Let $F$ be the linear hull of $P \cup G$. Since $P$ is sequentially dense in $E$, applying Theorem 1, it results that $F$ is a bornological space. Applying theorem $f)$ it results that $G$ is not barrelled and since $G$ is quasi-complete, then $G$ is not quasi-barrelled. Since $P$ has a countable basis, $G$ is a subspace of $F$, of countable codimension, and by $a)$ the codimension of $G$ is infinite. Therefore, $F$ is a bornological space, which has a subspace $G$, of infinite countable codimension, so that $G$ is not quasi-barrelled.

**Example 2.** — G. Kothe, [4, p. 433-434] gives an example of a Montel $\mathcal{DF}$-space, which has a closed subspace $L$, which is not a $\mathcal{DF}$-space. In this example, let $\{B_n\}_{n=1}^{\infty}$ be a fundamental sequence of bounded sets. Since $E$ is a Montel $\mathcal{DF}$-space, then $B_n$ is separable, $n = 1, 2, \ldots$. Let $A_n$ be a countable subset of $B_n$, dense in $B_n$, $n = 1, 2, \ldots$. Let $Q$ be the linear space generated by $\bigcup_{n=1}^{\infty} A_n$. Let $M$ be the linear hull of $Q \cup L$. Now, we shall prove that $M$ is quasi-barrelled. Indeed, given a closed, absolutely convex and bornivorous set $U$ in $M$, let $\overline{U}$ be its closure in $E$. If $x \in E$, there exists a positive integer $n_0$, such that $x \in B_{n_0}$ and, therefore, $x$ is in the closure of $A_{n_0}$. Hence, there exists a $\lambda \in K, \lambda > 0$, such that $x$ is in the closure of $A_{n_0}$. Therefore, $x \in \overline{U}$, i.e. $\overline{U}$ is a barrel in $E$, and therefore,
U = U ∩ M is a neighborhood of the origin in M. Since Q has a countable basis, L is a subspace of M, of countable codimension, and by d), the codimension of L is infinite. The space M is, therefore, an example of quasi-barrelled D-barrelled space which has a subspace L, of infinite countable codimension, so that L is not a D-barrelled space.

We say that a subspace E of F is locally dense if, for every x ∈ F, there exists a sequence \{x_n\}_{n=1}^\infty of points of E, which converges to x in the Mackey sense. In [9] we have proved the following result:

g) Let F be a locally convex space. If E is a bornological locally dense subspace of F, then F is bornological.

Theorem 2. — Let E be a bornological barrelled space which has a family \{E_n\}_{n=1}^\infty of subspaces, which satisfy the following conditions:

I. \bigcup_{n=1}^\infty E_n = E.

II. For every positive integer n, there exists a topology \mathcal{V}_n on E_n, finer than the initial one, so that E_n[\mathcal{V}_n] is a Fréchet space.

III. There exists in E a bounded set A, such that A ⊆ E_n, n = 1, 2, ...

Then there exists a bornological barrelled space F, which is not inductive limit of Baire spaces, so that E is a hyperplane of F.

Proof. — Let B be the closed, absolutely convex hull of A and let u be the canonical injection of E_B in E. If E_B is a Banach space, there exists, according to a theorem of Grothendieck, [4] or [5, p. 225], a positive integer n_1, such that u(E_B) = E_B ⊆ E_{n_1}, hence A ⊆ E_{n_1}, which is in contradiction with the condition III. We take in E_B a Cauchy sequence \{x_n\}_{n=1}^\infty which is not convergent. Let \hat{B} be the closure of B in the completion \hat{E} of E. Since the topology of the Banach space \hat{E}_B induces in E_B a topology coarser than the initial one, \{x_n\}_{n=1}^\infty converges in \hat{E}_B to an element
x. Since the set $M = \{x_1, x_2, \ldots\}$ is bounded in $E_B$, there exists a $\lambda \in K$, such that $M \subset \lambda B$ and, therefore, if $x \in E$, then $x \in \lambda B \subset E_B$, hence, according to a result of N. Bourbaki, [5, p. 210-211], $\{x_n\}_{n=1}^{\infty}$ converges to $x$ in $E_B$. This is a contradiction and, therefore, $x \notin E$. Let $F$ be the space generated by $E \cup \{x\}$, equipped with the topology induced by $\hat{E}$. Obviously $F$ is a barrelled space and, according to result $g)$, $F$ is bornological.

Finally we need to prove that $F$ is not inductive limit of Baire spaces. Suppose that there exists in $F$ a family $\{F_i: i \in I\}$ of subspaces, which union is $F$, so that for every $i \in I$, there exists a topology $\mathcal{U}_i$ on $F_i$, such that $F_i[\mathcal{U}_i]$ is a Baire space and $F$ is the locally convex hull of $\{F_i[\mathcal{U}_i]: i \in I\}$. Since $E$ is a dense hyperplane of $F$, there exists an index $i_0 \in I$, such that $E \cap F_{i_0}$ is a dense hyperplane of $F_{i_0}[\mathcal{U}_{i_0}]$. Let $G$ be the vector space $E \cap F_{i_0}$ with the topology induced by $\mathcal{U}_{i_0}$ and let $x_0$ be an element of $F_{i_0}$, which is not in $G$. If $\nu$ is the canonical injection of $G$ in $E$, $\nu$ is continuous. Let $G_n$ and $H_n$ be the spaces $G \cap \nu^{-1}(E_n)$ and that generated by $(G \cap \nu^{-1}(E_n)) \cup \{x_0\}$, respectively, equipped with the topologies induced by $\mathcal{U}_{i_0}$.

Obviously $F_{i_0} = \bigcup_{n=1}^{\infty} H_n$ and, therefore, there exists a positive integer $n_0$ such that $H_{n_0}$ is of the second category in $F_{i_0}[\mathcal{U}_{i_0}]$. If $\nu_{n_0}$ is the restriction of $\nu$ to $G_{n_0}$, the graph of $\nu_{n_0}$ is closed in $G_{n_0} \times E_{n_0}[\mathcal{U}_{n_0}]$ and, since $G_{n_0}$ is barrelled and $E_{n_0}[\mathcal{U}_{n_0}]$ is a Frechet space, $\nu_{n_0}$ is continuous from $G_{n_0}$ into $E_{n_0}[\mathcal{U}_{n_0}]$. If $\{y_m: m \in D\}$ is a net of elements of $G_{n_0}$, which converges to $y \in F_{i_0}[\mathcal{U}_{i_0}]$, then $\{\nu_{n_0}(y_m) = y_m: m \in D\}$ is a Cauchy net in the Frechet space $E_{n_0}[\mathcal{U}_{n_0}]$, which converges to $z$, hence $y = z$ and $G_{n_0}$ is closed in $F_{i_0}[\mathcal{U}_{i_0}]$. Also $H_{n_0}$ is closed in $F_{i_0}[\mathcal{U}_{i_0}]$ and since $H_{n_0}$ is of the second category in $F_{i_0}[\mathcal{U}_{i_0}]$, then $H_{n_0} = F_{i_0}[\mathcal{U}_{i_0}]$ and, therefore, $G_{n_0} = G$.

Finally, taking the net $\{y_m: m \in D\}$ converging to $x_0$, it results that $x_0 \in E$, which is not true. Hence $F$ is not inductive limit of Baire spaces. Q.E.D.

*Example 3.* — G. Kothe has given an example of a non-complete (LB)-space, which is defined by a sequence $\{E_n\}_{n=1}^{\infty}$
of Banach spaces, so that there exists a bounded set $A$ in $E$, which is not subset of $E_n$, $n = 1, 2, \ldots$. This example, and our Theorem 2, assure the existence of bornological barrelled spaces which are not inductive limits of Baire spaces.

BIBLIOGRAPHY


Manuscrit reçu le 22 juin 1971.
accepté par J. Dieudonné

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