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# SOME EXAMPLES ON QUASI-BARRELLED SPACES [1]

**by Manuel VALDIVIA**

J. Dieudonné has proved in [2] the following theorem:

*a) Let* E *be a bornological space. If* F *is a subspace of* E, *of finite codimension, then* F *is bornological.*

We have given in [6] and [7], respectively, the following results:

*b) Let* E *be a quasi-barrelled space. If* F *is a subspace of* E, *of finite codimension, then* F *is quasi-barrelled.*

*c) Let* E *be an ultrabornological space. If* F *is a subspace of* E, *of infinite countable codimension, then* F *is bornological.*

The results *a*), *b*) and *c*) lead to the question if the results *a*) and *b*) will be true in the case of being F a subspace of infinite countable codimension. In this paper we give an example of a bornological space E, which has a subspace F, of infinite countable codimension, such that F is not quasi-barrelled.

In [8] we have proved the two following theorems:

*d) Let* E *be a* $\mathfrak{DF}$*-space. If* G *is a subspace of* E, *of finite codimension, then* G *is a* $\mathfrak{DF}$*-space.*

*e) Let* E *be a sequentially complete* $\mathfrak{DF}$*-space. If* G *is a subspace of* E, *of infinite countable codimension, then* G *is a* $\mathfrak{DF}$*-space.*

Another question is if the result *d*) is also true for subspaces of infinite countable codimension. Here we give an example of a quasi-barrelled $\mathfrak{DF}$-space, which has a subspace G, of infinite countable codimension, which is not a $\mathfrak{DF}$-space.

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N. Bourbaki, [1, p. 35], notices that it is not known if every bornological barrelled space is ultrabornological. In [9] we have proved that if E is the topological product of an infinite family of bornological barrelled space, of non-zero dimension, there exists an infinite number of bornological barrelled subspaces of E, which are not ultrabornological. In this paper we give an example of a bornological barrelled space, which is not inductive limit of Baire spaces.

We use here vector spaces on the field K of real or complex numbers. The topologies on these spaces are separated.

In [10] we have proved the following result:

*f) Let* E *be a barrelled space. If* $\{E_n\}_{n=1}^{\infty}$ *is an increasing sequence of subspaces of* E, *such that* $\bigcup_{n=1}^{\infty} E_n = E$, *then* E *is the inductive limit of* $\{E_n\}_{n=1}^{\infty}$.

THEOREM 1. — *Let* E *be the strict inductive limit of an increasing sequence* $\{E_n\}_{n=1}^{\infty}$ *of metrizable locally convex spaces. Let* F *be a sequentially dense subspace of* E. *If* E *is barrelled, then* F *is bornological.*

*Proof.* — Let $\overline{E}_n$, $n = 1, 2, \ldots$, be the closure of $E_n$ in E. Obviously E is the strict inductive limit of the sequence $\{\overline{E}_n\}_{n=1}^{\infty}$. Let $F_n$ be the closure in E of $F \cap \overline{E}_n$, $n = 1, 2, \ldots$ If $x \in E$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points of F, which converges to $x$. Since the set of points of this sequence is bounded, there exists a positive integer $n_0$ such that $x_n \in \overline{E}_{n_0}$, $n = 1, 2, \ldots$, and, therefore, $x \in F_{n_0}$. Hence $E = \bigcup_{n=1}^{\infty} F_n$. Since E is barrelled, aplying the result $f$), we obtain that E is the strict inductive limit of the sequence $\{F_n\}_{n=1}^{\infty}$.

Given any Banach space L and a linear and locally bounded mapping $u$ from F into L, we must to prove that $u$ is continuous. Let $u_n$ be the restriction of $u$ to $F \cap \overline{E}_n$. Since $F \cap \overline{E}_n$ is a metrizable space and $u_n$ is locally bounded, $u_n$ is continuous. Let $v_n$ be the continuous extension of $u_n$ to $F_n$. Let $v$ be the linear mapping from E into L, which coincides with $v_n$ in $F_n$, $n = 1, 2, \ldots$ Since $v_n$ is equal to $\varphi_{n+1}$ on $F \cap \overline{E}_n$, then they are equal on $F_n$ and, therefore, $\varphi$ is well defined. Since E is the inductive limit of $\{F_n\}_{n=1}^{\infty}$ and since the restriction of $\varphi$ to $F_n$ is continuous, $n = 1, 2, \ldots$, then $\varphi$ is continuous. On other hand $u$ is the restriction of $\varphi$ to F and, therefore, $u$ is continuous. Q.E.D.

*Example* 1. — A. Grothendieck, [3], has given an example of a space E, which is strict inductive limit of an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of separable Frechet spaces, so that there exists in E a non-closed subspace G, such that $G \cap E_n$ in closed, $n = 1, 2, \ldots$ In this example let $A_n$. be a countable set of $E_n$, dense in $E_n$. Let P be the linear space generated by $\bigcup_{n=1}^{\infty} A_n$. Let F be the linear hull of $P \cup G$. Since P is sequentially dense in E, applying Theorem 1, it results that F is a bornological space. Applying theorem *f*) it results that G is not barrelled and since G is quasi-complete, then G is not quasi-barrelled. Since P has a countable basis, G is a subspace of F, of countable codimension, and by *a*) the codimension of G is infinite. Therefore, F is a bornological space, which has a subspace G, of infinite countable codimension, so that G is not quasi-barrelled.

*Example* 2. — G. Kothe, [4, p. 433-434] gives an example of a Montel $\mathfrak{DF}$-space, which has a closed subspace L, which is not a $\mathfrak{DF}$-space. In this example, let $\{B_n\}_{n=1}^{\infty}$ be a fundamental sequence of bounded sets. Since E is a Montel $\mathfrak{DF}$-space, then $B_n$ is separable, $n = 1, 2, \ldots$ Let $A_n$ be a countable subset of $B_n$, dense in $B_n$, $n = 1, 2, \ldots$ Let Q be the linear space generated by $\bigcup_{n=1}^{\infty} A_n$. Let M be the linear hull of $Q \cup L$. Now, we shall prove that M is quasi-barrelled. Indeed, given a closed, absolutely convex and bornivorous set U in M, let $\overline{U}$ be its closure in E. If $x \in E$, there exists a positive integer $n_0$, such that $x \in B_{n_0}$ and, therefore, $x$ is in the closure of $A_{n_0}$. Hence, there exists a $\lambda \in K, \lambda > 0$, such that $\lambda \times \chi \in \overline{U}$, i.e. $\overline{U}$ is a barrel in E, and therefore,

$U = \overset{\circ}{U} \cap M$ is a neighborhood of the origin in M. Since Q has a countable basis, L is a subspace of M, of countable codimension, and by *d*), the codimension of L is infinite. The space M is, therefore, an example of quasi-barrelled $\mathfrak{DF}$-space which has a subspace L, of infinite countable codimension, so that L is not a $\mathfrak{DF}$-space.

We say that a subspace E of F is locally dense if, for every $x \in F$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points of E, which converges to $x$ in the Mackey sense. In [9] we have proved the following result:

*g) Let* F *be a locally convex space. If* E *is a bornological locally dense subspace of* F, *then* F *is bornological.*

THEOREM 2. — *Let* E *be a bornological barrelled space which has a family* $\{E_n\}_{n=1}^{\infty}$ *of subspaces, which satisfy the following conditions:*

I. $\bigcup_{n=1}^{\infty} E_n = E.$

II. *For every positive integer* $n$, *there exists a topology* $\mathfrak{T}_n$ *on* $E_n$, *finer than the initial one, so that* $E_n[\mathfrak{T}_n]$ *is a Frechet space.*

III. — *There exists in* E *a bounded set* A, *such that* $A \not\subset E_n$, $n = 1, 2, \ldots$

*Then there exists a bornological barrelled space* F, *which is not inductive limit of Baire spaces, so that* E *is a hyperplane of* F.

*Proof.* — Let B be the closed, absolutely convex hull of A and let $u$ be the canonical injection of $E_B$ in E. If $E_B$ is a Banach space, there exists, according to a theorem of Grothendieck, [4] or [5, p. 225], a positive integer $n_1$, such that $u(E_B) = E_B \subset E_{n_1}$, hence $A \subset E_{n_1}$, which is in contradiction with the condition III. We take in $E_B$ a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ which is not convergent. Let $\hat{B}$ be the closure of B in the completion $\hat{E}$ of E. Since the topology of the Banach space $\hat{E}_{\hat{B}}$ induces in $E_B$ a topology coarser than the initial one, $\{x_n\}_{n=1}^{\infty}$ converges in $\hat{E}_{\hat{B}}$ to an element

$x$. Since the set $M = \{x_1, x_2, \dots\}$ is bounded in $E_B$, there exists a $\lambda \in K$, such that $M \subset \lambda B$ and, therefore, if $x \in E$, then $x \in \lambda B \subset E_B$, hence, according to a result of N. Bourbaki, [5, p. 210-211], $\{x_n\}_{n=1}^{\infty}$ converges to $x$ in $E_B$. This is a contradiction and, therefore, $x \notin E$. Let F be the space generated by $E \cup \{x\}$, equipped with the topology induced by $\hat{E}$. Obviously F is a barrelled space and, according to result *g*), F is bornological.

Finally we need to prove that F is not inductive limit of Baire spaces. Suppose that there exists in F a family $\{F_i : i \in I\}$ of subspaces, which union is F, so that for every $i \in I$, there exists a topology $\mathfrak{U}_i$ on $F_i$, such that $F_i[\mathfrak{U}_i]$ is a Baire space and F is the locally convex hull of $\{F_i[\mathfrak{U}_i] : i \in I\}$. Since E is a dense hyperplane of F, there exists an index $i_0 \in I$, such that $E \cap F_{i_0}$ is a dense hyperplane of $F_{i_0}[\mathfrak{U}_{i_0}]$. Let G be the vector space $E \cap F_{i_0}$ with the topology induced by $\mathfrak{U}_{i_0}$ and let $x_0$ be an element of $F_{i_0}$, which is not in G. If $\varphi$ is the canonical injection of G in E, $\varphi$ is continuous. Let $G_n$ and $H_n$ be the spaces $G \cap \varphi^{-1}(E_n)$ and that generated by $(G \cap \varphi^{-1}(E_n)) \cup \{x_0\}$, respectively, equipped with the topologies induced by $\mathfrak{U}_{i_0}$. Obviously $F_{i_0} = \bigcup_{n=1}^{\infty} H_n$ and, therefore, there exists a positive integer $n_0$ such that $H_{n_0}$ is of the second category in $F_{i_0}[\mathfrak{U}_{i_0}]$. If $\varphi_{n_0}$ is the restriction of $\varphi$ to $G_{n_0}$, the graph of $\varphi_{n_0}$ is closed in $G_{n_0} \times E_{n_0}[\mathfrak{T}_{n_0}]$ and, since $G_{n_0}$ is barrelled and $E_{n_0}[\mathfrak{T}_{n_0}]$ is a Frechet space, $\varphi_{n_0}$ is continuous from $G_{n_0}$ into $E_{n_0}[\mathfrak{T}_{n_0}]$. If $\{y_m : m \in D\}$ is a net of elements of $G_{n_0}$, which converges to $y \in F_{i_0}[\mathfrak{U}_{i_0}]$, then $\{\varphi_{n_0}(y_m) = y_m : m \in D\}$ is a Cauchy net in the Frechet space $E_{n_0}[\mathfrak{T}_{n_0}]$, which converges to $z$, hence $y = z$ and $G_{n_0}$ is closed in $F_{i_0}[\mathfrak{U}_{i_0}]$. Also $H_{n_0}$ is closed in $F_{i_0}[\mathfrak{U}_{i_0}]$ and since $H_{n_0}$ is of the second category in $F_{i_0}[\mathfrak{U}_{i_0}]$, then $H_{n_0} = F_{i_0}[\mathfrak{U}_{i_0}]$ and, therefore, $G_{n_0} = G$.

Finally, taking the net $\{y_m : m \in D\}$ converging to $x_0$, it results that $x_0 \in E$, which is not true. Hence F is not inductive limit of Baire spaces. Q.E.D.

*Example* 3. — G. Kothe has given an example of a non-complete (LB)-space, which is defined by a sequence $\{E_n\}_{n=1}^{\infty}$

of Banach spaces, so that there exists a bounded set A in E, which is not subset of $E_n$, $n = 1, 2, \ldots$ This example, and our Theorem 2, assure the existence of bornological barrelled spaces which are not inductive limits of Baire spaces.

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