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## ANALYSIS ON SOME LINEAR SETS

by Robert KAUFMAN

### 0.

Let  $F$  be a compact subset of  $(-\infty, \infty)$  and for each integer  $N \geq 1$  let  $\nu_N \equiv \nu(N; F)$  be the number of intervals  $[kN^{-1}, (k+1)N^{-1}]$  meeting  $F$ ;  $F$  is called *small* provided  $\log \nu_N = o(\log N)$ . The existence of small sets of « multiplicity » ( $M_0$ -sets in [6I, p. 344]) was proved in 1942 by Salem and used by Rudin [4, VIII]; a program somewhat analogous for locally compact abelian groups was completed by Varopoulos [5].

Does there exist a small set  $F$  with the property that both  $F$  and (say)  $F_2 = \{x^2 : x \in F\}$  are  $M_0$ -sets? The construction of these sets doesn't seem accessible by the method of Rudin and Salem [4], nor by the Brownian motion [3]. In this note an affirmative answer is given to a more general problem.

**THEOREM 1.** — *Let  $(h_n)$  be a sequence of real functions of class  $C^1(-\infty, \infty)$  with derivatives  $h'_n > 0$ . Then there is a small set  $F$  with the property that each  $h_n(F)$  is an  $M_0$ -set.*

Small sets occur naturally in the construction of independent sets [3, 4, 5]; after the metrical theory of Diophantine approximation a set  $F$  is called *metrically independent* if to each integer  $N \geq 1$  and each  $\varepsilon$  in  $(0, 1)$  there is a  $U_0$  so that the simultaneous inequalities

$$\left| \sum_{j=1}^N u_j x_j - \nu \right| < U^{-N-\varepsilon}, \quad U = \max(|u_1|, \dots, |u_N|) > U_0 \\ |x_i - x_j| \geq \varepsilon \quad \text{for} \quad 1 \leq i < j \leq N$$

have no solution in integers  $u_1, \dots, u_N, \nu$  and members  $x_1, \dots, x_N$  of  $F$ . Compare [1, VII].

Uncountable metrically independent subsets could perhaps be constructed by classical arguments, for example that of Perron [1, p. 79] or Davenport [2].

**THEOREM 2.** — *The set  $F$  determined in Theorem 1 can be required to have the property that each  $h_n(F)$  be metrically independent.*

**THEOREMS 1a, 2a.** — *Theorems 1 and 2 remain true provided each  $h_n$  is monotone-continuous and  $h'_n > 0$  almost everywhere.*

### 1.

In the proof of Theorem 1 we require two arrays of independent random variables  $(Y_{k,m})$  and  $(\xi_{k,m})$  defined on a space  $(\Omega, P)$  for  $1 \leq k < \infty$ ,  $1 \leq m \leq k^6$ . Each  $Y_k$  is uniformly distributed upon  $[0, 1]$  while

$$P\{\xi_{k,m} = 1\} = \pi_k = k^{-1} = 1 - P\{\xi_{k,m} = 0\}.$$

Suppose that  $f$  is a measurable function on  $(-\infty, \infty)$  and  $-1 \leq f \leq 1$ , and let  $\mu = \pi_k E(f(Y))$ ; elementary calculations show that

$$E(e^{t\xi_k f(Y_k)} e^{-t\mu}) \leq \exp \frac{1}{2} \pi_k t^2 \exp 0(\pi_k t^3)$$

with an '0' uniform for  $-1 \leq f \leq 1$ ,  $-1 \leq t \leq 1$ ,  $0 \leq \pi_k \leq 1$ . Hence for any  $z > 0$  and  $1 > t > 0$

$$P\left\{\left|\sum_m \xi_{k,m} - k^5\right| > zk^5\right\} \leq 2 \exp -zk^5 t \exp \frac{1}{2} k^6 \pi_k t^2 \exp 0(\pi_k k^6 t^3).$$

Choosing  $z = t = k^{-2}$  and using  $\pi_k = k^{-1}$  we obtain

$$P\left\{\left|\sum_m \xi_{k,m} - k^5\right| \geq k^3\right\} \leq 0(1) \exp -\frac{1}{2} k.$$

Thus

**LEMMA 1.** —  $\sum_{m=1}^{k^6} \xi_{k,m} = k^5 + 0(k^3)$  almost surely in  $\Omega$ .

A sequence of random measures  $\lambda_k$  is now determined as

follows: for any function  $g$  on  $(-\infty, \infty)$

$$\int g d\lambda_k = k^{-2}g(0) + k^{-5} \sum_m \xi_{n,m} g(e^{-k \log^2 k} Y_{k,m}).$$

Thus in every instance  $\lambda_k \geq 0$  and  $\|\lambda_k\| \geq k^{-2}$ ; moreover  $\|\lambda_k\| = 1 + O(k^{-2})$  almost surely. Because  $\sum e^{-k \log^2 k} < \infty$  the convolution  $\lambda = \pi * \lambda_k$  converges, and  $F$  is defined to be its closed support.  $F$  is contained in at most

$$\prod_{j=1}^k [j^5 + O(j^3)] = e^{O(k \log k)}$$

intervals of length  $e^{-k \log^2 k}$ .

Because  $(k+1) \log^2(k+1)/k \log^2 k \rightarrow 1$ , this is sufficient to obtain

LEMMA 2. —  $F$  is almost surely a small set.

LEMMA 3. — Let  $h \in C^1(-\infty, \infty)$  and  $h' > 0$ ; let  $(c_m)$ ,  $(u_m)$ ,  $(v_m)$  be sequences of real numbers such that

$$|c_m| + |v_m| = O(1) \quad \text{and} \quad |u_m v_m| \rightarrow \infty.$$

Then

$$\lim_{m \rightarrow \infty} \int_0^1 \exp iu_m h(c_m + v_m t) dt = 0.$$

*Proof.* — Let  $g$  denote the  $C^1$  function inverse to  $h$ , and let  $v_m > 0$ . The integral is transformed to

$$J = \int_{\alpha_m}^{\beta_m} g'(y) \exp iu_m y \cdot v_m^{-1} dy,$$

where  $\alpha_m = h(c_m)$ ,  $\beta_m = h(v_m + c_m)$ . A further substitution  $y = y_1 + \pi u_m^{-1}$  yields

$$J = \frac{1}{2} \int_{\alpha_m}^{\beta_m} g'(y) \exp iu_m y \cdot v_m^{-1} dy \\ - \frac{1}{2} \int_{\alpha_m - \pi u_m^{-1}}^{\beta_m - \pi u_m^{-1}} g'(y + \pi u_m^{-1}) \exp iu_m y \cdot v_m^{-1} dy.$$

This tends to 0 because  $\beta_m - \alpha_m = O(v_m)$  and  $v_m^{-1} u_m^{-1} = o(1)$ .

*Proof of Theorem 1.* — We show that for each function  $h_n$   $\lim_{u \rightarrow \infty} \int \exp iuh_n(s) \lambda(ds) = 0$ , almost surely. Then  $h_n(F)$  is an

$M_0$ -set; because  $h_n(F)$  is compact it is enough to prove

$$\lim_{r \rightarrow \infty} \int \exp ir^{\frac{1}{3}} h_n(s) \lambda(ds) = 0, \quad r = 1, 2, 3, \dots$$

To each integer  $r \geq 3$  we attach the integer  $k(r)$  defined by  $k(r) \leq \log^{\frac{1}{3}} r < k(r) + 1$  and write  $\lambda'_k = \prod_{j \neq k} * \lambda_j$ . Then

$$\int \exp ir^{\frac{1}{3}} h_n(s) \lambda(ds) = \iint \exp ir^{\frac{1}{2}} h_n(s + \omega) \lambda_k(ds) \lambda'_k(d\omega).$$

For each real number  $\omega$  in the support of  $\lambda'_k$  let  $m(\omega)$  be the expected value of  $\int \exp ir^{\frac{1}{2}} h_n(s + \omega) \lambda_k(ds)$ . Then

$$\left| \int \exp ir^{\frac{1}{2}} h_n(s) \lambda(ds) \right| \leq \int \left| \int \exp ir^{\frac{1}{2}} h_n(s + \omega) \lambda_k(ds) - m(\omega) \right| \lambda'_k(d\omega) + \|\lambda'_k\| \max |m(\omega)|.$$

The second integral, say I, can be handled by Jensen's inequality and the estimates at the beginning of 1. Let  $-1 < t < 1$  and  $\Phi(x) = e^{tx}$ . Then

$$\mathbb{E}(\Phi(\|\lambda'_k\|^{-1} k^5 \text{Re I})) \leq 2 \exp \frac{1}{2} k^5 t^2 0 (\exp k^5 t^3).$$

Choosing  $t = k^{-\frac{1}{2}}$  we observe

$$\begin{aligned} \mathbb{P}\left\{ |\text{Re I}| > \|\lambda'_k\| k^{-\frac{1}{2}} \right\} &= \mathbb{P}\left\{ \Phi(\|\lambda'_k\|^{-1} k^5 \text{Re I}) > \exp k^4 \right\} \\ &\leq 2 \exp \frac{1}{2} k^4 \exp 0(k^{7/2}) \exp -k^4. \end{aligned}$$

This is the general term of a convergent series, inasmuch as  $k = k(r) > -1 + \log^{\frac{1}{3}} r$ . Thus, almost surely in  $\Omega$ , for  $r > r_0$

$$\left| \text{Re} \int \exp ir^{\frac{1}{2}} h_n(s) \lambda(ds) \right| \leq k^{-\frac{1}{2}} \|\lambda'_k\| + \|\lambda'_k\| \max |m(\omega)|$$

and of course a similar statement holds for the imaginary part of the integral. Now

$$|m(\omega)| \leq k^{-2} + \left| \int_0^1 \exp ir^{\frac{1}{2}} h_n(e^{-k \log^2 kt} + \omega) dt \right|$$

with  $\omega = 0(1)$  and  $k = k(r)$ . To apply Lemma 3 we must

verify  $r^{\frac{1}{2}} e^{-k \log^2 k} \rightarrow \infty$  but this is plain from  $k(r) < \log^{\frac{1}{3}} r$ . Because  $\max_k \|\lambda'_k\| < \infty$  almost surely, the proof of Theorem 1 is complete.

## 2.

Theorem 2 requires the construction of a random function  $\varphi$  in  $C^\infty(-\infty, \infty)$ . Let  $\psi$  be a function in  $C^\infty(-\infty, \infty)$  with the properties

- (i)  $\psi = 0$  on  $[-\infty, -2]$ ,  $\psi = 3$  on  $[2, \infty]$ ,
- (ii)  $\psi' \geq 0$ , and  $\psi' > 1$  on  $(-1, 1)$ .

Let  $(a_p)$  be a sequence of real numbers such that every real number belongs to infinitely many of the intervals  $(a_p - p^{-1}, a_p + p^{-1})$ . Finally, let  $(Z_p)$  be a sequence of independent random variables on  $(\Omega, P)$ , uniformly distributed upon  $[0, 1]$ . We define

$$\varphi(x) = \sum_{p=1}^{\infty} e^{-p\psi} (p^{-1}Z_p + p^{\frac{1}{2}}(x - a_p)) + x.$$

To each compact set  $F$  and number  $\delta > 0$  there are numbers  $q_1$  and  $q_2$  so that

$$q_1 \geq 4, \quad q_1^{\frac{1}{2}} \delta \geq 5, \quad \bigcup_{p=q_1}^{q_2} (a_p - p^{-1}, a_p + p^{-1}) \supseteq F.$$

**THEOREM 3.** — *Let  $F$  be a small set and  $h \in C^1(-\infty, \infty)$ ,  $h' > 0$ ; then  $h\varphi(F)$  is almost surely metrically independent.*

For each integer  $U \geq 1$  we can choose a subset  $S(N, U)$  of  $R^N$  so that every point in  $F^N$  has distance  $< U^{-3N}$  from some point in  $S(N, U)$ , while  $\text{card } S(N, U) \leq \nu^N(NU^{3N}; F)$ .

Beginning with an inequality

$$\left| \sum_{j=1}^N u_j h\varphi(y_j) - \nu \right| < U^{-N-\epsilon}, \quad |h\varphi(y_j) - h\varphi(y_i)| > \epsilon \quad (i \neq j)$$

we conclude first that  $|y_i - y_j| > \eta$  for some fixed  $\eta > 0$ . Let  $(z_1, \dots, z_n)$  be the member of  $S(N, U)$  associated to

$(y_1, \dots, y_n)$ . Then

$$(1) \quad \left| \sum_{j=1}^N u_j h\varphi(z_j) - \nu \right| < U^{-N-\varepsilon} + 0(U \cdot U^{-3N}),$$

$$|z_i - z_j| > \eta - 2U^{-3N}.$$

For large  $U$  we can find  $\delta < \eta - 2U^{-3N}$  and corresponding numbers  $q_1, q_2$ . Let  $q_1 \leq p \leq q_2, |z_i - a_p| < p^{-1}$ .

$$\left| p^{-1}Z_p + p^{\frac{1}{2}}(z_i - a_p) \right| < p^{-1} + p^{-\frac{1}{2}} < 1,$$

$$|p^{-1}Z_p + p^{\frac{1}{2}}(Z_j - a_p)| > p^{\frac{1}{2}}\delta - p^{-1} - p^{-\frac{1}{2}} > 4, \quad \text{when } j \neq i.$$

Therefore  $\frac{\partial}{\partial Z_p} \sum_{j=1}^N u_j h\varphi(Z_j) = u_i \frac{\partial}{\partial Z_p} h\varphi(Z_i)$  exceeds  $\alpha|u_i|$  in modulus, with an  $\alpha > 0$  independent of  $u_1, \dots, u_n$ . Hence the probability of the inequality (1) is  $0(U^{-1} \cdot U^{-N-\varepsilon})$  for each  $(z_1, \dots, z_N)$ . The requirement  $U = \max(|u_1|, \dots, |u_n|)$  determines  $0(U^{N-1})$   $N$ -tuples and plainly  $\nu = 0(U)$ . Because  $F$  is a small set  $\nu^N(NU^{3N}; F) = U^{o(1)}$  as  $U \rightarrow \infty$ . Theorem 3 follows from this and  $\Sigma U^{-1-\varepsilon} U^{o(1)} < \infty$ .

*Proof of Theorem 2.* — Here we use the fact that  $F$  and  $\varphi$  depend on independent  $\sigma$ -fields.  $F$  is almost surely small, whence each  $h_n\varphi(F)$  is almost surely metrically independent, by Theorem 3. By Theorem 1, each  $h_n\varphi(F)$  is almost surely an  $M_0$ -set and Theorem 2 is proved.

### 3.

*Proof of Theorems 1a and 2a.* — According to a theorem of Marcinkiewicz [6II, pp. 73-77], to each  $\delta > 0$  there exist functions  $g_n$  in  $C^1(-\infty, \infty)$  so that

$$m(h_n \neq g_n) < \delta n^{-2}, \quad n = 1, 2, 3, \dots$$

At almost all points of density of the set  $(h_n = g_n), g'_n = h'_n > 0$ . Passing to a perfect subset of the set  $(g'_n > 0, g'_n = h'_n, g_n = h_n)$ , we can find a  $\tilde{g}_n$  in  $C^1(-\infty, \infty)$  such that

$$m(h_n \neq \tilde{g}_n) < 2\delta n^{-2}, \quad n = 1, 2, 3, \dots,$$

$\tilde{g}'_n > 0$  everywhere.

We observe next that to each  $\varepsilon > 0$  there is a constant  $B(\varepsilon)$  so that for all Borel sets  $S$

$$\int_{\Omega} \lambda(S) dP \leq \varepsilon + B(\varepsilon)m(S).$$

Thus to each  $\varepsilon > 0$  we can choose functions  $\tilde{g}_n$  by Marcinkiewicz' theorem, so that

$$P\{\lambda(x: \tilde{g}_n\varphi(x) \neq \tilde{h}_n\varphi(x) \text{ for some } n) > \varepsilon\} < \varepsilon.$$

In proving this implication it must be observed that  $\varphi$  and  $\lambda$  are stochastically independent and  $\varphi' > 1$ . Writing  $G$  for the inner set in the last inequality, we know that  $h_n\varphi(G' \cap F) = \tilde{g}_n\varphi(G' \cap F)$  is almost surely metrically independent and that  $h_n\varphi(G' \cap F)$  is almost surely an  $M_0$ -set, if only  $\lambda(G' \cap F) > 0$ ; and this holds for  $\|\lambda\| > \varepsilon$  excepting an event of probability  $< \varepsilon$ . Thus Theorems 1a and 2a are derived from Theorems 1 and 2.

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