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A HEREDITARY PROPERTY IN LOCALLY CONVEX SPACES (1)

by Manuel VALDIVIA

J. Dieudonné has given in [1] the two following theorems :

1) *If* F *is a subspace, of finite codimension, of a barrelled space* E, *then* F *is a barrelled space.*

2) *If* F *is a subspace, of finite codimension, of a hornological space, then* F *is a bornological space.*

In this paper we give a theorem analogous to the previous ones, but using infrabarrelled spaces instead of barrelled or bornological spaces. So we shall prove the following theorem : If F is a subspace, of finite codimension, of an infrabarrelled space E, then F is an infrabarrelled space.

Let K be the field of real or complex numbers. Let E be a locally convex topological vector space over the field K. If $\mathcal{B}$ is the family of all the absolutely convex, bounded and closed sets of E, we denote with E_B, $B \in \mathcal{B}$, the linear hull of E with the seminorm asociated to B. Let $\mathfrak{T}$ be the topology on E, so that $E[\mathfrak{T}]$ is the inductive limit of the family $\{E_B : B \in \mathcal{B}\}$.

Theorem. — *Let* F *be a subspace of* E, *with finite codimension. If* U *is a closed, bornivorous and absolutely convex set of* F, *then there exists in* E *an* U', *closed, bornivorous and absolutely convex set, such that* $U' \cap F = U$.

In particular, if E *is an infrabarrelled space, then* F *is also an infrabarrelled space.*

Proof. — Clearly, the $\mathfrak{T}$-topology is finer than the initial one on E. On the other hand, for every bounded set A, there exists a set $B \in \mathcal{B}$, such that $A \subset B$. Hence A is a bounded

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set of E_B, therefore A is a bounded set of $E[\mathfrak{T}]$. That is, the bounded sets of E and those of $E[\mathfrak{T}]$ are the same.

We denote with $F[\mathfrak{T}]$ the subspace F, equipped with the topology induced by $\mathfrak{T}$. Since $E[\mathfrak{T}]$ is the inductive limit of seminormed spaces, it is a bornological space and, according to theorem 2), $F[\mathfrak{T}]$ is a bornological space. Hence, U is a closed neighborhood of 0 in $F[\mathfrak{T}]$.

Clearly, it is sufficient to prove the theorem in the case of F being a vector subspace of E, with codimension one. So that we suppose that F is so.

Two cases are possible :

1º $F[\mathfrak{T}]$ being dense in $E[\mathfrak{T}]$. Let $\overline{U}$ and $\overline{U}^\star$ be the closures of U in E and $E[\mathfrak{T}]$ respectively. Since U is a neighborhood of 0 in $F[\mathfrak{T}]$, then $\overline{U}^\star$ is a neighborhood of 0 in $E[\mathfrak{T}]$, hence $\overline{U}^\star$ is a bornivorous set in the same space.

Furthermore, $\overline{U} \supset \overline{U}^\star$, then $\overline{U}$ is a bornivorous set in E. We can take $U' = \overline{U}$, then U' is a closed, bornivorous and absolutely convex set of E, such that $U' \cap F = U$.

2º $F[\mathfrak{T}]$ being closed in $E[\mathfrak{T}]$. If $U = \overline{U}$, we take a vector x such that $x \in E$ and $x \notin F$. Let C be the balanced hull of the set $\{x\}$, then $U + C$ is a closed set in E and $U + C$ is a neighborhood of 0 in $E[\mathfrak{T}]$, therefore, $U + C$ is bornivorous in E. If we take $U' = U + C$ the theorem is satisfied.

If $U \neq \overline{U}$, $\overline{U}$ is absorbing in E, hence there exists an element $z \in \overline{U}$ such as $z \notin F$. Let D be the balanced hull of $\{z\}$. $U + D$ is a neighbourhood of 0 in $E[\mathfrak{T}]$, hence it is bornivorous in E. Furthermore $\overline{U} \supset U$ and $\overline{U} \supset D$, then $2\overline{U} \supset U + D$, hence $\overline{U}$ is bornivorous in E. If we take $\overline{U} = U'$ the theorem is satisfied.

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