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GEOMETRY OF MANIFOLDS WHICH ADMIT CONSERVATION LAWS

by D. E. BLAIR [(1)] and A. P. STONE [(1, 2)]

1. Introduction.

A conservation law for an endomorphism $\underline{h}$ of the localized module of differential forms on an analytic manifold is a 1-form θ such that both θ and $\underline{h}\theta$ are exact. Conservation laws defined in this manner on manifolds can be quite easily related to conservation laws in the sense of physics. The problem of the existence of conservation laws on analytic manifolds in the case where h is cyclic or has distinct eigenvalues has been the subject of several papers ([4], [5], [6], and [7] for example). In these papers the differential concomitant $[\underline{h}, \underline{h}]$ of Nijenhuis has played an important role. The vanishing of $[\underline{h}, \underline{h}]$ is an integrability condition which guarantees the existence of certain local coordinates.

The existence of conservation laws on C^∞ manifolds is again a consequence of the condition that the Nijenhuis torsion $[\underline{h}, \underline{h}]$ vanish. The proof of this fact in the case that $\underline{h}$ has distinct eigenvalues is essentially given in [7] ; in the cyclic case use must be made of a theorem due to E.T. Kobayashi [3].

The study of the geometry of manifolds which admit conservation laws was initiated in [1]. Specifically the authors studied the holonomy group of a Riemannian manifold which carries a cyclic vector 1-form $\underline{h}$ which is covariant constant. In this paper the study of the geometry of Riemannian manifolds which carry conservation laws is continued. The manifolds which are investigated admit a structure $\underline{h}$

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with distinct eigenvalues. The integrability condition that $\underline{h}$ is covariant constant is also imposed. The more general case in which $\underline{h}$ is cyclic will be considered in a later paper.

Notation and definitions are given in Section 2. In Section 3 there is a brief discussion of the geometry of a manifold carrying a structure $\underline{h}$ as described in the previous paragraph. A result obtained in Theorem 3.1 is that such a manifold is flat. Section 4 is then devoted to a study of the structure induced by $\underline{h}$ on an immersed manifold.

2. Preliminaries.

Let M be an $(n + 1)$-dimensional Riemannian manifold and let $\widetilde{\nabla}$ denote covariant differentiation with respect to the Riemannian connexion on M. If $p \in \mathrm{M}$ the tangent space at p is denoted by M_p.

Since a structure $\underline{h}$ on a manifold M which admits conservation laws is only locally defined, we will consider an n-dimensional manifold N immersed (rather than imbedded) in M ; that is, there exists a mapping $\varphi : \mathrm{N} \longrightarrow \mathrm{M}$ such that for each $p \in \mathrm{N}$, the differential of the immersion $\mathrm{B} : \mathrm{N}_p \longrightarrow \mathrm{M}_{\varphi(p)}$ is one-to-one. If φ itself is one-to-one N is a submanifold and is a hypersurface of M.

The tensor field $\underline{h}$ can of course be viewed as an endomorphism of the localized module of vector fields E(M) as well as the localized module of 1-forms $\mathcal{E}$(M). No distinction in notation is made between $\underline{h}$ and its adjoint, and it should be clear from the context whether $\underline{h}$ is acting on vector fields or on 1-forms. The endomorphism $\underline{h}$ is often called a vector-valued 1-form.

An induced structure on the immersed manifold N is defined by setting

$$\underline{h}\mathrm{BX} = \mathrm{B}\underline{h}'\mathrm{X} + \alpha(\mathrm{X})\,\mathrm{C} \qquad (2.1)$$

where $\mathrm{X} \in \mathrm{N}_p$ and C is the unit normal to N in M. Thus (2.1) defines an endomorphism $\underline{h}'$ of E(N) and a 1-form $\alpha \in \mathcal{E}$(N). The manifold N is said to be *invariant* under $\underline{h}$ if $\mathrm{X} \in \mathrm{B}(\mathrm{N}_p)$ implies $\underline{h}\mathrm{X} \in \mathrm{B}(\mathrm{N}_p)$ also.

If h is singular at a point $p \in \mathrm{M}$, a non-singular transformation $\underline{h}^*$ can always be found such that $\widetilde{\nabla}_{\mathrm{X}}\underline{h} = \widetilde{\nabla}_{\mathrm{X}}\underline{h}^*$ and $[\underline{h}, \underline{h}] = [\underline{h}^*, \underline{h}^*]$

for every vector field X. One simply sets $\underline{h}^* = \underline{h} + a\mathrm{I}$ where a is a suitable non-zero constant.

The second fundamental form II on N and the Weingarten map W are given by the Gauss-Weingarten equations

$$\widetilde{\nabla}_{\mathrm{BX}}\mathrm{BY} = \mathrm{B}\nabla_{\mathrm{X}}\mathrm{Y} + \mathrm{II}(\mathrm{X}, \mathrm{Y})\,\mathrm{C} \tag{2.2}$$

$$\widetilde{\nabla}_{\mathrm{BX}}\mathrm{C} = -\,\mathrm{BWX}\ , \tag{2.3}$$

where X, Y $\in \mathrm{N}_p$ and ∇ denotes covariant differentiation with respect to the induced Riemannian connexion on N. If II or equivalently W vanishes we say that N is *totally geodesic* in M.

Covariant derivatives on M (or N) with respect to X_i or $\dfrac{\partial}{\partial u_i}$ will usually be denoted by $\widetilde{\nabla}_i$ (or ∇_i). Similarly partial derivatives such as $\dfrac{\partial a_j}{\partial u_i}$ will be denoted by $a_{j,i}$. The Riemannian curvature tensor and the connexion coefficients will be denoted by R and Γ^k_{ij} respectively.

The Nijenhuis torsion $[\underline{h}, \underline{h}]$ of h is given as in [2] by

$$[\underline{h}, \underline{h}](\mathrm{X}, \mathrm{Y}) = \underline{h}^2[\mathrm{X}, \mathrm{Y}] + [\underline{h}\mathrm{X}, \underline{h}\mathrm{Y}] - \underline{h}[\underline{h}\mathrm{X}, \mathrm{Y}] - \underline{h}[\mathrm{X}, \underline{h}\mathrm{Y}] \tag{2.4}$$

for any X, Y $\in$ E(M). Since $\widetilde{\nabla}$ is covariant differentiation with respect to the Riemannian connexion on M, it is easily verified that

$$[\underline{h}, \underline{h}](\mathrm{X}, \mathrm{Y}) = \{\underline{h}\widetilde{\nabla}_{\mathrm{Y}}\underline{h} - \widetilde{\nabla}_{h\mathrm{Y}}\underline{h}\}\mathrm{X} - \{\underline{h}\widetilde{\nabla}_{\mathrm{X}}\underline{h} - \widetilde{\nabla}_{h\mathrm{X}}\underline{h}\}\mathrm{Y} \tag{2.5}$$

for any X, Y $\in$ E(M).

A useful result from [5] is that if $\underline{h}$ has constant distinct eigenvalues and vanishing Nijenhuis torsion then there exist (locally) coordinates u_i, $i = 0, 1, \ldots, n$, such that the vector fields $\dfrac{\partial}{\partial u_i}$ form an eigenvector basis for E(M). Moreover this result yields a basis of conservation laws for $\mathcal{E}$(M).

3. The distinct eigenvalue case.

Suppose that $\underline{h}$ has distinct eigenvalues $\lambda_0, \ldots, \lambda_n$ which are non-zero at a point $p \in$ M. If $\widetilde{\nabla}_{\mathrm{X}}\underline{h} = 0$ for all X $\in$ E(M), then the existence of a basis for E(M) of eigenvectors $\frac{\partial}{\partial u_i}$, $0 \leqslant i \leqslant n$, is assured since $[\underline{h}, \underline{h}]$ vanishes as a consequence of formula (2.5). The following theorem summarizes several consequences of the integrability condition that $\widetilde{\nabla}_X \underline{h} = 0$, for $X \in E(M)$. The vector fields $\frac{\partial}{\partial u_j}$ will be denoted by Y_j and the Riemannian metric on M will be denoted by G with $G(Y_i, Y_j) \equiv G_{ij}$.

THEOREM 3.1. – *Let $\underline{h}$ have distinct non-zero eigenvalues $\lambda_0, \ldots, \lambda_n$ and suppose that $\widetilde{\nabla}_X h = 0$ for all $X \in E(M)$. Let $Y_i = \frac{\partial}{\partial u_i}$ denote the corresponding eigenvectors. Then the following results are obtained :*

(a) *the eigenvalues λ_i are all constant*

(b) $\widetilde{\nabla}_j Y_i = 0$, $i \neq j$.

(c) $\widetilde{\nabla}_j Y_j = \Gamma^j_{jj} Y_j$ *(no summation).*

(d) $\Gamma^j_{jj} = f(u_j)$

(e) $R = 0$

(f) $G_{ii,j} = G_{ij,k} = 0$, *i, j, k distinct positive integers between* 0 *and n.*

Proof. – Since coordinates $u_0, u_1, \ldots, u_n$ exist such that the $Y_i = \frac{\partial}{\partial u_i}$ and since each eigenvalue λ_i is a function of u_i alone (i.e., $\lambda_i = \lambda_i(u_i)$, $0 \leqslant i \leqslant n$) we have $\lambda_{i,j} = 0$ when $i \neq j$. Note then that $(\widetilde{\nabla}_i \underline{h})\, Y_j = \widetilde{\nabla}_i(\underline{h} Y_j) - \underline{h} \widetilde{\nabla}_i Y_j$ and if the condition $\widetilde{\nabla}_i Y_j = \widetilde{\nabla}_j Y_i$ is used in the preceding equation one obtains by routine calculations the results that

$$(\widetilde{\nabla}_i \underline{h})\, Y_j - (\widetilde{\nabla}_j \underline{h})\, Y_i = (\lambda_j - \lambda_i)\, \widetilde{\nabla}_j Y_i$$

and also

$$(\widetilde{\nabla}_j \underline{h})\, Y_j = (\lambda_j I - \underline{h})\, \widetilde{\nabla}_j Y_j + \lambda_{j,j} Y_j \;.$$

Thus the statements (a), (b), and (c) of the theorem are established. We remark that the result (a) was also obtained in [1].

The Riemannian curvature tensor R is given by

$$R(Y_i, Y_j)\, Y_k \equiv R_{ij}(Y_k) = \widetilde{\nabla}_i \widetilde{\nabla}_j Y_k - \widetilde{\nabla}_j \widetilde{\nabla}_i Y_k$$

and thus $R_{ij}(Y_k) = 0$ when i, j, k are distinct positive integers ranging over $(0, 1, \ldots, n)$: otherwise if $j = k$ and $i \neq j$, we have

$$R_{ij}(Y_j) = \tilde{\nabla}_i \tilde{\nabla}_j Y_j = \Gamma^j_{jj,i} Y_j \ .$$

Hence since $G(R_{ij}(Y_j), Y_j) = 0$, it follows that $\Gamma^j_{jj,i} = 0$ and $R = 0$ and the results (d) and (e) are established.

The result (f) is obtained by observing that

$$\tilde{\nabla}_i G_{jk} = G(\tilde{\nabla}_i Y_j, Y_k) + G(Y_j, \tilde{\nabla}_i Y_k)$$

and hence $G_{ii,j} = G_{ij,k} = 0$ when i, j, k are distinct positive integers ranging over $(0, 1, \ldots, n)$. Thus the $i - j$ entry in the matrix which represents G depends only on the variables u_i and u_j, and the proof of Theorem 3.1 is completed.

We remark that in Theorem 3.1 the condition that $\tilde{\nabla}_X \underline{h} = 0$ is equivalent to the condition that $\underline{h}$ commute with parallel transport. Specifically if $\gamma : [0,1] \longrightarrow M$ is a closed piecewise differentiable curve starting and ending at $P \in M$, and τ denotes parallel translation around γ with respect to the Riemannian connexion, then

$$\tilde{\nabla}_{\gamma^*} \underline{h} X = (\tilde{\nabla}_{\gamma^*} \underline{h}) X + h \tilde{\nabla}_{\gamma^*} X \ ,$$

where γ^* is the tangent vector field to γ. Hence if X_0 is a vector at $P \in M$, and $X = \tau X_0$, $Z = \tau Z_0$ with $Z_0 = \underline{h} X_0$, then $Z = \underline{h} X$ if and only if $\underline{h}$ is covariant constant. In particular, if Y_i is an eigenvector of $\underline{h}$ and $\underline{h}$ is covariant constant then $\underline{h} \tau Y_i = \tau \underline{h} Y_i = \tau \lambda_i Y_i = \lambda_i \tau Y_i$ and τY_i is an eigenvector of $\underline{h}$ corresponding to λ_i, and thus M is flat since the holonomy group at P is trivial.

4. The immersed manifold N.

We now turn our attention to a study of the geometry of the immersed manifold N. As indicated in Section 2 if $\underline{h}$ is a vector-valued 1-form on M, then locally an endomorphism $\underline{h}'$ and a 1-form α are defined on N by the relation

$$\underline{h} BX = B \underline{h}' X + \alpha(X) C$$

where X is tangent to N. The endomorphism $\underline{h}$ is assumed to be non-singular and have distinct eigenvalues. Let Y be an eigenvector of $\underline{h}$ with eigenvalue λ and let $Y = BX + bC$. Then

$$\underline{h}Y = \lambda BX + \lambda bC = B\underline{h}'X + \alpha(X)\,C + b\underline{h}C \;;$$

and hence X is an eigenvector of $\underline{h}'$ with eigenvalue λ if and only if $\underline{h}C$ is proportional to C. That is, X is an eigenvector of $\underline{h}'$ with eigenvalue λ if and only if $hC = \nu C$, for some $\nu \neq 0$. If now $\{Y_i\}$,

$$i = 0\,,1\,,\ldots,n$$

is a basis of eigenvectors (for M_p) of $\underline{h}$ with eigenvalues λ_i, then $C = \Sigma\, a_i Y_i$ and $\underline{h}C = \Sigma\, a_i\lambda_i Y_i = \Sigma\, \nu a_i Y_i$. Since the λ_i's are distinct, $\nu = \lambda_i$ for some i and the remaining a_j's vanish. That is, C is an eigenvector of $\underline{h}$ with eigenvalue ν and in this case $\alpha(X)$ is given by

$$\alpha(X) = b(\lambda - \nu)\ .$$

The preceding discussion can now be summarized by the following theorem.

THEOREM 4.1. – *Let $\underline{h}$ be non-singular and have distinct eigenvalues on* M *and suppose that* N *is immersed in* M. *Then there exists locally an endomorphism $\underline{h}'$ on* N *with n of the eigenvalues of $\underline{h}$ if and only if* C *is a fixed direction of $\underline{h}$.*

Now let $X_1,\ldots,X_n$ denote an eigenvector basis of $\underline{h}'$ for N_p with corresponding eigenvalues $\lambda_1,\ldots,\lambda_n$. The following proposition gives a necessary and sufficient condition that N be invariant under $\underline{h}$.

PROPOSITION 4.2. – N *is invariant under $\underline{h}$ if and only if every eigenvector* Y_i *of h,* $Y_i \neq C$, *is tangent to* N (i.e., $Y_i = BX_i$, $1 \leqslant i \leqslant n$).

Proof. – Suppose that $Y_i \in B(N_p)$, $i = 1\,,2\,,\ldots,n$. If $X \in N_p$, then $BX = \Sigma\, a_i Y_i$ since the Y_i's form a basis for $B(N_p)$ and hence $\underline{h}BX = \Sigma\, a_i\lambda_i Y_i$ is also in $B(N_p)$. Conversely if N is invariant under $\underline{h}$ then $\alpha = 0$. Hence if $Y = BX + bC$ is an eigenvector other than C then $0 = \alpha(X) = b(\lambda - \nu)$; but since $\lambda \neq \nu$ we must have $b = 0$ and hence $Y \in B(N_p)$.

In the remainder of this paper the condition that $\underline{h}$ is covariant constant will be denoted by $\widetilde{\nabla} h = 0$ instead of $\widetilde{\nabla}_X h = 0$ for every $X \in E(M)$ or for every $X \in M_p$.

THEOREM 4.3. – *Let* $\tilde{\nabla}\underline{h} = 0$ *and suppose* C *is an eigenvector of* $\underline{h}$ *; then* $\nabla\underline{h}' = 0$ *on* N *iff* N *is invariant under* $\underline{h}$ *or* N *is totally geodesic.*

Proof. – Let $\underline{h}\mathrm{C} = \nu\mathrm{C}$; then

$$\tilde{\nabla}_{\mathrm{BY}}(\underline{h}\mathrm{BX}) = (\tilde{\nabla}_{\mathrm{BY}}\underline{h})\,\mathrm{BX} + \underline{h}\mathrm{B}\,\nabla_{\mathrm{Y}}\mathrm{X} + \nu\mathrm{II}(\mathrm{Y}\,,\mathrm{X})\,\mathrm{C}$$

for any X, $\mathrm{Y} \in \mathrm{N}_p$. On the other hand $\tilde{\nabla}_{\mathrm{BY}}(\underline{h}\mathrm{BX})$ can also be computed from formula (2.1) and the two calculations then compared. The result is

$$(\tilde{\nabla}_{\mathrm{BY}}\underline{h})\,\mathrm{BX} = \{\mathrm{II}(\mathrm{Y}\,,\underline{h}'\mathrm{X}) - \nu\mathrm{II}(\mathrm{Y}\,,\mathrm{X}) + (\nabla_{\mathrm{Y}}\alpha)\,(\mathrm{X})\}\mathrm{C} + \mathrm{B}(\nabla_{\mathrm{Y}}\underline{h}')\,\mathrm{X} - \alpha(\mathrm{X})\,\mathrm{BWY}\ .$$

Hence if $\tilde{\nabla}\underline{h} = 0$ we must have

$$\nu\mathrm{II}(\mathrm{X}\,,\mathrm{Y}) - \mathrm{II}(\underline{h}'\mathrm{X}\,,\mathrm{Y}) = (\nabla_{\mathrm{Y}}\,\alpha)\,(\mathrm{X}) \tag{4.1a}$$

$$(\nabla_{\mathrm{Y}}\underline{h}')\,\mathrm{X} = \alpha(\mathrm{X})\,\mathrm{WY}\ , \tag{4.1b}$$

for all X, $\mathrm{Y} \in \mathrm{N}_p$, and the theorem is established.

Note that if $\underline{h}'\mathrm{X} = \lambda\mathrm{X}$, $\mathrm{X} \in \mathrm{N}_p$ then (4.1a) can be rewritten as

$$(\nu - \lambda)\,\mathrm{II}(\mathrm{X}\,,\mathrm{Y}) = (\nabla_{\mathrm{Y}}\,\alpha)\,(\mathrm{X})\ . \tag{4.1c}$$

The following corollaries are easily established.

COROLLARY 1. – *If* $\tilde{\nabla}\underline{h} = 0$, *then* α *is covariant constant iff* N *is totally geodesic.*

COROLLARY 2. – *If* $\tilde{\nabla}\underline{h} = 0$ *and* N *is invariant under* $\underline{h}$, *then* N *is totally geodesic.*

COROLLARY 3. – *If* $\tilde{\nabla}\underline{h} = 0$ *and* N *is invariant under* $\underline{h}$ *or* N *is totally geodesic, then* N *is flat.*

If the unit normal C in (2.1) is a fixed direction of $\underline{h}$ and if $\underline{h}$ is covariant constant and has distinct eigenvalues then conservation laws for $\underline{h}'$ exist on N when N is totally geodesic in M. These conservation laws $\theta \in \mathcal{E}(\mathrm{N})$ have the form $\theta = \Sigma\, a_i du_i$ where $a_i = a_i(u_i)$, a result which is obtained in [7]. One can also obtain the following theorem.

THEOREM 4.4. – *Let* $\underline{h}C = \nu C$; *if* $\widetilde{\nabla}\underline{h} = 0$ *and* N *is totally geodesic in* M, *then* α *is a conservation law for* h'.

Proof. – From the equation (4.1a) we have $\nabla_i \alpha = 0$ so that α is closed and hence locally exact. Since $\nabla \underline{h}' = 0$ there exist coordinates u_i, $1 \leqslant i \leqslant n$, such that $X_i = \dfrac{\partial}{\partial u_i}$ is an eigenvector basis for E(N). Hence if $\alpha = \sum_1^n f_i du_i$ one obtains the result that $f_{i,j} = 0$ when $i \neq j$ as a consequence of Theorem 3.1 and the relation

$$X_i \alpha(X_j) = \alpha(\nabla_i X_j) + (\nabla_i \alpha)(X_j) \ .$$

Thus $f_i = f_i(u_i)$ and moreover the f_i are given explicitly by

$$f_{i,i} = X_i \alpha(X_i) = \alpha(\nabla_i X_i) = \Gamma^i_{ii} f_i \ ,$$

and consequently $\underline{h}'\alpha$ is also exact.

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