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## ON CONTINUOUS COLLECTIONS OF MEASURES

by R. M. BLUMENTHAL and H. H. CORSON

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### 1. Introduction.

Let  $M$  be a compact space and  $X$  a complete metric space. By a probability measure on  $X$  we mean a positive regular Borel measure of total mass 1; let  $P(X)$  denote the collection of these. Let  $C(X)$  denote the set of bounded continuous real valued functions on  $X$ ,  $C(M, X)$  the set of continuous functions from  $M$  to  $X$ . Let  $P(X)$  have the weak topology as functionals on  $C(X)$  and give  $C(M, X)$  the topology of uniform convergence.

There are many commonly used objects in mathematics which can be viewed as continuous functions from  $M$  to  $P(X)$ . We will first give some examples of these.

*Example 1.* — Any continuous  $T$  from  $M$  to  $X$  may be viewed in this way, considering  $x \in X$  as a point mass.

*Example 2.* — If  $T$  is a continuous function from  $M$  into a finite dimensional simplex or simplicial complex, then  $T$  may be viewed as a continuous map from  $M$  to  $P(X)$  where  $X$  is the set of vertices under the discrete topology, since there is an obvious choice for the measure corresponding to  $T(t)$ .

*Example 3.* — Let  $X$  be compact, and let  $T$  be a positive linear operator from  $C(X)$  to  $C(M)$  such that  $T(1) = 1$ . Then the Riesz representation theorem implies that there is a unique continuous map from  $M$  to  $P(X)$  corresponding

to  $T$ ; in fact it is just the adjoint of  $T$ . Conversely, any continuous map from  $M$  to  $P(X)$  corresponds to such an operator. Note that  $T$  is multiplicative if and only if  $T$  corresponds to a function from  $M$  into  $X$ , as in Example 1.

*Example 4.* — Let  $X$  be a partition of unity of  $M$  and give  $X$  the discrete topology. Define  $T$  from  $M$  to  $P(X)$  by  $T(t)g = g(t)$  for  $g$  in  $X$ . Then  $T$  is continuous. This is similar to the situation immediately following the proof of the Lemma in Section 2.

Before giving the last, basic example we need more notation: Let  $Y$  and  $X$  be complete metric, and let  $\pi$  be a continuous function from  $Y$  into  $X$ . Then  $\pi$  induces a mapping, also denoted by  $\pi$ , from  $P(Y)$  to  $P(X)$  and defined by  $\pi\mu(E) = \mu(\pi^{-1}(E))$ . Also, for  $t \in M$  denote simply by  $t$  the mapping  $f \rightarrow f(t)$  from  $C(M, X)$  into  $X$ . Hence we see that  $\mu \rightarrow t\mu$  from  $P(C(M, X))$  to  $P(X)$  is defined and continuous for any  $X$  by taking  $Y$  to be  $C(M, X)$  and  $t$  to be  $\pi$  in the definition of  $\mu \rightarrow \pi\mu$  above.

*Example 5.* — For a fixed  $\mu \in P(C(M, X))$ , the mapping  $t \rightarrow t\mu$  is continuous from  $M$  to  $P(X)$ .

In this paper we show that any continuous map from  $M$  to  $P(X)$  is of the form given in Example 5, provided that  $M$  is totally disconnected. As we see from Example 3, this establishes an integral representation theorem which generalizes the theorem of F. Riesz for compact metric  $X$ : For any such  $X$  and any positive linear operator  $T$  from  $C(X)$  to  $C(M)$  with  $T(1) = 1$  and  $M$  totally disconnected, there is a regular Borel measure  $\mu$  on the space of multiplicative operators under the strong operator topology such that  $Tf(t) = \int Sf(t) d\mu(S)$ . Obviously the Riesz theorem is the case where  $M$  is a space with just one element.

The most obvious shortcomings of this statement are first that  $M$  is very special (although it is clear that the restriction on  $M$  is essential for a conclusion in this generality) and second that it is not clear which of the  $\mu$  on  $C(M, X)$  are to be preferred, since several of them can give rise to the same mapping from  $M$  into  $P(X)$ . As far as the connectivity of  $M$  goes, see e.g. [3] for a discussion of its significance.

## 2.

**THEOREM.** — *Let  $M$  be compact and totally disconnected and let  $X$  be a complete metric space. Then for each continuous function  $T$  from  $M$  into  $P(X)$  there is a  $\mu$  in  $P(C(M, X))$  such that  $t\mu = T(t)$  for all  $t$  in  $M$ .*

We will first prove a lemma which treats a special case and provides some additional information for use in the general case. Before stating it we need one more piece of notation. Let  $X$  and  $Y$  be complete metric spaces and  $\pi : Y \rightarrow X$  a continuous mapping. Then  $\pi$  induces a continuous mapping from  $C(M, Y)$  to  $C(M, X)$  by  $(\pi\varphi)(t) = \pi(\varphi(t))$ ,  $t \in M$ ,  $\varphi \in C(M, Y)$ . We denote this mapping by  $\pi$  also. If  $\mu$  is a measure on  $C(M, Y)$  then it is simply an exercise in unscrambling the notation to check that the measures  $t\pi\mu$  and  $\pi t\mu$  on  $X$  are the same. In fact,

$$\begin{aligned} t\pi\mu(E) &= \pi\mu\{f \in C(M, X) : tf = f(t) \in E\} \\ &= \mu\{f \in C(M, Y) : \pi f(t) \in E\}, \end{aligned}$$

and

$$\pi t\mu(E) = t\mu\{y \in Y : \pi y \in E\} = \mu\{f \in C(M, Y) : \pi f(t) \in E\}.$$

**LEMMA.** — *Let  $X$  and  $Y$  be discrete spaces and let  $\pi$  be a continuous mapping from  $Y$  onto  $X$ . Let  $T$  be a continuous mapping from  $M$  into  $P(Y)$  and let  $\mu$  be a measure on  $C(M, X)$  such that  $t\mu = \pi T(t)$ . Then there is a measure  $\nu$  in  $P(C(M, Y))$  such that (1)  $t\nu = T(t)$  for all  $t$  and (2)  $\pi\nu = \mu$ .*

*Proof.* — Consider a positive regular Borel measure  $\theta$  on  $C(M, Y)$  having mass  $\leq 1$  (but perhaps not a probability measure) and such that (1)  $t\theta \leq T(t)$  and (2)  $\pi\theta \leq \mu$  (an inequality between positive measures means set-wise inequality). Let  $A$  denote the set of all such measures. Of course,  $A$  is non-empty, for example the 0 measure is in  $A$ , and we will now show that there is a non-zero measure in  $A$ . Indeed  $C(M, X)$  is a discrete space and so there is an element  $f \in C(M, X)$  such that  $\mu(\{f\}) = \varepsilon > 0$ . Suppose that  $f$

takes on values  $x_1, \dots, x_n$  on sets  $M_1, \dots, M_n$  respectively and let  $Y_i = \pi^{-1}(x_i)$ ,  $1 \leq i \leq n$ . For each  $y \in Y$  the function  $t \rightarrow T(t)(y)$  is continuous, and for each  $i$   $\sum_{y \in Y_i} T(t)(y) \geq \varepsilon$  for all  $t \in M_i$ . Since  $M_i$  is compact and totally disconnected, each point has arbitrarily small open and closed neighborhoods [2, page 20]. Hence we may find a finite cover  $\mathcal{U}_i$  of  $M_i$  and a  $\delta_i > 0$  such that each element of  $\mathcal{U}_i$  is open and closed and for  $U \in \mathcal{U}_i$  there is a  $y_U \in Y_i$  such that  $T(t)(y_U) \geq \delta_i$  for all  $t \in U$ . In fact we may clearly choose  $\mathcal{U}_i$  to be a partition of  $M_i$ . For each  $i$  and each  $U \in \mathcal{U}_i$  define  $g(t) = y_U$  for  $t \in U$ . If  $\delta$  is the minimum of the  $\delta_i$  then  $\delta > 0$  and  $T(t)(g(t)) \geq \delta$  for all  $t \in M$ . If we let  $\theta$  be the measure putting mass  $(\delta \wedge \varepsilon)$  at the point  $g \in C(M, Y)$  then  $\theta$  is non-zero and is an element of  $A$ . Now we return to the proof of the lemma. The set  $A$  is inductively ordered: indeed if  $K$  is a totally ordered subset of  $A$  and we take  $\mu_1 \leq \mu_2 \leq \dots$  from  $K$  such that  $\lim_n \mu_n(C(M, Y)) = \sup_{\mu \in K} \mu(C(M, Y))$  then  $\alpha = \lim_n \mu_n$  is an element of  $A$  and  $\alpha \geq \rho$  for every  $\rho \in K$ . Let  $\theta$  be a maximal element of  $A$ . If  $\theta$  has total mass 1 then  $t\theta = T(t)$  and  $\pi\theta = \mu$ . If  $\theta$  has mass  $\eta < 1$  then we may apply the first part of the proof to the mapping

$$T'(t) = (T(t) - t\theta)/1 - \eta$$

and measure  $\mu' = (\mu - \pi\theta)/1 - \eta$ . This will yield a strictly positive measure  $\theta'$  with  $t\theta' \leq T'(t)$  and  $\pi\theta' \leq \mu'$  and then  $\theta + (1 - \eta)\theta'$  will be an element of  $A$  strictly exceeding  $\theta$ . This completes the proof.

Now we return to the proof of the theorem. For each  $n \geq 0$  let  $F_n$  be a partition of unity on  $X$  subordinate to a cover of diameter less than  $1/n$ . Give  $F_n$  the discrete topology. We take  $F_0$  to be the trivial partition consisting of the function 1. Let  $X_n$  be the subspace of  $F_0 \times \dots \times F_n$  consisting of all  $(g_0, \dots, g_n)$  such that  $g_0 g_1 \dots g_n$  is not identically 0. Let  $\pi_n: X_n \rightarrow X_{n-1}$  be the mapping that sends  $(g_0, \dots, g_{n-1}, g_n)$  to  $(g_0, \dots, g_{n-1})$ . Let

$$G = \{(g_0, g_1, \dots) \in F_0 \times F_1 \times \dots : (g_0, \dots, g_n \in X_n) \text{ for all } n\},$$

and let  $\tilde{G} = C(M, G)$ . Then  $\tilde{G}$  is simply

$$\{(f_0, f_1, \dots) \in C(M, F_0) \times \dots : (f_0, \dots, f_n) \in C(M, X_n) \text{ for all } n\}.$$

$G$  and  $\tilde{G}$  are closed subsets of  $\prod F_i$  and  $\prod C(M, F_i)$  respectively.

Let  $T$  be a continuous mapping from  $M$  into  $P(X)$ . Then for each  $n$ ,  $T$  induces a continuous mapping  $T_n$  of  $M$  into  $P(X_n)$  by the rule  $T_n(t)[(g_0, \dots, g_n)] = \int g_0 g_1 \dots g_n dT(t)$ . Clearly,  $\pi_n T_n(t) = T_{n-1}(t)$ . When  $n = 0$  we have of course the trivial measure  $\mu_0$  putting mass 1 on the one point of  $C(M, X_0)$  so that  $t\mu_0 = T_0(t)$  for all  $t$ . Consequently by repeatedly applying the lemma we obtain a sequence of measures  $\mu_n \in P(C(M, X_n))$  such that  $t\mu_n = T_n(t)$  for all  $t$  and  $\pi_n \mu_n = \mu_{n-1}$  for all  $n$ . By Kolomogorov's Consistency Theorem [1, Th. 5, 11, page 120] there is a measure  $\tilde{\mu}$  in  $P(\tilde{G})$  such that  $P_n \tilde{\mu} = \mu_n$  for all  $n$ . (Here  $P_n$  stands for the natural projection of  $\tilde{G}$  onto  $X_n$  or for any of its other interpretations as a mapping of continuous functions or of measures.)

Since  $X$  is complete we can define a continuous function  $\varphi : G \rightarrow X$  by taking  $\varphi(g_0, g_1, \dots)$  to be the unique point  $x \in X$  such that  $x \in \text{supp}(g_n)$  for all  $n$ . As usual  $\varphi$  may be regarded also as a continuous function from  $\tilde{G}$  to  $C(M, X)$ . Let  $\mu = \varphi \tilde{\mu}$  so that  $\mu \in P(C(M, X))$ . We will complete the proof by showing that  $t\mu = T(t)$  for all  $t$ . Let  $K$  be a closed subset of  $X$ . Write  $K_n$  for  $\{(g_0, \dots, g_n) \in X_n \text{ such that } \text{supp}(g_0 g_1 \dots g_n) \cap K \neq \emptyset\}$  and  $h_n$  for  $\sum g_0 \dots g_n$ , the sum being over all  $(g_0, \dots, g_n) \in K_n$ . Then  $h_n \rightarrow I_K$  boundedly as  $n \rightarrow \infty$  and  $P_n^{-1}(K_n)$  decreases to  $\varphi^{-1}(K)$ . Since  $t\varphi \tilde{\mu} = \varphi t\tilde{\mu}$  and  $t\mu_n = P_n t\tilde{\mu}$  we have

$$\begin{aligned} T(t)(K) &= \lim_n \int h_n dT(t) \\ &= \lim_n \sum_{K_n} \int (g_0 \dots g_n) dT(t) \\ &= \lim_n t\mu_n(K_n) = \lim_n t\tilde{\mu}(P_n^{-1}(K_n)) \\ &= t\tilde{\mu}(\varphi^{-1}(K)) = \varphi t\tilde{\mu}(K) = t\varphi \tilde{\mu}(K) = t\mu(K). \end{aligned}$$

A measure on a metric space is determined by its values on closed sets, so the proof is complete.

## 3.

In this section we will give two corollaries. The first is simply Prohorov's theorem on tightness of compact sets of measures.

**COROLLARY 1.** — *Let  $X$  be complete metric and  $K$  a compact subset of  $P(X)$ . Then for every  $\varepsilon > 0$  there is a compact subset  $K_\varepsilon$  of  $X$  such that  $\nu(K_\varepsilon) \geq 1 - \varepsilon$  for every  $\nu \in K$ .*

*Proof.* — There is a totally disconnected compact space  $M$  and a continuous mapping  $T: M \rightarrow P(X)$  such that  $T(M) = K$ . Such an  $M$  may be constructed by letting  $M$  be the Stone-Cech compactification of the set  $K$  under the discrete topology. The identity map extends continuously over  $M$ , and it is a simple, well known exercise to check that  $M$  has the required properties. Let  $\mu$  be a measure in  $P(C(M, X))$  such that  $t\mu = T(t)$  for all  $t \in M$ . By the regularity of  $\mu$  there is a compact subset  $L_\varepsilon$  of  $C(M, X)$  such that  $\mu(L_\varepsilon) \geq 1 - \varepsilon$ . Then  $K_\varepsilon = \{f(t) : f \in L_\varepsilon, t \in M\}$  satisfies the conclusion of the theorem.

As stated above, it is not possible to find many measures on  $C(M, X)$  if  $M$  is not totally disconnected. The reason for this is that there are not many functions from  $M$  to  $X$  that are continuous. However, our theorem gives some information in this situation, if we allow more functions.

In fact, let  $\mathcal{R}(M)$  be any collection of functions on  $M$ , and let  $rM$  be the set  $M$  with the weakest topology such that each  $f \in \mathcal{R}(M)$  is continuous. Suppose that the Stone-Cech compactification  $\beta rM$  is totally disconnected. If  $T$  is a continuous function from  $rM$  to  $PX$  such that  $T(rM)$  is contained in a compact subset of  $PX$ , then  $T$  may be extended over  $\beta rM$  and our theorem gives a measure on  $C(\beta rM, X)$ . However,  $C(\beta rM, X)$  may be considered as a collection of functions from  $M$  to  $X$ , and by a suitable choice of  $\mathcal{R}(M)$  one gets results such as the next corollary.

**COROLLARY 2.** — *Let  $I$  denote the unit interval, and let  $T$  be a continuous function from  $I$  to  $P(X)$  or more generally a right continuous function from  $I$  to  $PX$  such that  $T(I)$*

is contained in a compact subset of  $P(X)$ . Then there is a regular Borel probability measure  $\mu$  on the space of right continuous functions from  $I$  to  $X$  under the uniform topology such that  $t\mu = T(t)$  for all  $t$  in  $I$ .

*Proof.* — Pick  $\mathcal{R}(I)$  to be the right continuous functions on  $I$ , and proceed as above.

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