ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 19, nº 2 (1969), p. 371-417

<http://www.numdam.org/item?id=AIF_1969__19_2_371_0>

Ann. Inst. Fourier, Grenoble
19, 2 (1969), 371-417.

FLUX IN AXIOMATIC POTENTIAL THEORY II: DUALITY

by Bertram WALSH (1).

0. Introduction.

This paper reports a continuation of the investigations of [21]. There, given a pair (W, $\mathcal{H}$) consisting of a locally compact Hausdorff space W and a complete presheaf $\mathcal{H}$ of vector spaces of continuous scalar-valued functions over W that satisfies the axioms of [4], we constructed a fine resolution of the sheaf (associated with) $\mathcal{H}$ that gave us a way to determine analytically the sheaf cohomology groups of $\mathcal{H}$ (at least in the presence of certain other assumptions). Here, we shall examine the relations between the fine resolution and cohomology groups of [21, § 2] and the theory of adjoint presheaves introduced by R.-M. Hervé in [9]. These relations turn out to be as close as one might reasonably expect to those that subsist in the classical adjoint theory of second-order elliptic differential equations with smooth coefficients, and again indicate the power of the axiomatic approach to the study of these equations.

While this paper is a continuation of [21], not all the results of that paper underlie those of the present one; the notion of normal structure is used only in a note, and § 4 below is the only part of this paper that employs the material of [21, §§ 3 and 4]. The material below is organized as follows: § 1 consi-

(1) Partially supported by NSF GP-11625.

ders the two related topics of patching local adjoint sheaves together to give an adjoint theory in situations where the assumptions of the Hervé adjoint-sheaf theory are satisfied only locally, and replacement of the rather ungainly sheaf $\mathcal{Q}$ of [21] by a very natural-looking sheaf of measures derived from the Hervé integral representation theory. Then § 2 establishes a duality relation between $H^1_K(W, \mathcal{H})$ (the subscript K denoting compact supports) and $\Gamma(W, \mathcal{H}^*) = \mathcal{H}^*_W$, and § 3 investigates more deeply the properties of this duality—notably, the question of whether it is separated. That question is related to certain questions of approximation that have been investigated by A. de la Pradelle, and the most easily verified sufficient conditions for the duality to be separated are related to his work [19]. § 4 considers the consequences of the preceding §§ in the case where both $\mathcal{H}$ and $\mathcal{H}^*$ satisfy the hypothesis of proportionality and the separatedness condition, and relates the flux theory of [21, §§ 3 and 4] to duality and the adjoint sheaf. Finally, § 5 offers notes on the material of its predecessors.

That the notation of this paper will be the same as that of [21] goes without saying, and we shall maintain its standing hypotheses : a pair $(W, \mathcal{H})$ satisfying the Brelot axioms will be fixed throughout the paper, and the base space W will be assumed to possess a countable basis for its topology. W may be compact. Real scalars are used throughout this paper. We feel we should warn the reader that the space of all scalar-valued functions on an open set $U \subseteq W$ that belong to $\mathcal{H}$ may be denoted by $\mathcal{H}_U$, $\Gamma(U, \mathcal{H})$ or $H^0(U, \mathcal{H})$, depending on the emphasis that seems appropriate in a particular situation.

1. A new resolution for $\mathcal{H}$; global adjoint sheaves.

Here we « patch together » local integral representations for local potentials to give a global integral representation theory, thus making it possible to replace the sheaf $\mathcal{Q}$ of [21] by a more tractable sheaf of measures; similarly, we patch together local adjoint sheaves to give a satisfactory adjoint theory in the situation where the essentially local assumptions of the Hervé theory are satisfied, but no global potential necessarily exists on W.

DEFINITION 1.1. — *A region* $V \subseteq W$ *will be called small if* $\mathcal{H}|V$ *possesses a nonzero potential. A set* $A \subseteq W$ *will be called small if it is contained in a small region.*

It is well known and easy to verify that every point of W has a neighborhood V such that $\mathcal{H}|V$ possesses a nonzero potential; consequently, there is a basis for the topology of W consisting of small regions. Incidentally, since there is a nonzero potential on a region U with support all of U whenever there is a nonzero potential on U at all [9, p. 429, N° 1], if a region is a small set then it is a small region.

While the Hervé theory of integral representations of potentials [9] is a global theory, only one of its underlying hypotheses is a global one (in addition to the hypothesis that the topology of W has a countable basis, which is a standing hypothesis in this paper), namely, that any nonzero potentials exist on W at all. Indeed, the Brelot axioms are obviously of a local character, and it is a consequence of the Hervé extension theorem that the hypothesis of proportionality of potentials with common one-point support is a local hypothesis [9, Thm. 16.4, p. 470]. We have just observed, however, that the « global » hypothesis of existence of a nonzero potential is satisfied in neighborhoods of any point in W; thus if we henceforth make the assumption that the hypothesis of proportionality holds locally for $(W, \mathcal{H})$, then we shall know that in any small region in W the entire integral-representation theory of [9] is available for the restriction of $\mathcal{H}$ to that region. We shall now investigate the relations between these integral representations on overlapping regions in W. The following definition will be convenient.

DEFINITION 1.2. — *If* V *is a small region in* W, *by* [9, Thm. 18.1, p. 479] *for each* $y \in V$ *one can choose a potential* $p_y(\bullet)$ *on* V *in such a way that* p_y *has support* $\{y\}$ *and that the function* $y \to p_y(x)$ *is continuous in* $V \setminus \{x\}$ *for each* $x \in V$, *and that the function* $(x, y) \to p_y(x)$ *on* $V \times V$ *is a lower-semicontinuous* $\overline{R}^+$*-valued function, continuous off the diagonal* [9, Prop. 18.1, p. 480]. *Such a function* $(x, y) \to p_y(x)$ *will be called a kernel on* V.

The simplest form of overlapping is containment; if we take a small region V, a subregion U, and a kernel $p_y(\bullet)$ on V, a seemingly reasonable choice of a kernel on U with the « same singularity » at y would be $q_y(\bullet) = p_y(\bullet) - M[p_y]$, where for any function $s \in \overline{\mathcal{H}}_U^+$ we denote by $M[s]$ its greatest harmonic minorant in U. It will be necessary, however, to prove that $q_y(\bullet)$ is a kernel on U; in the absence of other hypotheses, the easiest way to do this seems to be *a posteriori* verification (cf. Lemma 1.12 below, however).

PROPOSITION 1.3. — *With* V, U, *and* $p_y(\bullet)$ *as above, the function* $q_y(\bullet) = p_y(\bullet) - M[p_y]$ *is a kernel on* U.

Proof. — The results of [9] guarantee that kernels exist on U; let $s_y(\bullet)$ be such a kernel. By the Hervé extension theorem [9, Thm. 13.2, pp. 458-459], for each $y \in U$ there is a unique potential $t_y(\bullet)$ on V which differs from $s_y(\bullet)$ on U by a harmonic function. If $y_0 \in U$ and B is a compact neighborhood of y_0 in U, then by [21, Cor. (3.5)] the functions $t_y(\bullet)$ converge uniformly on compacta in $V \setminus B$ to $t_{y_0}(\bullet)$ as $y \to y_0$ in B. Consequently, $(x, y) \to t_y(x)$ is jointly continuous off the diagonal of $V \times U$. Since $t_y(\bullet)$ is a potential on V with support $\{y\}$, the hypothesis of proportionality guarantees the existence of a number $\varphi(y) > 0$ such that $t_y(\bullet) = \varphi(y).p_y(\bullet)$ for $y \in U$; if $y_0 \in U$, B is a compact neighborhood of y_0 and $x_0 \in V \setminus B$, then $\varphi(y) = p_y(x_0)^{-1}.t_y(x_0)$, a fact that shows that $\varphi(y)$ depends continuously on y in B and thus that φ is a continuous function on U. Consequently $\varphi(y)^{-1}.s_y(\bullet)$ is also a kernel on U; but since $\varphi(y)^{-1}.t_y = p_y$, it is also true that $\varphi(y)^{-1}.s_y = q_y$, Q.E.D.

The next proposition shows that the relation between p_y and q_y is inherited by their integrals–which means all the potentials on V.

PROPOSITION 1.4. — *Let* V, U, p_y *and* q_y *be as above. If* p *is a potential on* V *and* λ *is the measure on* V *(which exists by* [9, Thm. 18.2 (2), pp. 481-482]) *for which* $p(\bullet) = \int p_y(\bullet)\, d\lambda(y)$, *then the potential* $p|U - M[p|U]$ *on* U *is given by* $\int q_y(\bullet)\, d[\chi_U.\lambda]$; *if* $\lambda = \chi_U.\lambda$ *(i.e.,* λ *does not* *charge* $V \setminus U$) *then it is also true that*

$$M[p|U] = \int M[p_y](\bullet)\, d\lambda(y).$$

Proof. — If $\lambda_U p$ is the specific restriction of p to U (which, it should be remembered, is also a potential on V), then p and $\lambda_U p$ differ by a harmonic function on U by [9, Thm. 12.2, p. 456]. Consequently we might as well deal with $\lambda_U p$ as with p; but $\lambda_U p = \int p_y\, d[\chi_U.\lambda]\,(y)$ by [9, Thm. 18.3, p. 482], so we also might as well assume that λ puts no mass in $V \setminus U$ and work only with λ.

Suppose further, for the moment, that λ has compact support in U. Let $\{U_n\}_{n=1}^{\infty}$ be an exhaustion of U by regular inner regions, and for any function $s \in \overline{\mathcal{H}}_U^+$ define $M_n[s]$ to be the function equal to s in $U \setminus U_n$ and to $H(s|\partial U_n, U_n)$ in U_n; then $M[s]$ is the decreasing limit of the $M_n[s]$. (The existence of such an exhaustion is guaranteed by [10, § 4, p. 183 ff.] and the existence of a countable base for W.) If $x \in U$ and N is so large that $x \in U_N$ and $\operatorname{Supp} \lambda \subseteq U_N$, then for $n \geqslant N$ we have

$$\begin{aligned} M_n[p](x) &= \int_{\partial U_n} \Big[\int p_y(t)\, d\lambda(y)\Big]\, d\rho_x^{U_n}(t) \\ &= \int \Big[\int_{\partial U_n} p_y(t)\, d\rho_x^{U_n}(t)\Big]\, d\lambda(y) \\ &= \int M_n[p_y](x)\, d\lambda(y) \end{aligned}$$

since $M_n[p_y](x)$ equals the inside integral except for y outside U_n–but $V \setminus U_n$ is a set of λ-measure zero. Taking the limit on $n \geqslant N$ gives

$$M[p](x) = \int M[p_y](x)\, d\lambda(y)$$

by the Lebesgue monotone convergence theorem. Finally, then,

$$\begin{aligned} p(x) - M[p](x) &= \int p_y(x)\, d\lambda(y) - \int M[p_y]\, d\lambda(y) \\ &= \int [p_y(x) - M[p_y](x)]\, d\lambda(y) \\ &= \int q_y(x)\, d\lambda(y). \end{aligned}$$

In the case where $\operatorname{Supp} \lambda$ is not compact, one can always write $\lambda = \sum_{j=1}^{\infty} \lambda_j$ where each measure λ_j does have compact support in U. We then have

$$p(\bullet) = \sum_{j=1}^{\infty} \int p_y(\bullet)\, d\lambda_j(y)$$
$$= \sum_{j=1}^{\infty} \mathrm{M}\left[\int p_y\, d\lambda_j(y)\right](\bullet) + \sum_{j=1}^{\infty} \int q_y(\bullet)\, d\lambda_j(\mathrm{Y}).$$

The first of these sums consists of harmonic functions, the second consists of potentials, and both are bounded above by the superharmonic function $p|\mathrm{U}$. The first sum is thus (by Harnack's principle) a harmonic function and the second sum a potential on U. From the uniqueness of decomposition of the superharmonic function $p|\mathrm{U}$ into its harmonic part and its potential part, it follows that the first sum must be the greatest harmonic minorant on U of $p|\mathrm{U}$, and so we have

$$\mathrm{M}[p|\mathrm{U}] = \sum_{j=1}^{\infty} \int \mathrm{M}[p_y|\mathrm{U}]\, d\lambda_j(y) = \int \mathrm{M}[p_y|\mathrm{U}]\, d\lambda(y)$$

as desired, and

$$\sum_{j=1}^{\infty} q_y(\bullet)\, d\lambda_j(y) = \int q_y(\bullet)\, d\lambda(y)$$

is $p|\mathrm{U} - \mathrm{M}[p|\mathrm{U}]$, Q.E.D.

LEMMA 1.5. — *Let* V_i *and* V_j *be two small regions in* W *for which* $\mathrm{V}_i \cap \mathrm{V}_j \neq \emptyset$; *let* p_y^i *and* p_y^j *be kernels for each of those regions, and let* U *be a region contained in the intersection. Then there exists a uniquely determined continuous positive real-valued function* φ_{ij} *on* U *such that*

$$(1) \qquad q_y^j(x) = \varphi_{ij}(y) \cdot q_y^i(x)$$

for $(x, y) \in \mathrm{U} \times \mathrm{U}$, *where* $q_y^k = p_y^k - \mathrm{M}[p_y^k|\mathrm{U}]$ *is the kernel induced on* U *by* p_y^h *as in the discussion above* $(k = i, j)$. *Equivalently,*

$$(2) \quad p_y^j(x) + \varphi_{ij}(y) \cdot \mathrm{M}[p_y^i|\mathrm{U}](x) = \varphi_{ij}(y) \cdot p_y^i(x) + \mathrm{M}[p_y^j|\mathrm{U}](x)$$

for all $(x, y) \in \mathrm{U} \times \mathrm{U}$, *and in particular* p_y^j *and* $\varphi_{ij}(y) \cdot p_y^i$ *differ by a harmonic function for all* $y \in \mathrm{U}$.

Proof. — There is very little to prove; the existence of a function for which (1) is true is a consequence of [9, p. 480, Remarque], and (2) is equivalent to (1) by the definition of the q^k_y, $k = i, j$, Q.E.D.

THEOREM 1.6. — *Let* $\{V_i\}_{i\in I}$ *be a cover of* W *by small regions. There exists a corresponding set of kernels* $\{p^i_y(\bullet)\}_{i\in I}$ *such that for each ordered pair* (i, j) *of indices with* $V_i \cap V_j \neq \emptyset$ *and each region* $U \subseteq V_i \cap V_j$, *the relations* (1) *and* (2) *of Lemma* 1.5 *above hold with* $\varphi_{ij} \equiv \mathbf{1}$. *In particular* p^i_y *and* p^j_y *differ by a harmonic function in* $V_i \cap V_j$.

Proof. — Begin by choosing some set of kernels $\{P^i_y\}_{i\in I}$ on the respective V_i. Applying (1) of the lemma to any component U of $V_i \cap V_j$, we find that there is a uniquely determined φ_{ij} defined on U for which (1) and (2) of the lemma hold with the p^k_y replaced by the P^k_y, and so φ_{ij} is defined, component-by-component, on all of $V_i \cap V_j$. By interchanging the rôles of i and j we see that $\varphi_{ij}.\varphi_{ji} \equiv \mathbf{1}$, and it is similarly easy to verify that if $V_h \cap V_i \cap V_j \neq \emptyset$, then the 1-cocycle relation $\varphi_{hi}.\varphi_{ij}.\varphi_{jh} \equiv \mathbf{1}$ must hold on that intersection. We thus have a 1-cocycle on $N(\{V_i\}_{i\in I})$ with coefficients in the (complete pre-) sheaf under multiplication of continuous positive real-valued functions on open sets in W. It is well known that this sheaf is fine, and so we can determine a Čech 0-cochain $\{\varphi_i\}_{i\in I}$ of continuous positive real-valued functions on the respective V_i for which $\varphi_{ij} = \varphi_i.\varphi_j^{-1}$ (cf. the argument in [8, Ch. I, Sec. E, p. 31 ff.]). Now (2) of Lemma 1.5 above is equivalent to saying that $\varphi_i(y).P^i_y(\bullet)$ and $\varphi_j(y).P^j_y(\bullet)$ differ by a harmonic function on $V_i \cap V_j$. Consequently the kernels

$$\{p^i_y(\bullet) = \varphi_i(y).P^i_y(\bullet)_{i\in I}\}$$

satisfy the specifications of the theorem, Q.E.D.

DEFINITION 1.7. — *A pair* $(\{V_i\}_{i\in I}, \{p^i_y(\bullet)\}_{i\in I})$ *satisfying the specifications of Theorem* 1.6 *above is called a normalization for* $\mathcal{H}$. *If* ψ *is a continuous positive real-valued function on* W, *then the pair* $(\{V_i\}_{i\in I}, \{\psi(y)^{-1}.p^i_y(\bullet)\}_{i\in I})$, *which clearly also satisfies the specifications of Theorem* 1.6, *is a renormalization of the pair originally given.*

It is clear that normalizations exist–we have given a construction for them–and that the construction is at least ambiguous up to renormalization, which is simply the replacement of the cochain $\{\varphi_i\}_{i\in I}$ of the construction by something cohomologous. The fact that this is all the ambiguity involved in a choice of normalization can be deduced from the following proposition. We have another use for the proposition, and so we leave the classification of ambiguity to the reader.

PROPOSITION 1.8. — *Let* $(\{V_i\}_{i\in I}, \{p_y^i\}_{i\in I})$ *be a normalization, and let* V *be a small region in* W. *Then for any kernel* p_y^o *on* V *there exists a unique continuous positive real-valued function* $\varphi(y)$ *on* V *such that for every point* $y \in V$ *and every* $i \in I$ *with* $y \in V_i$ *there exists a neighborhood of* y *in* $V \cap V_i$ *on which* p_y^i *and* $\varphi(y).p_y^o$ *differ by a harmonic function. Equivalently, there is a kernel* p_y *on* V *such that for every* $y \in V$ *and every* $i \in I$ *with* $y \in V_i$ *there exists a neighborhood of* y *in* $V \cap V_i$ *on which* p_y *differs from* p_y^i *by a harmonic function. Clearly* p_y *is unique.*

Proof. — This is about the same as the proof of Theorem 1.6. For a fixed index i and a fixed region $U \subseteq V \cap V_i$, Lemma 1.5 provides a continuous positive real-valued function $\varphi_{i,U}$ on U such that p_y^i and $\varphi_{i,U}(y).p_y^o$ differ by a harmonic function on U for $y \in U$, and it is easy to see that $\varphi_{i,U}$ does not change if U is replaced by a smaller region. Thus $\varphi_{i,U}$ also does not change if U is replaced by a larger region, and we may think of $\varphi_{i,U} = \varphi_i$ as defined on all of $V \cap V_i$. Given another index j we may repeat the performance; however, it will have to be true that both the difference of p_y^i and $\varphi_i(y).p_y^o$ and the difference of p_y^j and $\varphi_j(y).p_y^o$ can be extended to be harmonic on $V \cap V_i \cap V_j$. But then $\varphi_i(y).p_y^o$ and $\varphi_j(y).p_y^o$ differ by a harmonic function in a neighborhood of y, and so they must be equal. This can happen only if $\varphi_i(y) = \varphi_j(y)$ in that intersection, and so we can unambiguously define φ by setting $\varphi(y) = \varphi_i(y)$ for $y \in V_i$, for each $i \in I$. This function φ clearly satisfies the specifications of the proposition, and $p_y = \varphi(y).p_y^o$ is a kernel that satisfies the specifications given for it, Q.E.D.

DEFINITION 1.9. — *Let* $\mathfrak{L}$ *be the complete presheaf of signed measures on* W *satisfying the following conditions: if* X *is an* *open set in* W, *a measure* μ *in* $\mathcal{M}(X)$ *belongs to* $\mathfrak{L}$ *if and only if for every* $x \in X$ *there is a small neighborhood* V *of* x *and a neighborhood* K *of* x *with* $K \subseteq V$, *such that* $\chi_K . |\mu|$ *has the property that for some kernel* p_y *on* V *the function* $x \to \int p_y(x)\, d[\chi_K . |\mu|](y)$ *is a continuous potential on* V.

As an easy consequence of the fact that whenever the sum of two potentials is continuous both the summands are continuous, we see that if $\int p_y(\bullet)\, d[\chi_K . |\mu|](y)$ is a continuous potential on V, then the same is true if $|\mu|$ is replaced by any $\lambda \in \mathcal{M}^+(X)$ with $\lambda \leqslant k . |\mu|$ for some k. Thus K can be replaced by any smaller set–in particular, by a relatively compact one–and therefore p_y can be replaced by $\psi(y) . p_y$, the most general replacement possible, without changing the meaning of the definition. The fact that K can be shrunk if necessary insures that the defining condition for $\mathfrak{L}$ is a local condition; we also see that $\mathfrak{L}$ is a sheaf of modules over the algebra $\mathfrak{B}$ of bounded Borel functions on W (where f acts on μ by multiplication) and thus *a fortiori* a sheaf of modules over the subalgebra $\mathfrak{S}$ of simple functions.

There is a homomorphism from the presheaf Q of [21, § 2] to $\mathfrak{L}$ which we now proceed to construct. Select some normalization $(\{V_i\}_{i \in I}, \{p_y^i(\bullet)\}_{i \in I})$ for $\mathcal{H}$ on W. For each open region V in W, let $\mathfrak{P}_V$ denote the cone of continuous potentials on V, as in [21, § 2], and let $\mathfrak{Q}_V = \mathfrak{P}_V - \mathfrak{P}_V$. If $\mathfrak{P}_V = \{0\}$, set $Z_V : \mathfrak{Q}_V \to \mathfrak{L}_V$ equal to the zero transformation. Otherwise (i.e., if V is small), for each $p \in \mathfrak{P}_V$ let $Z_V p$ be the unique measure on V for which

$$p(\bullet) = \int p_y(\bullet)\, d[Z_V p](y),$$

where the kernel p_y on V is constructed as in Proposition 1.8 above. The mapping $p \to Z_V p$ is clearly additive and positively homogeneous, so it can be extended uniquely to a linear mapping $Z_V : \mathfrak{Q}_V \to \mathcal{M}(V)$, and it is clear that Z_V takes its values in $\mathfrak{L}_V$. Moreover, Z_V carries the action of specific restriction on $\mathfrak{Q}_V$ to the action of the usual restriction of measures on $\mathfrak{L}_V$: if $Y \subseteq W$ is an open set and $p \in \mathfrak{P}_V$, then [9, Thm. 18.3, pp. 482-483] shows precisely that

$$\lambda_Y[p] = \int p_y(\bullet)\, d[\chi_{Y \cap V} . Z_V p](y)$$

and consequently that the same equation holds if Y is replaced by a closed set K. Thus for any Borel set E, compact K and open Y with $K \subseteq E \subseteq Y$ we have

$$\lambda_K[p] \leqslant \lambda_E[p] \leqslant \lambda_Y[p]$$

and

$$\lambda_K[p] \leqslant \int p_y(\bullet)\, d[\chi_{E \cap V} . Z_V p](y) \leqslant \lambda_Y[p],$$

and taking limits from both sides we get

$$\lambda_E[p] = \int p_y(\bullet)\, d[\chi_{E \cap V} . Z_V p].$$

Suppose $U \subseteq V$ is an open region. By the definition of the linking maps in the presheaf Q [21, Def. (2.7)] and Proposition 1.4 above, we have

$$\begin{aligned} r_{UV}[p] &= p(\bullet) - M[p|U](\bullet) \\ &= \int p_y(\bullet)\, d[Z_V p](y) - \int M[p_y|U](\bullet)\, d[Z_V p](y) \\ &= \int q_y(\bullet)\, d[\chi_U . Z_V p](y) \end{aligned}$$

where, as before, $q_y = p_y|U - M[p_y|U]$, a kernel on U. Since the kernel q_y has the property of Proposition 1.8 above with respect to the region U, the measure $\chi_U . Z_V p$ is $Z_U[r_{UV} p]$ by definition, and so $Z_U \circ r_{UV} = \chi_U . Z_V$. Since multiplication by χ_U is the restriction mapping on measures, this shows that the family of linear maps $\{Z_V : V$ a region in $W\}$ defines a homomorphism of presheaves from Q to $\mathfrak{L}$; we shall denote that homomorphism of presheaves by Z and its corresponding homomorphism of sheaves by ζ. It is clear that ζ preserves the action of specific restriction over Borel sets $E \subseteq W$; equivalently, ζ is a homomorphism of sheaves of $\mathfrak{S}$-modules.

Our interest in $\mathfrak{L}$ and ζ lies precisely in the following fact.

THEOREM 1.10. — *The homomorphism* ζ *is an isomorphism of* $\mathfrak{Q}$ *and* $\mathfrak{L}$.

Proof. — Little is left to prove. The definition of $\mathfrak{L}$ insists that the elements of the stalk $\mathfrak{L}_x$ at $x \in W$ are just the (germs of) measures of the form $Z_V[p^+ - p^-]$, where p^+

is the potential $\int p_y(\bullet)\, d[\chi_K . \mu^+](y)$ and p^- is defined similarly–V being some small region containing the point x. The fact that ζ is 1-1 on stalks is just a restatement of the uniqueness of the representing measure in the Hervé integral representation. Suppose $x \in W$ and V is a region containing x on which there is a function $p_1 - p_2 \in \mathfrak{D}_V$ (with each $p_i \in \mathfrak{P}_V$, $i = 1, 2$) such that $Z_V[p_1 - p_2]$ is a measure whose germ lies in the zero of the stalk $\mathcal{L}_x$. That means precisely that there is a neighborhood U of x in which $Z_V[p_1 - p_2]$ places no mass, and without loss of generality one can take U to be connected. With p_y and q_y being the usual things, that means that

$$r_{UV}[p_1 - p_2] = \int q_y(\bullet)\, d[\chi_U . Z_V[p_1 - p_2]](y) = 0,$$

so $p_1 - p_2$ determines the zero of the stalk $\mathcal{L}_x$. Thus ζ is 1-1 and onto, Q.E.D.

What this theorem means, of course, is that whenever the hypothesis of uniqueness is satisfied (locally) on W, one may forget about the sheaf $\mathcal{Q}$ of [21] : one merely replaces the sheaf $\mathcal{Q}$ by $\mathcal{L}$, the homomorphism Δ of [21, Thm. (2.11)] by $\zeta \circ \Delta$ (and immediately agrees to call the composite Δ), and the resolution $0 \to \mathcal{H} \to \mathfrak{R} \xrightarrow{\Delta} \mathcal{Q} \to 0$ of [21] by a new resolution $0 \to \mathcal{H} \to \mathfrak{R} \xrightarrow{\Delta} \mathcal{L} \to 0$. The new sheaf and homomorphism are much pleasanter to work with, if only in that a sheaf of (germs of) measures is much more susceptible to analysis than a sheaf which can only be treated by dealing with the presheaf that generates it–a presheaf whose linking maps are not « restrictions » in any common sense of the word. It is immediately clear, and useful, that if $p \in \mathfrak{P}_V$ where V is a small region, then $Z_V[p] = \Delta p$ with this new definition of Δ, so the relation between a potential on a small region V and the measure that represents it with respect to a kernel $p_y(\bullet)$ on V satisfying the specifications of Proposition 1.8 above is represented by the satisfyingly classical formula $p = \int p_y(\bullet)\, d[\Delta p](y)$. There is a loss of naturality in the functorial sense, in that the sheaf $\mathcal{Q}$ is determined directly from knowing $\mathcal{H}$ (as is $\mathfrak{R}$), while the construction of $\mathcal{L}$ seems to depend on the normalization $(\{V_i\}_{i \in I}, \{p^i_y\}_{i \in I})$.

However, it is easy to verify that if this normalization is renormalized by a function ψ, the result is to replace $Z_V[p]$ by $\psi(\bullet).Z_V[p]$ for each region V and $p \in \mathfrak{P}_V$; thus the elements of $\mathcal{L}$ are replaced by their multiples by ψ and (since $\mathcal{L}$ is closed under multiplication by [locally] bounded Borel functions, as is easily verified) $\mathcal{L}$ is unchanged. The homomorphism ζ, however, is altered by « postmultiplication by ψ », and that means that after $\mathcal{Q}$ has been identified with $\mathcal{L}$, Δ is identified with « Δ followed by multiplication by ψ ». Thus the constructions for varying normalizations are naturally equivalent, with the equivalence being a multiplicative one.

The usefulness of kernels satisfying the condition of Proposition 1.8 above suggests a formal definition.

Definition 1.11. — *If* $V \subseteq W$ *is a small region and* $(\{V_i\}_{i \in I}, \{p_y^i\}_{i \in I})$ *is a normalization of* $\mathcal{H}$ *on* W, *then a kernel* p_y *on* V *satisfying the condition of Proposition* 1.8 *above that* $p_y - p_y^i$ *have a harmonic extension to a neighborhood of* y *for each* $i \in I$ *with* $y \in V_i$ *will be called a normalized kernel (with respect to the given normalization).*

In addition to the assumption that the hypothesis of proportionality holds (locally, if you wish) for $\mathcal{H}$, the theory of adjoint sheaves makes the assumption that there is a basis $\mathfrak{D}$ for the topology of W composed of « completement determinant » (henceforth, *c.d.*) domains. This again is a local hypothesis, and if we make it (as we shall, henceforth, for the rest of this paper) for $(W, \mathcal{H})$ the entire adjoint-sheaf theory of [9, Ch. vi] is available for the restriction of $\mathcal{H}$ to any small domain in W. However, one gets different sheaves for different small regions (and for different choices of kernels on those regions); one needs to know the way in which these sheaves are interrelated.

To this end, let V be a small region, $U \subseteq V$ a subregion, and $p_y(\bullet)$ a kernel on V; let $q_y = p|U - M[p|U]$ as in 1.3 through 1.6 above, let $\{U_n\}_{n=1}^{\infty}$ be an exhaustion of U by regular inner regions, and let $M_n[\bullet]$ have the same meaning for superharmonic nonnegative functions on U as it did in 1.4 above. Using the kernel p_y we can define an adjoint sheaf on V, which we shall denote by $(\mathcal{H}|V)^*$ since there is no danger of ambiguity as yet; similarly, we shall denote by $(\mathcal{H}|U)^*$ the adjoint sheaf defined on U by the kernel q_y.

LEMMA 1.12. — *With* U, V, p_y *and* q_y *as above, the function* $(x, y) \to M[p_y](x)$ *is jointly continuous on* $U \times U$; *moreover, it belongs to* $\Gamma(U, \mathcal{H})$ *in the first variable for fixed values of the second and to* $\Gamma(U, (\mathcal{H}|V)^*)$ *in the second variable for fixed values of the first.*

Proof. — The function $(x, y) \to M[p_y](x)$ is the decreasing limit of the functions $(x, y) \to M_n[p_y](x)$ pointwise on $U \times U$. Select a compact $K \subseteq U$, and think of (x, y) as restricted to $K \times K$. If N is a sufficiently large number that $K \subseteq U_N$, then $M_N[p_y](x) = \int p_y(t)\, d\rho_x^{U_N}(t)$ where ρ denotes as usual the representing measure on ∂U_N for $\mathcal{H}$. Since $(t, y) \to p_y(t)$ is jointly continuous on $\partial U_N \times K$, there is a constant $k \geqslant p_y(t)$ for all $(t, y) \in \partial U_N \times K$; thus if $h(x) = \int k\, d\rho_x^{U_N}$, then for all $n \geqslant N$ the inequality $M_n[p_y](x) \leqslant M_N[p_y](x) \leqslant h(x)$ holds for all $(x, y) \in U_N \times K$. By Harnack's principle, the family

$$\{M_n[p_y] : n \geqslant N,\ y \in K\}$$

is an equicontinuous family of harmonic functions on U_N; consequently the family of pointwise limits

$$\{M[p_y](\bullet) = \lim_n M_n[p_y](\bullet) : y \in K\}$$

is also equicontinuous and harmonic on U_N. It thus suffices to show that $y \to M[p_y](x_0)$ is continuous for each $x_0 \in K$ in order to make $(x, y) \to M[p_y](x)$ jointly continuous on $K \times K$. But $t \to (y \to p_y(t))$ is a continuous $\mathcal{C}(U_n)$-valued function defined on ∂U_n, taking its values in the closed subspace $\Gamma(U_n, (\mathcal{H}|V)^*) \subseteq \mathcal{C}(U_n)$ by [9, Example, p. 537]–and for $x_0 \in U_n$, $y \to M_n[p_y](x_0) = \int p_y(t)\, d\rho_{x_0}^{U_n}(t)$ is just the vector integral of this $\Gamma(U, (\mathcal{H}|V)^*)$-valued function, whence it too belongs to $\Gamma(U, (\mathcal{H}|V)^*)$. Consequently the limit function $y \to M[p_y](x_0)$ belongs to $\Gamma(U, (\mathcal{H}|V)^*)$ by the Harnack principle for the adjoint sheaf [9, Cor. p. 540]. This establishes that the function is adjoint-harmonic in y for fixed x and *a fortiori* proves the separate continuity that was needed to make $(x, y) \to \mathrm{M}[p_y](x)$ jointly continuous, Q.E.D.

We insert the following lemma now, though we shall not need it for a while, because its proof is so similar to that of 1.12 above. Indeed, we only sketch the proof.

LEMMA 1.13. — *Let* U, V, *and* p_y *be as above. Suppose* $(x, y) \to s_y(x)$ *is a function defined and continuous off the diagonal of* $\mathrm{U} \times \mathrm{U}$, *such that*

(1) *for each* $y \in \mathrm{U}$, $p_y(\bullet) - s_y(\bullet)$ *can be extended (i.e., defined at* y*) in such a way as to belong to* $\Gamma(\mathrm{U}, \mathscr{H})$;

(2) *for each* x, $y \to s_y(x)$ *is in* $\Gamma(\mathrm{U}, (\mathscr{H}|\mathrm{V})^*)$. *Then*

(*A*) *for each* x, $y \to p_y(x) - s_y(x)$ *has an extension to* x *belonging to* $\Gamma(\mathrm{U}, (\mathscr{H}|\mathrm{V})^*)$;

(*B*) *the extended function* $(x, y) \to p_y(x) - s_y(x)$ *is jointly continuous on* $\mathrm{U} \times \mathrm{U}$;

(*C*) *if* λ *is a measure of compact support in* U, *then* $\int p_y(\bullet)\, d\lambda(y)$ *and* $\int s_y(\bullet)\, d\lambda(y)$ *differ by a harmonic function in* U.

Proof sketch. — Since all the conclusions given above are of a local character and U has an exhaustion by regular inner regions which must eventually contain $\mathrm{Supp}\,\lambda$, there is no loss of generality in giving a proof for inner-regular U. Condition (1) above then shows that

$$s_y - \mathrm{M}[s_y] = q_y = p_y - \mathrm{M}[p_y]$$

for each $y \in \mathrm{U}$; thus (A) and (B) will both follow if one can show that $(x, y) \to \mathrm{M}[s_y](x)$ is jointly continuous and harmonic and $*$-harmonic in x and y separately. But this is even easier to see than it was for p_y in 1.12, because one can write $\mathrm{M}[s_y](x) = \int s_y(t)\, d\rho_x^{\mathrm{U}}(t)$ and use the joint continuity and separate harmonicity of $s_y(x)$ with no need to take a limit over an exhaustion. We therefore omit the details. Similarly,

$$\begin{aligned}\int s_y(x)\, d\lambda(y) &= \int q_y(x)\, d\lambda(y) + \int \mathrm{M}[s_y](x)\, d\lambda(y)\\ &= \int p_y(x)\, d\lambda(y) + \int [\mathrm{M}[s_y](x) - \mathrm{M}[p_y](x)]\, d\lambda(y)\end{aligned}$$

and checking that the second integral is harmonic is again simply a matter of looking at it as the vector integral of the $\Gamma(U, \mathcal{H})$-valued function $y \to (M[s_y] - M[p_y])$ with respect to λ. This suffices to prove (C), Q.E.D.

The next proposition is precisely that the two competing ways to define the adjoint-harmonic functions on U are consistent.

PROPOSITION 1.14. — *If* $(\mathcal{H}|V)^*$ *is the adjoint sheaf formed on* V *using* p_y *and* $(\mathcal{H}|U)^*$ *is the adjoint sheaf formed on* U *using* q_y, *then* $(\mathcal{H}|V)^*|U = (\mathcal{H}|U)^*$.

It is convenient to separate one part of the proof as the following lemma.

LEMMA 1.15. — *Let* K *be a compact subset of* U *and* X *be a relatively compact open set with* $K \subseteq X \subseteq \overline{X} \subseteq U$. *Let* s *be a potential on* V *whose support is contained in* K. *Denote the operation of forming the reduced function (« reduite ») on* V *and* U *by the letters* R *and* P *respectively. Then* $\hat{R}_s^{V\setminus X}|U = \hat{P}_{s|U}^{V\setminus X}$.

Proof. — Set $\mathfrak{T}_V = \{t_V \in \overline{\mathcal{H}}_V^+ : t_V|V\setminus X = s|V\setminus X$ and $t_V|X \leqslant s|X\}$ and $\mathfrak{T}_U = \{t_U \in \overline{\mathcal{H}}_V^+ : t_U|U\setminus X = s|U\setminus X$ and $t_U|X \leqslant s|X\}$; then $R_s^{V\setminus X} = \inf \mathfrak{T}_V$ and $P_{s|U}^{U\setminus X} = \inf \mathfrak{T}_U$. Any such superharmonic function t_V clearly restricts to such a t_U; on the other hand, any such t_U can be extended in a canonical way to the rest of V, since one may simply set it equal to s on $V\setminus U$. (The resulting extension will then belong to $\mathfrak{T}_V$.) Thus the infima that define R and P are the same, so the values of R and P are the same on U, and this equality passes over to the lower-semicontinuous regularizations since U is open.

Proof of 1.14. — Begin by making the observation that all elements of $\mathfrak{T}_V$ above have the greatest harmonic minorant $M[s|U]$ on U, since their greatest harmonic minorants can be calculated using an exhaustion of U all of whose elements contain $\overline{X}$–and all elements of $\mathfrak{T}_V$ equal s outside $\overline{X}$. Consequently, $\hat{P}_{(s|U-M[s|U])}^{U\setminus X} = \hat{P}_{s|U}^{U\setminus X} - M[s|U]$, because if $t_U \in \mathfrak{T}_U$ then

$$t_U - M[s|U] = t_U - M[t_U] \geqslant 0$$

and so $t_U - M[s|U]$ is a competitor for the infimum that defines $P^{U\setminus X}_{(s|U-M[s|U])}$; this makes $P^{U\setminus X}_{(s|U-M[s|U])} + M[s|U] \leqslant P^{U\setminus X}_{s|U}$, but the reverse inequality is obvious, as is the fact that the equality passes over to the lower-semicontinuous regularizations. Now if X is a c.d. open set with compact closure contained in U, let $y \in X$ be fixed and let σ^X_y and τ^X_y respectively denote the measures defined in [9, Def. 1, p. 537] relative to $(\mathcal{H}|V)^*$ and p_y and relative to $(\mathcal{H}|U)^*$ and q_y respectively; that is, $\hat{R}^{V\setminus X}_{p_y}(\bullet) = \int p_z(\bullet)\, d\sigma^X_y(z)$ in V and $\hat{P}^{U\setminus X}_{q_y}(\bullet) = \int q_z(\bullet)\, d\tau^X_y(z)$ in U, respectively. By 1.15 above, we know that

$$\hat{R}^{V\setminus X}_{p_y}(x) = \hat{P}^{U\setminus X}_{p_y|U}(x)$$

for all $x \in U$. The measure σ^X_y has compact support contained in U; therefore for all $x \in U$

$$\begin{aligned}
\hat{R}^{V\setminus X}_{p_y}(x) &= \int p_z(x)\, d\sigma^X_y(z) \\
&= \int q_z(x)\, d\sigma^X_y(z) + \int M[p_z](x)\, d\sigma^X_y(z) \\
&= \int q_z(x)\, d\sigma^X_y(z) + M\Big[\int p_z(\bullet)\, d\sigma^X_y(z)\Big](x) \\
&= \int q_z(x)\, d\sigma^X_y(z) + M[\hat{R}^{V\setminus X}_{p_y}|U](x) \\
&= \int q_z(x)\, d\sigma^X_y(z) + M[p_y|U](x),
\end{aligned}$$

the third equality being a consequence of 1.4 above and the last a consequence of the fact that p_y is supported in X. On the other hand, for all $x \in U$ we have

$$\begin{aligned}
\hat{P}^{U\setminus X}_{p_y|U}(x) &= \hat{P}^{U\setminus X}_{(p_y|U-M[p_y|U])}(x) + M[p_y|U](x) \\
&= \hat{P}^{U\setminus X}_{q_y}(x) + M[p_y|U](x) \\
&= \int q_z(x)\, d\tau^X_y(z) + M[p_y|U](x).
\end{aligned}$$

Combining those two relations with that of 1.15 gives

$$\int q_z(\bullet)\, d\sigma^X_y(z) = \int q_z(\bullet)\, d\tau^X_y(z)$$

on U, and by uniqueness of integral representations of potentials on U with respect to the kernel $z \to q_z(\bullet)$, we have $\sigma^X_y = \tau^X_y$. Since those are the representing measures for the regular* set X with respect to the adjoint sheaves $(\mathcal{H}|V)^*$ and $(\mathcal{H}|U)^*$ respectively and X and y were arbitrary, the restriction of the former sheaf to U is the latter, Q.E.D.

It is quite easy to see what this fact implies in the situations considered in 1.5 and 1.6 above. (The number (3) for the condition is chosen to continue the numbering of 1.5.)

THEOREM 1.16. — *If* V_i *and* V_j *are two small regions in* W *with* $V_i \cap V_j \neq \emptyset$ *and* p_y^i *and* p_y^j *are two kernels on* V_i *and* V_j *respectively, and if* φ_{ij} *is the function of* 1.5 *above, then*

$$(3) \qquad (\mathcal{H}|V_j)^*|(V_i \cap V_j) = \varphi_{ij}.(\mathcal{H}|V_i)^*|(V_i \cap V_j)$$

where $(\mathcal{H}|V_k)^*$ *is the adjoint sheaf formed on* V_k *using the kernel* p_y^k, $k = i, j$. *In particular, if* $(\{V_i\}_{i\in I}, \{p_y^i\}_{i\in I})$ *is a normalization of* $\mathcal{H}$ *on* W, *then*

$$(3') \qquad (\mathcal{H}|V_j)^*|(V_i \cap V_j) = (\mathcal{H}|V_i)^*|(V_i \cap V_j)$$

for any indices i *and* j.

Proof. — There is little left to prove: given any region $U \subseteq V_i \cap V_j$, we know by 1.16 above that $(\mathcal{H}|V_k)^*|U$ is the adjoint sheaf induced on U by the kernel q_y^k, $k = i, j$, where the kernel q_y^k is $p_y^k - M[p_y^k|U]$ as usual. Since $q_y^j(\bullet) = \varphi_{ij}(y).q_y^i(\bullet)$ by 1.6 above, we have (3) by [9, Remarque, p. 536]; and (3′) is a special case of (3), Q.E.D.

DEFINITION 1.17. — *Given a normalization* $(\{V_i\}_{i\in I}, \{p_y^i\}_{i\in I})$ *of* $\mathcal{H}$ *on* W, *the global adjoint sheaf it defines is the (complete pre-) sheaf of vector spaces of continuous scalar-valued functions* $\mathcal{H}^*$ *defined by the condition that for any open* $X \subseteq W$, $f \in \mathcal{C}(X)$ *belongs to* $\mathcal{H}^*_X$ *if and only if*

$$f|V_i \cap X \in (\mathcal{H}|V_i)^*_{V_i\cap X}$$

for each $i \in I$ *for which* $V_i \cap X \neq \emptyset$.

By virtue of 1.16 above, this is a well-defined complete presheaf whose restriction to any V_i, $i \in I$, is precisely the adjoint sheaf induced on V_i by p_y^i.

The following proposition is frequently useful. We omit its proof, because the proof is essentially the same as that of 1.16 above.

PROPOSITION 1.18. — *Let a normalization of* $\mathcal{H}$ *on* W *and the global adjoint sheaf* $\mathcal{H}^*$ *it defines as in* 1.17 *above be given. If* V *is a small region in* W *and* p_y *is the normalized kernel on* V *given by* 1.8 *above, then the adjoint sheaf induced on* V *by* p_y *is precisely* $\mathcal{H}^*|\mathrm{V}$.

It is easy to see that the ambiguity in the definition of the global adjoint sheaf is the same ambiguity that is present in the definition of the sheaf homomorphism ζ of the discussion preceding 1.10 above. That is, if $(\{\mathrm{V}_i\}_{i\in\mathrm{I}}, \{p_y^i\}_{i\in\mathrm{I}})$ is renormalized by multiplication by $\psi^{-1}(y)$ as in 1.7 above, then $\mathcal{H}^*$ is replaced by the multiplicatively equivalent sheaf $\psi^{-1}.\mathcal{H}^*$. This ambiguity will in general be a source of no concern, since in everything that follows we shall have begun by choosing and fixing a normalization of $\mathcal{H}$ on W. However, there will be one special case in § 4 below in which it will be desirable and natural to replace $\mathcal{H}^*$ by a certain multiplicatively equivalent sheaf, and it is desirable to note at this point that this replacement is effected simply by renormalizing $\mathcal{H}$.

2. The fundamental duality relation.

This section will be concerned with the analytical details of establishing a duality relation between the spaces $\mathrm{H}^1_{\mathrm{K}}(\mathrm{W}, \mathcal{H})$ (the K denoting compact supports, as it will in all contexts henceforth) and $\mathrm{H}^0(\mathrm{W}, \mathcal{H}^*) = \mathcal{H}^*_{\mathrm{W}}$; the question of whether this duality is separated, and the consequences of its being separated, will be dealt with in the next section. We shall assume throughout that a normalization of $\mathcal{H}$ on W has been chosen and fixed, and denote by $\mathcal{H}^*$ the global adjoint sheaf defined by that normalization. We shall also regularly use p_y to denote the unique normalized kernel on a given small region V, as constructed in 1.8 above.

To begin, we give the following useful lemma.

LEMMA 2.1. — *Let* V *be a small region and* p_y *the normalized kernel thereon. If* $\lambda \in \Gamma_{\mathrm{K}}(\mathrm{V}, \mathcal{L})$ *(i.e.,* λ *is a measure in* $\mathcal{L}$ *of compact support in* V*) then there is a difference of potentials* $q \in \mathfrak{Q}_{\mathrm{V}}$ *for which* $\Delta q = \lambda$. *If* $g \in \Gamma_{\mathrm{K}}(\mathrm{V}, \mathcal{R})$ *(i.e.,* g *is a* *function of compact support in* V *which belongs to* $\mathfrak{R}$) *then* $g = \int p_y(\bullet)\, d[\Delta g](y)$.

Proof. — To prove the first assertion it will suffice to show that if λ is positive, then $p = \int p_y(\bullet)\, d\lambda(y)$ is continuous; for p is a potential, and if it is continuous then $\Delta p = \lambda$. Given $x_0 \in \mathrm{V}$, by definition of $\mathfrak{L}$ there is a connected open neighborhood U of x_0 and a smaller neighborhood Y of x_0 for which $p_0 = \int q_y(\bullet)\, d[\chi_{\mathrm{Y}}.\lambda](y)$ is a continuous potential on U. (Here and below q_y is the unique normalized kernel on U, which [M denoting as usual the operator that takes the greatest harmonic minorant of nonnegative superharmonic functions on U] is just $p_y - \mathrm{M}[p_y]$.) By 1.4 above and [9, Thm. 18.3, p. 482] the specific restriction of p to Y is $\lambda_{\mathrm{Y}} p = \int p_y(\bullet)\, d[\chi_{\mathrm{Y}}.\lambda](y)$ and

$$\lambda_{\mathrm{Y}} p - \mathrm{M}[\lambda_{\mathrm{Y}} p \,|\, \mathrm{U}] = \int q_y(\bullet)\, d[\chi_{\mathrm{Y}}.\lambda](y) = p_0$$

on U. Thus $p = \lambda_{\mathrm{V}\setminus\mathrm{Y}} p + p_0 + \mathrm{M}[\lambda_{\mathrm{Y}} p \,|\, \mathrm{U}]$, which shows that p is continuous at the point x_0, which was arbitrary.

For the second assertion, observe that we already know that $\int p_y(\bullet)\, d[\Delta g](y)$ has the same Laplacian as g. Consequently the function $g - \int p_y(\bullet)\, d[\Delta g](y)$ has zero Laplacian on V, and so is harmonic on V. On the other hand, since g has compact support and the potential $p = \int p_y(\bullet)\, d|\Delta g|(y)$ on V has no zeros, one can easily see that

$$g - \int p_y(\bullet)\, d[\Delta g](y)$$

is both majorized and minorized by appropriate scalar multiples of p. Consequently, $g - \int p_y(\bullet)\, d[\Delta g](y) = 0$, Q.E.D.

We should perhaps remark that the adjoint–sheaf machinery, and the hypothesis of a basis for the topology of W consisting of c.d. sets, was not used in 2.1 above.

PROPOSITION 2.2. — *Let* V *be a small region,* $\mathrm{U} \subseteq \mathrm{V}$ *an open set that is relatively compact in* V, *and let* $\mu \in \mathcal{M}^+(\mathrm{U})$ *be a measure of finite total mass. Let* σ_y^{U} *be the measure (supported by* $\partial\mathrm{U}$) *for which* $\hat{\mathrm{R}}_{p_y}^{\mathrm{V}\setminus\mathrm{U}} = \int p_y\, d\sigma_y^{\mathrm{U}}$.

Then

(1) *the measure-valued function* $y \to \sigma_y^U$ *is (scalarly) integrable with respect to* μ;

(2) *if* $\lambda = \int \sigma_y^U \, d\mu(y)$, *then for any nonnegative lower-semicontinuous function* f *on* ∂U, $\int^* f \, d\lambda = \int^* H^*(f) \, d\mu$, *where* $H^*(f)$ *is the upper (Perron) solution of the Dirichlet problem for* $\mathcal{H}^*$ *on* U;

(3) *if* μ *has compact support and* U *is regular for* $\mathcal{H}$, *then* $\lambda \in \Gamma_K(V, \mathcal{L})$;

(4) *if* μ *has compact support and belongs to* $\mathcal{L}$, *then the integral* $p = \int p_y \, d\mu(y)$ *defines a continuous potential on* V *with* $\hat{R}_p^{V \setminus U} = \int p_z \, d\lambda(z)$;

(5) *if* $g \in \Gamma(V, \mathcal{R})$ *and* $\operatorname{Supp} g \subseteq U$ *is compact, then* $\int H^*(f) \, d[\Delta g] = 0$ *for any nonnegative lower-semicontinuous* f *on* ∂U *which is integrable with respect to* $*$*-harmonic measure. In particular,* $\int f \, d[\Delta g] = 0$ *for any* $f \in \Gamma(V, \mathcal{H}^*)$.

Proof. — Regardless of whether U is regular for $\mathcal{H}^*$, σ_y^U is the representing measure for $\mathcal{H}^*$ at y in U [9, Cor. 1, p. 549], so for any lower-semicontinuous nonnegative f on ∂U, $H^*(f)(y) = \int^* f(t) \, d\sigma_y^U(t)$; since there exist positive $*$-harmonic functions defined in neighborhoods of the compact set $\overline{U}$, it is easy to see that the finite-total-mass assumption on μ is necessary and sufficient for $y \to \sigma_y^U$ to be scalarly integrable. That proves (1); (2) is just [17, Cor. p. 19], because it says that $\int^* f \, d\left[\int \sigma_y^U \, d\mu(y)\right] = \int^* \left[\int^* f \, d\sigma_y^U\right] d\mu(y)$. (Essential upper integrals do not enter the picture because all topological spaces present have countable bases.) To prove (3), observe that the Harnack principle for $\mathcal{H}^*$ implies the existence of a constant k for which $\sigma_y^U \leqslant k.\sigma_{y_0}^U$ holds for all $y \in \operatorname{Supp} \mu$ with respect to a fixed $y_0 \in \operatorname{Supp} \mu$, because $\operatorname{Supp} \mu$ is assumed to be compact. One thus has $k.\sigma_{y_0}^U = (k.\sigma_{y_0}^U - \sigma_y^U) + \sigma_y^U$ for $y \in \operatorname{Supp} \mu$, with both these measures being positive and scalarly continuously dependent on y. For any fixed $x \in V$

we thus have

$$\begin{aligned}\|\mu\| .k.\hat{R}_{p_{y_0}}^{V\setminus U}(x) &= k.\int p_z(x)\, d\sigma_{y_0}^U(z).\int d\mu(y)\\ &= \int p_z(x)\, d\Big[\int (k.\sigma_{y_0}^U - \sigma_y^U)\, d\mu(y)\Big](z)\\ &\quad+ \int p_z(x)\, d\lambda(z)\end{aligned}$$

by the same Bourbaki corollary cited earlier, because $z \to p_z(x)$ is nonnegative and lower-semicontinuous on V. Since the left side of this equation is a continuous potential (because U was taken to be regular for $\mathcal{H}$) and both terms on the right side are potentials, both terms on the right are continuous; in particular $\int p_z(\bullet)\, d\lambda(z)$ is continuous and its Laplacian λ belongs to $\Gamma_K(V, \mathfrak{L})$. (4) follows from the fact that by [9, Cor. 2, p. 552]

$$\hat{R}_p^{V\setminus U}(\bullet) = \int \hat{R}_{p_y}^{V\setminus U}(\bullet)\, d\mu(y)$$

while by definition of σ_y^U and that same Bourbaki corollary

$$\begin{aligned}\int \hat{R}_{p_y}^{V\setminus U}(\bullet)\, d\mu(y) &= \int \Big[\int p_z(\bullet)\, d\sigma_y^U(z)\Big]\, d\mu(y)\\ &= \int p_z(\bullet)\, d\Big[\int \sigma_y^U\, d\mu(y)\Big](z)\\ &= \int p_z(\bullet)\, d\lambda(z).\end{aligned}$$

Finally, to prove (5) let μ^+ and μ^- be the positive and negative parts respectively of Δg. Since by 2.1 above

$$g = \int p_y\, d\mu^+(y) - \int p_y\, d\mu^-(y),$$

the potentials $p^\pm = \int p_y\, d\mu^\pm(y)$ are equal everywhere but on the compact support of g. Consequently $\hat{R}_{p^+}^{V\setminus U} = \hat{R}_{p^-}^{V\setminus U}$ everywhere in V. Set $\lambda^+ = \int \sigma_y^U\, d\mu^+(y)$ and define λ^- similarly; then by (4) above

$$\int p_z(\bullet)\, d\lambda^+(z) = \hat{R}_{p^+}^{V\setminus U} = \hat{R}_{p^-}^{V\setminus U} = \int p_z(\bullet)\, d\lambda^-(z)$$

and so $\lambda^+ = \lambda^-$. Thus for any nonnegative lower semicontinuous f on ∂U we have

$$\begin{aligned}\int H^*(f)\, d[\Delta g] &= \int H^*(f)\, d\mu^+ - \int H^*(f)\, d\mu^-\\ &= \int f\, d\lambda^+ - \int f\, d\lambda^- = 0, \quad \text{Q.E.D.}\end{aligned}$$

COROLLARY 2.3. — *Every element of the dual space of* $\Gamma(\mathrm{W}, \mathcal{H}^*)$ *(topologized by uniform convergence on compacta) can be realized in the form* $h^* \to \int h^*\, d\lambda$, *where* λ *belongs to* $\Gamma_{\mathrm{K}}(\mathrm{W}, \mathfrak{L})$. *Moreover, the space* $\Delta[\Gamma_{\mathrm{K}}(\mathrm{W}, \mathfrak{R})]$ *is contained in the annihilator of* $\Gamma(\mathrm{W}, \mathcal{H}^*)$, *so there is a surjection of* $\mathrm{H}^1_{\mathrm{K}}(\mathrm{W}, \mathcal{H}) = \Gamma_{\mathrm{K}}(\mathrm{W}, \mathfrak{L})/\Delta[\Gamma_{\mathrm{K}}(\mathrm{W}, \mathfrak{R})]$ *onto the dual of* $\Gamma(\mathrm{W}, \mathcal{H}^*)$.

Proof. — Since $\Gamma(\mathrm{W}, \mathcal{H}^*)$ is topologized as a subspace of $\mathcal{C}(\mathrm{W})$, the Hahn-Banach theorem guarantees that every element of $\Gamma(\mathrm{W}, \mathcal{H}^*)$, can be given in the form $h^* \to \int h^*\, d\mu$, where μ is a measure (far from uniquely determined, of course) of compact support on W. Since any measure of compact support can be written as a linear combination of measures of small support, we may as well assume that μ and V are as in the proposition, and U is an inner-regular set for $\mathcal{H}$ that contains Supp μ. But then (3) of the proposition produces a measure $\lambda \in \Gamma_{\mathrm{K}}(\mathrm{V}, \mathfrak{L})$ for which $\int h^*\, d\mu = \int h^* d\lambda$ when $h^* \in \Gamma(\mathrm{V}, \mathcal{H}^*)$. A similar argument produces the second assertion of the corollary: any element of $\Gamma_{\mathrm{K}}(\mathrm{W}, \mathfrak{R})$ can be written as a sum of elements with small compact supports, so it suffices to show that Δg annihilates $\Gamma(\mathrm{V}, \mathcal{H}^*)$ when $g \in \Gamma_{\mathrm{K}}(\mathrm{V}, \mathfrak{R})$, which is (5) of the proposition, Q.E.D.

We now have a pairing of

$$\Gamma(\mathrm{W}, \mathcal{H}^*) \qquad \text{and} \qquad \Gamma_{\mathrm{K}}(\mathrm{W}, \mathfrak{L})/\Delta[\Gamma_{\mathrm{K}}(\mathrm{W}, \mathfrak{R})]$$

that identifies the dual of the Fréchet space $\Gamma(\mathrm{W}, \mathcal{H}^*)$ in a natural way with a quotient of $\mathrm{H}^1_{\mathrm{K}}(\mathrm{W}, \mathcal{H})$, and it is reasonable to look for a topology on $\mathrm{H}^1_{\mathrm{K}}(\mathrm{W}, \mathcal{H})$ the dual of which can be identified under this pairing with $\Gamma(\mathrm{W}, \mathcal{H}^*)$. (The weak topology induced by the pairing, an obvious choice, is not yet seen to be sufficiently natural from the point of view of $\mathcal{H}$.) A good candidate for such a topology can be generated by recalling the connection between Čech cohomology with compact supports and coefficients in $\mathcal{H}$ and the cohomology derived from the resolution $0 \to \mathcal{H} \to \mathfrak{R} \xrightarrow{\Delta} \mathfrak{L} \to 0$. Suppose (A, U) is a Cousin pair in the sense of [21, § 3], such that $\overline{\mathrm{U}}$ is compact and small. Then given $s \in \mathcal{H}_{\mathrm{U} \setminus \mathrm{A}}$, one can find functions $g_s \in \mathfrak{R}_{\mathrm{U}}$ and $f_s \in \mathfrak{R}_{\mathrm{W} \setminus \mathrm{A}}$ such that g_s and f_s depend linearly on s and $s = g_s - f_s$ in $U \setminus A$. For example, suppose that $1 \in \mathcal{H}_V$, so $\mathcal{R}_V$ is a ring admitting partitions of unity [21, Prop. (2.2) and Cor. (2.4)], and let $h_1, h_2 \in \mathcal{R}_V$ be such that $\operatorname{Supp} h_1 \subseteq U$, $\operatorname{Supp} h_2 \subseteq W \setminus A$, both are nonnegative and the sum $h_1 + h_2 = 1$ on W. Set $g_s = s.h_2$ in $U \setminus A$ and 0 in A, and set $f_s = -s.h_1$ in $U \setminus A$ and 0 in $W \setminus U$. It is easy to see that g_s and f_s belong to the desired spaces, and clearly $s = g_s - f_s$ in $U \setminus A$. In the general situation where 1 may not be harmonic on V, one can go through this construction in $h^{-1}.(\mathcal{H}|V)$ and $h^{-1}.(\mathcal{H}|V)$ where $h \in \mathcal{H}_V$ is a positive harmonic function (existing by [9, Thm. 16.1, p. 468]), then multiply by h again to yield the desired result. It is clear that $s = g_s - f_s$ in $U \setminus A$, and so one can extract an element of $\Gamma_K(W, \mathcal{L})$ from s by setting $\mu = \Delta f_s$ in $W \setminus A$ and $\mu = \Delta g_s$ in U; these measures surely belong to $\mathcal{L}$, and they are the same measure in $U \setminus A$ because $g_s - f_s = s$ is harmonic in $U \setminus A$. Since $f_s = 0$ outside U, μ has compact support. For fixed h_1, h_2 the construction of f_s and g_s is linear in s, so we have a linear map $\tilde{j}_{(A,U)} : \mathcal{H}_{U \setminus A} \to \Gamma_K(W, \mathcal{L})$ and thus a linear map $j_{(A,U)} : \mathcal{H}_{U \setminus A} \to H^1_K(W, \mathcal{H})$. If (A_1, U_1) is a refinement of (A, U) and $s = g_1 - f_1$ in $U_1 \setminus A_1$ where $g_1 \in \mathcal{R}_{U_1}$ and $f_1 \in \mathcal{R}_{W \setminus A_1}$, then one can define a measure $\mu_1 \in \Gamma_K(W, \mathcal{L})$ by taking $\mu_1 = \Delta f_1$ in $W \setminus A_1$ and $\mu_1 = \Delta g_1$ in U_1, as before; but the function φ defined as $g_s - g_1$ in U_1 and $f_s - f_1$ in $W \setminus A_1$ is a well-defined element of $\Gamma_K(W, \mathcal{R})$, since in $U_1 \setminus A_1$ one has $g_s - f_s = s = g_1 - f_1$; thus μ_1 and μ differ by $\Delta\varphi \in \Delta[\Gamma_K(W, \mathcal{R})]$ and thus define the same cohomology class in $H^1_K(W, \mathcal{H})$. Thus $j_{(A,U)}$ is defined independently of the choice of h_1 and h_2 above (even though $\tilde{j}_{(A,U)}$ wasn't), and the $j_{(\cdot,\cdot)}$'s respect refinement. It is obvious that if $s \in \mathcal{H}_{U \setminus A}$ has an extension to all of U which is harmonic there, then s determines the zero element of $H^1_K(W, \mathcal{H})$; in other words, the map from $\mathcal{H}_{U \setminus A}$ to $H^1_K(W, \mathcal{H})$ is well defined up to coboundaries for any pair (A, U).

We may now observe that as (A, U) ranges over all Cousin pairs with $\overline{U}$ compact and small, the ranges of the mappings $j_{(A,U)}$ generate $H^1_K(W, \mathcal{H})$. Indeed, suppose μ is an element of $\Gamma(W, \mathcal{L})$ with small compact support, and let V be a small region containing $\operatorname{Supp} \mu$, p_y the normalized kernel on V.

Setting $A = \operatorname{Supp} \mu$ and taking U to be an inner-regular region of V containing A, we see that

$$s = \int q_y(\bullet)\, d\mu(y) | U \setminus A \in \mathcal{H}_{U \setminus A},$$

where q_y is the normalized kernel on U, and we have an immediate and natural way to write $s = g - f$: set $g = \int q_y \, d\mu(y)$ and $f = 0$.

Definition 2.4. — *The inductive topology on* $H^1_K(W, \mathcal{H})$ *is the inductive (locally convex) topology* [14, p. 54] *generated by the Fréchet spaces* $\mathcal{H}_{U \setminus A}$ *and the maps*

$$j_{(A, U)} : \mathcal{H}_{U \setminus A} \to H^1_K(W, \mathcal{H})$$

as (A, U) *ranges over all Cousin pairs in* W *for which* $\overline{U}$ *is compact and small.*

There is no good *a priori* reason that the inductive topology should be a Hausdorff topology. Nonetheless, we can characterize its topological dual, which consists precisely of the linear functionals induced by the pairing we have already constructed between $H^1_K(W, \mathcal{H})$ and $\Gamma(W, \mathcal{H}^*)$.

Proposition 2.5. — *For every linear functional* F *in the topological dual of the space* $H^1_K(W, \mathcal{H})$ *equipped with the inductive topology, there is an element* $h^*_F \in \Gamma(W, \mathcal{H}^*)$ *that induces* F *with respect to the pairing of* 2.3 *above. Conversely, each linear functional* F *induced by an element* $h^* \in \Gamma(W, \mathcal{H}^*)$ *with respect to that pairing is continuous in the inductive topology, and the correspondence is* $1 - 1$.

Proof. — Let F be an element of the topological dual of $H^1_K(W, \mathcal{H})$. If V is a relatively compact small region and p_y is the normalized kernel on V, then for any compact $A \subseteq V$ we claim that $y \to F[j_{(A, V)}(p_y | V \setminus A)]$ defines an element of $\mathcal{H}^*$ on A^o. Indeed, $(y, x) \to p_y(x)$ is a continuous function on $A^o \times (V \setminus A)$ that can be construed as a $\Gamma(A^o, \mathcal{H}^*)$-valued function of $x \in V \setminus A$, the functional $F \circ j_{(A, V)} = j_{(A, V)'}[F]$ is a continuous linear functional on $\mathcal{H}_{V \setminus A}$ that can be given as a measure μ of compact support in $V \setminus A$, and construing $(y \to F[j_{(A, V)}(p_y | V \setminus A)]) = \left(y \to \int p_y \, d\mu\right)$ as a vector integral establishes the claim. It is a routine verification, using the fact that the mappings $j_{(\cdot,\cdot)}$ respect refinement and are zero on coboundaries, to show that if U were another relatively compact small set with normalized kernel q_y and $y \to \mathrm{F}[j_{(\mathrm{B},\mathrm{U})}(q_y|\mathrm{U}\diagdown\mathrm{B})]$ were constructed similarly, then the two functions so constructed would agree on $\mathrm{A}^{\mathrm{o}} \cap \mathrm{B}^{\mathrm{o}}$; from this it follows that we can define h_{F}^{*} on all of W by simply setting it equal to each of these functions $y \to \mathrm{F}[j_{(\mathrm{A},\mathrm{V})}(p_y|\mathrm{V}\diagdown\mathrm{A})]$ on its domain. Now given any $\lambda \in \Gamma_{\mathrm{K}}(\mathrm{W}, \mathcal{L})$, if λ has small compact support A and V is a small relatively compact region containing A, we may set $s = \int p_y \, d\lambda(y)$ on $\mathrm{V}\diagdown\mathrm{A}$; then

$$
\begin{aligned}
\mathrm{F}[\lambda] &= \mathrm{F}[j_{(\mathrm{A},\mathrm{V})}s] \\
&= [j_{(\mathrm{A},\mathrm{V})}'\mathrm{F}]\left(\int p_y \, d\lambda(y)\right) \\
&= \int ([j_{(\mathrm{A},\mathrm{V})}'\mathrm{F}](p_y|\mathrm{V}\diagdown\mathrm{A}))\, d\lambda(y) = \int h_{\mathrm{F}}^{*}(y)\, d\lambda(y)
\end{aligned}
$$

and that suffices to prove the first part of the proposition.

For the other half of the proposition we shall need the following little approximation lemma.

LEMMA 2.6. — *Let* V *be a small region and* U *an open relatively compact subset of* V. *Let* $\mathrm{E} \subseteq \Gamma(\mathrm{U}, \mathcal{H}^*)$ *denote the closed linear subspace spanned by the functions*

$$\{(y \to p_y(x))\,|\,\mathrm{U} : x \in \mathrm{V}\diagdown\overline{\mathrm{U}}\}.$$

Then $\Gamma(\mathrm{V}, \mathcal{H}^*)|\mathrm{U} \subseteq \mathrm{E}$.

Proof. — By applying 2.3 above component by component in U, it is easy to see that any element of the annihilator of E in $\Gamma(\mathrm{U}, \mathcal{H}^*)'$ is represented by some measure $\lambda \in \Gamma_{\mathrm{K}}(\mathrm{U}, \mathcal{L})$. If λ is such a measure, it can be construed as an element of $\Gamma_{\mathrm{K}}(\mathrm{V}, \mathcal{L})$, and the function $g(x) = \int p_y(x)\, d\lambda(y)$ belongs to $\Gamma(\mathrm{V}, \mathcal{R})$. By the choice of λ, the values $g(x)$ are zero for $x \in \mathrm{V}\diagdown\overline{\mathrm{U}}$, so g has compact support (contained in $\overline{\mathrm{U}}$). By 2.3 above applied to V, the fact that $\lambda = \Delta g$ is cohomologous to zero in $\mathrm{H}_{\mathrm{K}}^{1}(\mathrm{V}, \mathcal{H})$ implies that

$$\int h^* \, d\lambda = \int h^* \, d[\Delta g] = 0$$

for all $h^* \in \Gamma(V, \mathcal{H}^*)$, i.e., λ belongs to the annihilator of $\Gamma(V, \mathcal{H}^*)|U$, Q.E.D.

Conclusion of proof of 2.5. — Let $h^* \in \Gamma(W, \mathcal{H}^*)$ be given. Suppose (A, V) is a Cousin pair with $\overline{V}$ compact and small, and with no loss of generality (since the topology of uniform convergence on compacta on $\mathcal{H}_{V \setminus A}$ can be formed component by component) assume that V is connected. One can then find a refinement (B, U) of (A, V) with $A \subseteq B^o$ and with U a regular (for $\mathcal{H}$) relatively compact region in V. Let $H(s)$ denote $H(U, s|\partial U)$ for $s \in \mathcal{H}_{V \setminus A}$. For fixed $s \in \mathcal{H}_{V \setminus A}$ we can find a function $r \in \Gamma(V, \mathcal{R})$ with $r = s$ in a neighborhood of $\overline{V \setminus B}$, using the fineness of $\mathcal{R}$; then Δr belongs to the cohomology class given by $j_{(B, U)}[s|U \setminus B] = j_{(A, V)}s$. Since $r - \int q_y \, d[\Delta r](y)$ is harmonic in U (where q_y, as usual, is the normalized kernel on U) and $H(U, r|\partial U) = H(s)$ because $r|\partial U = s|\partial U$, the function

$$r - H(s) - \int q_y \, d[\Delta r](y)$$

is simultaneously harmonic and a linear combination of potentials on U; that is, it is zero, and

$$\int q_y \, d[\Delta r](y) = r - H(s).$$

Now by the lemma we can find a sequence of functions on B^o of the form

$$y \to \sum_{j=1}^{n_k} \alpha_{jk} q_y(x_{jk}) \quad (k = 1, 2, \ldots)$$

where $\{x_{jk}\}_{j=1}^{n_k}$ belong to $U \setminus B^{o-}$ and $\{\alpha_{jk}\}_{j=1}^{n_k}$ are scalars, that converge uniformly on compacta in B^o to $h^*|B^o$ as $k \to \infty$. Since Δr is a measure of compact support contained in B^o, we have

$$\begin{aligned}\int \sum_{j=1}^{n_k} \alpha_{jk} q_y(x_{jk}) \, d[\Delta r](y) &= \sum_{j=1}^{n_k} \alpha_{jk} \int q_y(x_{jk}) \, d[\Delta r](y) \\ &= \sum_{j=1}^{n_k} \alpha_{jk} . (r(x_{jk}) - H(s)(x_{jk})) \\ &= \sum_{j=1}^{n_k} \alpha_{jk} . (s(x_{jk}) - H(s)(x_{jk})),\end{aligned}$$

the last equality because $r = s$ in a neighborhood of $\overline{U \setminus B^o}$. Taking limits on both sides as $k \to \infty$, we have

$$\int h^*(y)\, d[j_{(A,V)}s] = \int h^*(y)\, d[\Delta r]$$
$$= \lim_k \sum_{j=1}^{n_k} \alpha_{jk} . (s(x_{jk}) - H(s)(x_{jk})).$$

Neither the left- nor the right-hand side of this equation depends on the choice of r, and each of the limitands on the right-hand side clearly depends continuously on $s \in \mathcal{H}_{V \setminus A}$. Thus the functional on $H^1_K(W, \mathcal{H})$ induced by h^* with respect to the natural pairing has the property that the functional it induces on $\mathcal{H}_{V \setminus A}$ (its image under $j_{(A,V)'}$) is the simple limit of a sequence of continuous linear functionals. By the Banach-Steinhaus theorem [14, Corollary, p. 86] the induced functional is continuous on $\mathcal{H}_{V \setminus A}$, and thus by the definition of the inductive topology the functional h^* determined on $H^1_K(W, \mathcal{H})$ is continuous.

We have thus shown that the subspace of the algebraic dual of $H^1_K(W, \mathcal{H})$ given by $\Gamma(W, \mathcal{H}^*)$ with respect to the natural pairing of the two spaces is precisely the topological dual of $H^1_K(W, \mathcal{H})$ equipped with the inductive topology. The remaining assertion of the proposition follows from the pairing's being separated with respect to $\Gamma(W, \mathcal{H}^*)$, which means precisely that the mapping of $\Gamma(W, \mathcal{H}^*)$ into the algebraic dual of $H^1_K(W, \mathcal{H})$ given by the pairing is $1 - 1$, Q.E.D.

3. Approximation theorems; further duality theorems.

At this point it would be desirable to find sufficient conditions for the natural duality between $H^1_K(W, \mathcal{H})$ and $\Gamma(W, \mathcal{H}^*)$ to be separated in $H^1_K(W, \mathcal{H})$. We begin with the case in which W is noncompact, because a knowledge of this case aids us in constructing Leray covers when W is compact. The following proposition, which shows that an approximation property for $\mathcal{H}^*$ is necessary for the space $H^1_K(W, \mathcal{H})$ to be separated, is a good starting point.

PROPOSITION 3.1. — *Let* W *be noncompact, and let* $U_1 \subseteq U_2 \subseteq \cdots$ *be an exhaustion of* W *by subregions. Suppose*

$H^1_K(W, \mathcal{H})$ is a Hausdorff LTS in the inductive topology. Then for every compact $A \subseteq W$ there exists an index n for which $A \subseteq U_n$ and every element of $\Gamma(U_n, \mathcal{H}^*)$ can be approximated uniformly on A by restrictions of elements of $\Gamma(W, \mathcal{H}^*)$.

Proof. — Obviously we can assume that $A \subseteq U_1$. Let $\| \ \|_A$ denote the seminorm on $\Gamma(U_1, \mathcal{H}^*)$ that gives « uniform convergence on A », i.e., $\|h^*\|_A = \sup \{|h^*(x)| : x \in A\}$; let E_A be the Banach space consisting of the elements of the topological dual $\Gamma(U_1, \mathcal{H}^*)'$ that are $\| \ \|_A$-continuous, under the norm dual to $\| \ \|_A$, and let $F \subseteq E_A$ be the annihilator in E_A of $\Gamma(W, \mathcal{H}^*)|U_1$. By 2.3 above, for every linear functional $\Phi \in F$ we can find a measure $\lambda_\Phi \in \Gamma_K(U_1, \mathcal{L})$ for which $\Phi = \left(h^* \to \int h^* \, d\lambda_\Phi\right)$. However, since $H^1_K(W, \mathcal{H})$ is assumed to be Hausdorff and its topological dual is given by $\Gamma(W, \mathcal{H}^*)$, there must exist some $g \in \Gamma_K(W, \mathcal{R})$ for which $\Delta g = \lambda_\Phi$ (otherwise λ_Φ could not annihilate $\Gamma(W, \mathcal{H}^*)$). Because g has compact support, one sees that actually $g \in \Gamma_K(U_k, \mathcal{R})$ for some index k. Let F_k denote the set (obviously a subspace) of elements of F that can be given by the Laplacian of some element of $\Gamma_K(W, \mathcal{R})$ whose support is contained in U_k, $k = 1, 2, \ldots$. Clearly $F_1 \subseteq F_2 \subseteq \cdots$ and $F = \bigcup_{k=1}^{\infty} F_k$, so applying the Baire category theorem to F we see that some $\overline{F}_n$ has interior points (where both the interior and the closure are taken in the norm on E_A dual to $\| \ \|_A$); since F_n is a subspace, $\overline{F}_n = F$. By 2.3 above, F_n is contained in the annihilator in E_A of $\Gamma(U_n, \mathcal{H}^*)|U_1$, and since that annihilator is norm-closed, $\overline{F}_n = F$ is contained in that annihilator. By the bipolar theorem (or, simply, the Hahn-Banach theorem), $\Gamma(U_n, \mathcal{H}^*)|U_1$ is in the $\| \ \|_A$-closure of $\Gamma(W, \mathcal{H}^*)|U_1$, Q.E.D.

COROLLARY 3.2. — *If W is noncompact and $H^1_K(W, \mathcal{H})$ is a Hausdorff LTS in the inductive topology, then every exhaustion of W by relatively compact subregions $\{U_k\}_{k=1}^{\infty}$ possesses a subsequence $\{U_{k_i}\}_{i=1}^{\infty}$ such that $\overline{U}_{k_i} \subseteq U_{k_{i+1}}$ and every element of $\Gamma(U_{k_{i+1}}, \mathcal{H}^*)$ can be approximated uniformly on $\overline{U}_{k_i}$ by restrictions of elements of $\Gamma(W, \mathcal{H}^*)$.*

We shall now see that in certain circumstances the approximation condition given above is sufficient for $H^1_K(W, \mathcal{H})$ to be a Hausdorff LTS. In order to give a useful sufficient condition for this approximation criterion, let us recall the definition of quasi-analyticity given by A. de la Pradelle [19, p. 383] : in the presence of the hypothesis of proportionality on $\mathcal{H}^*$-potentials with point support, the sheaf $\mathcal{H}$ will be said to have the property (A) or (A*) respectively if the condition of quasi-analyticity holds for $\mathcal{H}$ or $\mathcal{H}^*$ respectively, i.e., if for any region $U \subseteq W$ and any $h \in \mathcal{H}_U$ or $\mathcal{H}^*_U$ respectively, h vanishes on an open set in U if and only if it vanishes identically on U. Note that since the property (A) or (A*) is present if and only if it is present for the restrictions of $\mathcal{H}$ and $\mathcal{H}^*$ to some neighborhood of each point in W (an easy consequence of the connectedness assumed of U above), these properties are essentially local and do not depend, for example, on the selection of the global adjoint sheaf $\mathcal{H}^*$.

Proposition 3.3. — *Let* W *be noncompact. Each of the following conditions implies its successor:*

(*a*) *The hypothesis of proportionality of $\mathcal{H}^*$-potentials with point support and the property* (A) *of* [19] *hold, and* W *possesses an exhaustion* $\{U_i\}_{i=1}^{\infty}$ *by small regions (in particular,* W *may be small)* ;

(*b*) W *possesses an exhaustion by small open sets* $\{U_i\}_{i=1}^{\infty}$ *such that* $\overline{U}_i \subseteq U_{i+1}$ *and every element of* $\Gamma(U_{i+1}, \mathcal{H}^*)$ *can be uniformly approximated on* $\overline{U}_i$ *by the restrictions of elements of* $\Gamma(U_{i+2}, \mathcal{H}^*)$ *to* U_{i+1}, *with each* $\overline{U}_i$ *compact;*

(*c*) $H^1_K(W, \mathcal{H})$ *is Hausdorff in the inductive topology.*

Proof. — That $(a) \Longrightarrow (b)$ follows readily from results of de la Pradelle [19, § 3, p. 395 ff.], applied to $\mathcal{H}^*$; for if U possesses an exhaustion by small regions, then by replacing each of these regions by its envelope [19, p. 395] and passing to a subsequence if necessary, we can assume that no U_i possesses relatively compact complementary components. The « Extension du théorème précédent » of [19, p. 397], applied to $\mathcal{H}^*$, then shows that $\Gamma(U, \mathcal{H}^*)|U_i$ is dense in

$\Gamma(U_i, \mathcal{H}^*)$ for the topology of uniform convergence on compacta.

Condition (b) is in fact equivalent to the apparently weaker condition

(b') W possesses an exhaustion by small open sets $\{U_i\}_{i=1}^{\infty}$ such that $\overline{U}_i \subseteq U_{i+1}$ and every element of $\Gamma(U_{i+1}, \mathcal{H}^*)$ can be uniformly approximated on $\overline{U}_i$ by the restrictions of elements of $\Gamma(W, \mathcal{H}^*)$ to U_{i+1}, with each $\overline{U}_i$ compact.

This equivalence, which has nothing to do with adjoint sheaves or small sets, will be proved as a Remark following 3.7 below. Granting this equivalence, we prove that $(b') \Longrightarrow (c)$. We know by 2.5 above that showing that $H^1_K(W, \mathcal{H})$ is Hausdorff is equivalent to showing that if $\lambda \in \Gamma_K(W, \mathcal{L})$ and $\int h^* \, d\lambda = 0$ for every $h^* \in \Gamma(W, \mathcal{H}^*)$, then $\lambda = \Delta g$ for some $g \in \Gamma_K(W, \mathcal{R})$. Given such a λ, take n so large that $\operatorname{Supp} \lambda \subseteq U_n$, and then take a small region V containing $\overline{U}_{n+1}$. Let p_y be a normalized kernel for V and form $g(\bullet) = \int p_y(\bullet) \, d\lambda(y) \in \Gamma(V, \mathcal{R})$. For each $x \in V \setminus U_{n+1}$ the function $y \to p_y(x)$ belongs to $\Gamma(U_{n+1}, \mathcal{H}^*)$, and since every element of $\Gamma(U_{n+1}, \mathcal{H}^*)$ can be approximated uniformly on $\overline{U}_n$ by restrictions of elements of $\Gamma(W, \mathcal{H}^*)$ and $\int h^* \, d\lambda = 0$ for $h^* \in \Gamma(W, \mathcal{H}^*)$, we see that $g(x) = \int p_y(x) \, d\lambda(y) = 0$ for $x \in V \setminus U_{n+1}$. Thus $\operatorname{Supp} g \subseteq \overline{U}_{n+1}$, a compact subset of V. Since $\Delta g = \lambda$ and g can be extended trivially to be zero outside V, we have shown that $\lambda \in \Delta[\Gamma_K(W, \mathcal{R})]$, Q.E.D.

When $H^1_K(W, \mathcal{H})$ is Hausdorff, it has a pleasant characterization in terms of $\Gamma(W, \mathcal{H}^*)$ and the natural pairing:

Proposition 3.4. — *When $H^1_K(W, \mathcal{H})$ is a Hausdorff LTS and is identified with $\Gamma(W, \mathcal{H}^*)'$ by the natural pairing, the inductive topology of $H^1_K(W, \mathcal{H})$ is identified with the topology $\tau(\Gamma(W, \mathcal{H}^*)', \Gamma(W, \mathcal{H}^*))$. In particular, $H^1_K(W, \mathcal{H})$ is nuclear, reflexive, and the dual of a Fréchet space when equipped with the inductive topology.*

Proof. — That $\Gamma(W, \mathcal{H}^*)$ is a nuclear Fréchet space in the topology of uniform convergence on compacta is by now well known; see, e.g., [18, § 6.3, p. 266 ff.]. For any small Cousin pair (A, U) the natural mapping of $\mathcal{H}_{U\setminus A} \to H^1_K(W, \mathcal{H})$ is continuous by definition of the inductive topology; therefore this mapping is also

$$\sigma(\mathcal{H}_{U\setminus A}, (\mathcal{H}_{U\setminus A})') - \sigma(H^1_K(W, \mathcal{H}), \Gamma(W, \mathcal{H}^*))\text{-continuous},$$

since $\Gamma(W, \mathcal{H}^*)$ is identified with the dual of $H^1_K(W, \mathcal{H})$; by a standard theorem of duality [14, Thm. 7.4, p. 158] the mapping is also continuous from $\tau(\mathcal{H}_{U\setminus A}, (\mathcal{H}_{U\setminus A})')$ to $\tau(H^1_K(W, \mathcal{H}), \Gamma(W, \mathcal{H}^*))$. Since $\tau(\mathcal{H}_{U\setminus A}, (\mathcal{H}_{U\setminus A})')$ is the (Fréchet) topology of uniform convergence on compacta, we see that $\tau(H^1_K(W, \mathcal{H}), \Gamma(W, \mathcal{H}^*))$ is coarser than the inductive topology; but since $\Gamma(W, \mathcal{H}^*)$ is identified with the dual of $H^1_K(W, \mathcal{H})$, the (Arens-Mackey) characterization of the topology τ shows that τ is precisely the inductive topology [14, Thm. 3.2, Cor. 1, p. 131]. Thus $H^1_K(W, \mathcal{H})$ is identified as the dual of a nuclear Fréchet space, and in particular is nuclear and reflexive [14, Thm. 9.6, p. 172], Q.E.D.

A rather trivial corollary of this proposition is that if W is compact and $H^1_K(W, \mathcal{H})$ is separated by elements of $\Gamma(W, \mathcal{H}^*)$, then it is finite-dimensional (because the space $\Gamma(W, \mathcal{H}^*)$ is a nuclear Banach space and therefore finite-dimensional). This, however, would be easy to see directly.

Classically, approximation conditions like (*b*) of 3.3 above are used to prove that $H^1(W, \mathcal{H}) = 0$ (without compact supports). Results of this kind are also available in the axiomatic theory:

PROPOSITION 3.5. — *Let* W *be noncompact. Each of the following conditions implies its successor*:

(*a*) *The hypothesis of proportionality of* $\mathcal{H}^*$*-potentials with point support and the property* (A*) *of* [19] *hold, and* W *possesses an exhaustion* $\{U_i\}_{i=1}^{\infty}$ *by small relatively compact regions (in particular,* W *may be small)*;

(*b*) W *possesses an exhaustion by small relatively compact open sets* $\{U_i\}_{i=1}^{\infty}$ *such that* $\overline{U}_i \subseteq U_{i+1}$ *and every element of* $\Gamma(U_{i+1}, \mathcal{H})$ *can be uniformly approximated on* $\overline{U}_i$ *by restrictions of functions in* $\Gamma(U_{i+2}, \mathcal{H})$ *to* U_{i+1};

(*c*) $H^1(W, \mathcal{H}) = 0$.

Proof. — The proof that $(a) \Longrightarrow (b)$ is the same as that of the corresponding implication in 3.3 above, except that the results of [19] are applied to $\mathcal{H}$. That $(b) \Longrightarrow (c)$ follows from an essentially classical argument [7, Thm. 4, pp 42-44] which we merely summarize. Given a measure $\lambda \in \Gamma(\mathrm{W}, \mathcal{L})$, by an inductive process we can determine a sequence of functions $\{f_i\}_{i=1}^{\infty}$ with the properties

(i) $f_i \in \Gamma(\mathrm{U}_i, \mathcal{R})$;

(ii) $\Delta f_i = \chi_{\mathrm{U}_i}.\lambda$;

(iii) $|f_i(x) - f_{i-1}(x)| \leqslant 1/2^i$ for all $x \in \overline{\mathrm{U}}_{i-2}$. For $i = 1, 2$ one may simply select a region V_2 which is small and contains $\overline{\mathrm{U}}_2$, let p_y^2 be the normalized kernel on V_2, and take $f_i = \int p_y^2(\bullet)\, d[\chi_{\mathrm{U}_2}.\lambda](y)|\mathrm{U}_i$; (iii) will be satisfied vacuously. Suppose $f_1, f_2, \ldots, f_{n-1}$ have been constructed. If V_n is a small region that contains $\overline{\mathrm{U}}_n$ and p_y^n is the normalized kernel on V_n, then the function

$$g_n = \int p_y^n(\bullet)\, d[\chi_{\mathrm{U}_n}.\lambda](y)|\mathrm{U}_n \in \Gamma(\mathrm{U}_n, \mathcal{R})$$

has the property that $\Delta g_n = \chi_{\mathrm{U}_n}.\lambda$, and since

$$\Delta f_{n-1} = \lambda|\mathrm{U}_{n-1} = (\Delta g_n)|\mathrm{U}_{n-1},$$

the function $g_n - f_{n-1}$ is harmonic in U_{n-1}. By the approximation assumption there is a function $h \in \Gamma(\mathrm{U}_n, \mathcal{H})$ for which $|g_n(x) - f_{n-1}(x) - h(x)| < 1/2^n$ for all $x \in \overline{\mathrm{U}}_{n-2}$, and the function $f_n = g_n - h$ then satisfies (i), (ii), and (iii). For each $x \in \mathrm{U}_k$ the sequence $\{f_i(x)\}_{i=k}^{\infty}$ converges and thus defines a limit function f on all of W; since one has

$$f(x) = f_k(x) + \sum_{i>k} [f_i(x) - f_{i-1}(x)]$$

for all $x \in \mathrm{U}_k (k \geqslant 2)$ and the series converges uniformly on U_k to a harmonic function, we have $f|\mathrm{U}_k \in \Gamma(\mathrm{U}_k, \mathcal{R})$, $f \in \Gamma(\mathrm{W}, \mathcal{R})$, $\Delta f|\mathrm{U}_k = \Delta f_k|\mathrm{U}_k = \lambda|\mathrm{U}_k$ for all $k \geqslant 2$ and therefore $\Delta f = \lambda$, Q.E.D.

Corollary 3.6. — *If the hypothesis of proportionality of $\mathcal{H}^*$-potentials with point support and the property* (A*) *hold on* W, *then* $\mathrm{H}^q(\mathrm{U}, \mathcal{H}) = 0$ *for all* $q \geqslant 1$ *for every small region* U, *and therefore for every small open set* U, *in* W.

Consequently, W *possesses Leray covers (cf.* [8, p. 46]*), namely, covers by small open sets.*

Indeed, $H^q(U, \mathcal{H}) = 0$ for all $q \geqslant 2$ in any event [21, Thm. (2.11)], while the present proposition takes care of $H^1(U, \mathcal{H}) = 0$ for small regions U and therefore for small open sets U, since one needs $\Delta[\Gamma(U, \mathcal{R})] = \Gamma(U, \mathcal{L})$ and Laplacians are computed locally.

Corollary 3.7. — *If* W *is noncompact,* W *has an exhaustion* $\{U_i\}_{i=1}^{\infty}$ *by small subregions, the hypothesis of proportionality of* $\mathcal{H}^*$*-potentials holds and* $H^1_K(W, \mathcal{H})$ *is a Hausdorff LTS in the inductive topology, then* $H^1(W, \mathcal{H}^*) = 0$.

Indeed, smallness of sets means the same thing for $\mathcal{H}^*$ as it does for $\mathcal{H}$, and consequently one can apply 3.2 above and the implication $(b) \Longrightarrow (c)$ of the present proposition to $\mathcal{H}^*$.

Remark. — The condition that W possess an exhaustion $\{U_i\}_{i=1}^{\infty}$ such that $\overline{U}_i \subseteq U_{i+1}$ and every element of $\Gamma(U_{i+1}, \mathcal{H})$ be uniformly approximable on $\overline{U}_i$ by restrictions of elements of $\Gamma(U_{i+2}, \mathcal{H})$, $i = 1, 2, \ldots$, which occurs in 3.3 and 3.5 above, is in fact equivalent to the condition that every element of $\Gamma(U_{i+1}, \mathcal{H})$ be uniformly approximable on $\overline{U}_i$ by restrictions of elements of $\Gamma(W, \mathcal{H})$, $i = 1, 2, \ldots$. This can be proved by the same inductive technique used in the proof of 3.5 above: given a fixed set U_i and $h_1 \in \Gamma(U_{i+1}, \mathcal{H})$, for any $\varepsilon > 0$ one may choose $h_2, h_3, \ldots$ inductively with $h_n \in \Gamma(U_{i+n}, \mathcal{H})$ having the property that $|h_n(x) - h_{n-1}(x)| < \varepsilon/2^{n-1}$ for all $x \in \overline{U}_{i+n-1}$, $n = 2, 3, \ldots$. Again it is clear that the sequence $\{h_k(x)\}_{k=n}^{\infty}$ converges to a limit function $h(x)$ for $x \in U_{i+n}$, with h being defined on all of W, and since the convergence is uniform on a neighborhood of each $x \in W$ the limit function $h \in \Gamma(W, \mathcal{H})$; clearly one has $|h(x) - h_1(x)| \leqslant \Sigma_{n>1}|h_n(x) - h_{n-1}(x)| < \Sigma_{n>1}\varepsilon/2^{n-1} = \varepsilon$ for any $x \in \overline{U}_i$.

For compact W, the situation is quite classical.

Proposition 3.8. — *If* W *is a compact space, the following conditions are equivalent:*

(*a*) $H^1(W, \mathcal{H})$ *is a Hausdorff LTS in the inductive topology;*

(*b*) $H^1(W, \mathcal{H})$ *is finite-dimensional*;

(*c*) *There is a cover* $\mathfrak{U}$ *of* W *for which the natural map* $H^1(N(\mathfrak{U}), \mathcal{H}) \to H^1(W, \mathcal{H})$ *is surjective.*

Proof. — Since the dual $\Gamma(W, \mathcal{H}^*)$ of $H^1(W, \mathcal{H})$ is a nuclear Banach space and therefore finite-dimensional, (*a*) implies that $H^1(W, \mathcal{H})$ is in separated duality with a finite-dimensional space, which implies (*b*). That (*b*) $\Longrightarrow$ (*c*) follows from the fact that for all covers $\mathfrak{U}$ the natural mappings $H^1(N(\mathfrak{U}), \mathcal{H}) \to H^1(W, \mathcal{H})$ give the space $H^1(W, \mathcal{H})$ as their inductive limit: since the limit space is finitely generated, it must be equal to one of the limitands.

Finally, (*c*) $\Longrightarrow$ (*a*): since the natural map

$$H^1(N(\mathfrak{U}), \mathcal{H}) \to H^1(W, \mathcal{H})$$

can be factored through the natural map

$$H^1(N(\mathfrak{V}), \mathcal{H}) \to H^1(W, \mathcal{H})$$

for any refinement $\mathfrak{V}$ of $\mathfrak{U}$, one can assume that $\mathfrak{U}$ is finite; one can then take a refinement $\mathfrak{V}$ of $\mathfrak{U}$ such that every $V \in \mathfrak{V}$ has the property that $\overline{V} \subseteq U$ for some $U \in \mathfrak{U}$. The natural map $H^1(N(\mathfrak{V}), \mathcal{H}) \to H^1(W, \mathcal{H})$ is still onto, and the natural refinement map $H^1(N(\mathfrak{U}), \mathcal{H}) \to H^1(N(\mathfrak{V}), \mathcal{H})$ is therefore $1-1$ and onto, because all the natural maps $H^1(N(\mathfrak{W}), \mathcal{H}) \to H^1(W, \mathcal{H})$ are $1-1$ [7, p. 47]. The rest of the argument is the classical one used in the situation where $\mathfrak{U}$ is a Leray cover [8, p. 245], so we merely summarize it briefly. If $C^0(N(\mathfrak{V}), \mathcal{H})$ is the space of 0-cochains on $\mathfrak{V}$, the fact that the natural restriction map $H^1(N(\mathfrak{U}), \mathcal{H}) \to H^1(N(\mathfrak{V}), \mathcal{H})$ is an isomorphism implies that the map

$$u : Z^1(N(\mathfrak{U}), \mathcal{H}) \oplus C^0(N(\mathfrak{V}), \mathcal{H}) \to Z^1(N(\mathfrak{V}), \mathcal{H})$$

defined by

$$u : f \oplus g \to \operatorname{Rest} f + \delta g$$

is surjective, and so (as in the classical theorem cited above) implies that the map $\delta : C^0(N(\mathfrak{V}), \mathcal{H}) \to Z^1(N(\mathfrak{V}), \mathcal{H})$ is a perturbation of a surjective mapping of Fréchet spaces by a compact mapping (viz., the natural map

$$Z^1(N(\mathfrak{U}), \mathcal{H}) \to Z^1(N(\mathfrak{V}), \mathcal{H}))$$

and therefore has closed finite-codimensional range. We have

$$H^1(W, \mathcal{H}) \cong H^1(N(\mathfrak{V}), \mathcal{H}) \cong Z^1(N(\mathfrak{V}), \mathcal{H})/\delta[C^0(N(\mathfrak{V}), \mathcal{H})]$$

and the unique Hausdorff topology on $H^1(W, \mathcal{H})$ is the quotient topology. Now given a small Cousin pair (A, U) in W, we could clearly have chosen $\mathfrak{V}$ in the argument above to be a refinement of $\{W\setminus A, U\}$, whereupon we would have had a natural mapping

$$\mathcal{H}_{U\setminus A} = H^1(\{W\setminus A, U\}, \mathcal{H}) \to H^1(N(\mathfrak{V}), \mathcal{H}) \cong H^1(W, \mathcal{H}).$$

The last isomorphism is a topological isomorphism for the unique Hausdorff topology on the finite-dimensional space $H^1(W, \mathcal{H})$, and the mapping represented by the arrow (being a refinement map) is continuous. Thus the natural map from Cousin data on $U\setminus A$ into $H^1(W, \mathcal{H})$ equipped with its unique Hausdorff topology is a continuous map, and the inductive topology of $H^1(W, \mathcal{H})$ is Hausdorff, Q.E.D.

COROLLARY 3.9. — *If the hypothesis of proportionality of potentials with point support and the condition of quasianalyticity hold for* $\mathcal{H}^*$, *then the equivalent conditions of* 3.8 *hold.*

Indeed, 3.6 above guarantees the existence of Leray covers for W, and the Leray theorem [8, p. 189] guarantees that (*c*) of 3.8 holds.

4. Second duals; the case in which $1 \in \overline{\mathcal{H}}$.

In conclusion, we consider two subjects: first, the duality theory obtained when $\mathcal{H}^*$ also satisfies the hypothesis of proportionality of potentials with common point support, so that $\mathcal{H}^*$ possesses an adjoint sheaf which can be identified with $\mathcal{H}$, and second, the meaning of the results of [21, § 4] in the case in which $\mathcal{H}$ admits an adjoint sheaf.

In order to study $(\mathcal{H}^*)^*$, we assume (unless explicit mention is made to the contrary) throughout the following discussion that $\mathcal{H}^*$ satisfies the hypothesis of proportionality. In the situation originally considered in [9], $\mathcal{H}^*$ is constructed from a single kernel on the entire space W, no normalization considerations appear, and the verification that $(\mathcal{H}^*)^* = \mathcal{H}$

is relegated to a remark [9, p. 559]. Given a normalization $(\{V_i\}_{i\in I}, \{p_y^i\}_{i\in I})$ of $\mathcal{H}$ on W, this result of [9] implies directly that $((\mathcal{H}|V_i)^*)^* = \mathcal{H}|V_i$ for each $i \in I$, so $\mathcal{H}$ can be viewed as the result of patching together the second adjoints on the open sets $\{V_i\}_{i\in I}$. In order to make the considerations of the preceding §§ directly applicable, however, it is desirable to know that the kernels $\{(y \to p_y^i(x))\}_{i\in I}$ constitute a normalization of $\mathcal{H}^*$ that determines $\mathcal{H}$ as $(\mathcal{H}^*)^*$, so that $\mathcal{H}$ can be viewed as having been constructed from $\mathcal{H}^*$ as in § 1 above, only using the « transposed kernels ». For this we need the following proposition.

PROPOSITION 4.1. — *Let* $(\{V_i\}_{i\in I}, \{p_y^i\}_{i\in I})$ *be a normalization of* $\mathcal{H}$. *Then the kernels* $\{(y \to p_y^i(x))\}_{i\in I}$ *satisfy conditions* (1) *and* (2) *of* 1.5 *above with* $\varphi_{ij} = \mathbf{1}$, *and* (3′) *of* 1.16 *above, relative to the sheaf* $\mathcal{H}^*$. *Moreover, if* U *is a (nonempty) region contained in* $V_i \cap V_j$ *and* q_y *is the (normalized) kernel induced on* U *(equivalently by* p_y^i *or* p_y^j*), then* $(y \to q_y(x))$ *is equal to either of the kernels* $y \to (p_y^k(x) - M^*[z \to p_z(x)](y))$ *induced on* U *by* $y \to p_y^k(x)$, $k = i, j$.

Proof. — The fact that the $\{p_y^i\}_{i\in I}$ are part of a normalization and are therefore themselves normalized insures that $p_y^k(x) = q_y(x) + M[p_y^k|U](x)$ for $x, y \in U$, $k = i, j$. By 1.12 above, the functions $y \to M[p_y^k|U](x)$ belong to

$$(\mathcal{H}|V_k)^* = \mathcal{H}^*|V_k \qquad \text{for each} \qquad x \in U, \quad k = i, j;$$

consequently, the $\mathcal{H}^*$-potential parts of the $\mathcal{H}^*$-superharmonic functions $y \to p_y^k(x)$ on U are equal for each $x \in U$. But $y \to q_y(x)$ is a $(\mathcal{H}|V_k)^*|U = \mathcal{H}^*|U$-potential on U by [9, Prop. 30.1, p. 544], and thus it is the $\mathcal{H}^*$-potential part of either of the $y \to p_y^k(x)$, $k = i, j$. That gives us the equality of the kernels $y \to p_y^k(x) - M^*[(z \to p_z(x)](y)$, Q.E.D.

Essentially the same argument gives us the following proposition, whose proof we omit:

PROPOSITION 4.2. — *Let* $(\{V_i\}_{i\in I}, \{p_y^i\}_{i\in I})$ *be a normalization of* $\mathcal{H}$, *let* U *be a small region in* W, *and let* p_y *be a normalized kernel on* U. *Then* $y \to p_y(x)$ *is a normalized kernel on* U *for the normalization of* $\mathcal{H}^*$ *given by*

$$(\{V_i\}_{i\in I}, \{(y \to p_y(x))\}_{i\in I}).$$

Of course, normalized kernels are just as unique for $\mathcal{H}^*$ as they were for $\mathcal{H}$.

Using the normalization $(\{V_i\}_{i\in I}, \{(y \to p_y^i(x))\}_{i\in I})$, we can define a fine resolution

$$0 \to \mathcal{H}^* \to \mathcal{R}^* \xrightarrow{\Delta^*} \mathcal{L}^* \to 0$$

of $\mathcal{H}^*$ by the same considerations we used in defining the resolution $0 \to \mathcal{H} \to \mathcal{R} \xrightarrow{\Delta} \mathcal{L} \to 0$. All the duality theorems above will have dual versions, obtainable simply by interchanging starred and unstarred objects throughout. While there is no good reason to transcribe the duals of all the theorems, the following dual statements would seem to be of particular interest.

PROPOSITION 4.3 ($=$ (2.3)*). — *Every element of the dual space of* $\Gamma(W, \mathcal{H})$ *(topologized by uniform convergence on compacta) can be realized in the form* $h \to \int h\, d\lambda^*$, *where* $\lambda^* \in \Gamma_K(W, \mathcal{L}^*)$. *Moreover, the space* $\Delta^*[\Gamma_K(W, \mathcal{R}^*)]$ *is contained in the annihilator of* $\Gamma(W, \mathcal{H})$, *so there is a surjection of* $H_K^1(W, \mathcal{H}^*) = \Gamma_K(W, \mathcal{L}^*)/\Delta^*[\Gamma_K(W, \mathcal{R}^*)]$ *onto the dual of* $\Gamma(W, \mathcal{H})$.

PROPOSITION 4.4 ($=$ (3.3)*). — *Let* W *be noncompact. Each of the following conditions implies its successor*:

(*a*) *The property* (A*) *of* [19] *holds, and* W *possesses an exhaustion* $\{U_i\}_{i=1}^{\infty}$ *by small regions (in particular,* W *may be small)*;

(*b*) W *possesses an exhaustion by small open sets* $\{U_i\}_{i=1}^{\infty}$ *such that* $\overline{U}_i \subseteq U_{i+1}$ *and every element of* $\Gamma(U_{i+1}, \mathcal{H})$ *can be uniformly approximated on* $\overline{U}_i$ *by the restrictions of elements of* $\Gamma(U_{i+2}, \mathcal{H})$ *to* U_{i+1}, *with each* $\overline{U}_i$ *compact*;

(*c*) $H_K^1(W, \mathcal{H}^*)$ *is Hausdorff in the inductive topology.*

If $H_K^1(W, \mathcal{H}^*)$ is known to be Hausdorff, that fact implies an approximation property like (*b*) for $\mathcal{H}$, by the dual of 3.1 above, and we also have

PROPOSITION 4.5 ($=$ (3.4)*). — *When* $H_K^1(W, \mathcal{H}^*)$ *is a Hausdorff LTS and is identified with* $\Gamma(W, \mathcal{H})'$ *by the natural pairing, the inductive topology of* $H_K^1(W, \mathcal{H}^*)$ *is identified with* *the topology* $\tau(\Gamma(W, \mathcal{H})', \Gamma(W, \mathcal{H}))$. *In particular,* $H^1_K(W, \mathcal{H}^*)$ *is nuclear, reflexive, and the dual of a Fréchet space when equipped with the inductive topology.*

Thus, with certain approximation or quasi-analyticity assumptions, we can characterize the strong dual of $\mathcal{H}_W$ as $H^1_K(W, \mathcal{H}^*)$ equipped with the inductive topology. The quasi-analyticity condition is certainly satisfied in the (self-adjoint) case of harmonic functions on a Riemann surface or Riemannian manifold, and so we have a generalization of some results of Tillmann [20]. See (C) in § 5 below.

The dualities considered here take a familiar form if we note that the following relation holds:

LEMMA 4.6. — *Let* $f \in \Gamma(W, \mathfrak{R})$ *and* $g \in \Gamma(W, \mathfrak{R}^*)$, *and suppose that one of them has compact support. Then* $\int f\, d[\Delta^* g]$ *and* $\int g\, d[\Delta f]$ *both are defined, and they are equal.*

Proof. — Since one or the other of the function and the measure in each integral has compact support, the integrals are surely defined. Suppose f has compact support. Then without loss of generality we can assume that f has small compact support, and we can write $g = g_1 + g_2$, where g_1 is supported in a small region containing the support of f and g_2 vanishes in a neighborhood of the support of f. It will clearly suffice to prove $\int f d[\Delta^* g_1] = \int g_1\, d[\Delta f]$. But if V is a small region containing the supports of f and g_1, then we have $f = \int p_y(\bullet)\, d[\Delta f](y)$ and

$$g_1(y) = \int p_y(x)\, d[\Delta^* g_1](x)$$

by 2.1 above (and its dual). But

$$\iint p_y(x)\, d[\Delta f](y)\, d[\Delta^* g_1](x) = \iint p_y(x)\, d[\Delta^* g_1](x)\, d[\Delta f](y)$$

by the Fubini theorem, Q.E.D.

Now let us put $\Gamma(W, \mathfrak{R})$ and $\Gamma_K(W, \mathfrak{L}^*)$ in duality under the bilinear form $\langle f, \lambda^* \rangle = \int f d\lambda^*$. It is easy to check that this duality is separated: if $\lambda^* \neq 0$ has small support, the fact that the differences of $\mathcal{H}$-potentials on any small region V form a positively rich subspace of $\mathcal{K}(V)$ [9, Cor.,

p. 38] insures that some f in $\Gamma_K(W, \mathcal{R})$ can be found for which $\int f\,d\lambda^* > 0$, and an easy partitioning argument handles the general case. The lemma above can now be expressed in the form

$$\langle f, \Delta^* g\rangle = \langle \Delta f, g\rangle$$

if we put $\Gamma(W, \mathcal{L})$ and $\Gamma_K(W, \mathcal{R}^*)$ in duality by setting $\langle \mu, g\rangle = \int g\,d\mu$. Again it is clear that this is a separated duality, and Δ^* is revealed as the adjoint of

$$\Delta : \Gamma(W, \mathcal{R}) \to \Gamma(W, \mathcal{L})$$

relative to the duality of those spaces with $\Gamma_K(W, \mathcal{L}^*)$ and $\Gamma_K(W, \mathcal{R}^*)$ respectively. The duality between $\Gamma(W, \mathcal{H})$ and $\Gamma_K(W, \mathcal{L}^*)/\Delta^*[\Gamma_K(W, \mathcal{R}^*)] = H^1_K(W, \mathcal{H}^*)$, therefore, is simply the duality induced on a subspace with respect to the quotient by its annihilator; or from the linear-transformation standpoint, is the duality between Ker Δ and the quotient of the dual $\Gamma_K(W, \mathcal{L}^*)$ by $\operatorname{Im}\Delta^*$.

If W is compact, the restrictions to compact supports of the considerations above are taken care of automatically.

Suppose W is compact. The results of [21, §§ 3 and 4] can be interpreted in the present setting as relating positivity properties of $\mathcal{H}$ and $\mathcal{H}^*$ for compact W. We consider first the case where $\mathbf{1} \in \overline{\mathcal{H}}_W$ but $\mathbf{1} \notin \mathcal{H}_W$, and in 4.7 through 4.9 below we do not assume the hypothesis of proportionality for $\mathcal{H}^*$.

Proposition 4.7. — *Suppose* W *is compact and* $\mathbf{1}$ *is* $\mathcal{H}$*-superharmonic but not* $\mathcal{H}$*-harmonic. Then* $\Gamma(W, \mathcal{H}^*) = 0$; *moreover, the normalization* $(\{V_i\}_{i\in I}, \{p^i_y\}_{i\in I})$ *that determines* $\mathcal{H}^*$ *also determines a unique « normalized global kernel »* $P(x, y)$ *which is* $\overline{R}^+$*-valued and lower-semicontinuous on* $W \times W$, *continuous off the diagonal, and differs from* $p^i_y(x)$ *by a harmonic function for* $x, y \in V_i$ *and* y *fixed.* $P(x, y)$ *is harmonic and* $*$*-harmonic in* x *and* y *separately off the diagonal of* $W \times W$, *and as a consequence* $\mathcal{H}^*$ *possesses global positive superharmonic functions.*

Proof. — By [21, Thm. (4.4)], $H^1(W, \mathcal{H}) = 0$; since the duality between $H^1(W, \mathcal{H})$ and $\Gamma(W, \mathcal{H}^*)$ always separates points of the latter, $\Gamma(W, \mathcal{H}^*) = 0$. To construct $P(x, y)$, take a region V_i and a point $y \in V_i$, let A be an outer-regular set with $y \in A^o \subseteq A \subseteq U \subseteq \overline{U} \subseteq V_i$, where U is an inner-regular region with compact closure contained in V_i. Then by applying [21, Thm. (3.2)] with A and U as above and $V = W \setminus A$, $M = H(\bullet, W \setminus A)$, we can find a function $P(\bullet, y)$, defined (with S as in the theorem of [21]) as $S[p^i_y | \overline{U \setminus A}]$ outside A and as

$$p^i_y - H[(p^i_y - S[p^i_y|\overline{U\setminus A}])|\partial U, U) = q_y - H(S[p^i_y|\overline{U\setminus A}]|\partial U, U)$$

inside U. (Here q_y is the normalized kernel on U.) An easy vector-integration argument shows that the function $(x, y) \to S[p^i_y|\overline{U\setminus A}](x)$ is jointly continuous and separately harmonic and $*$-harmonic for $(x, y) \in (W\setminus A) \times A^o$; a similar argument yields a similar conclusion for

$$(x, y) \to H(S[p^i_y|\overline{U\setminus A}]|\partial U, U)(x) \qquad \text{on} \qquad U \times A^o.$$

That suffices to establish the assertions made above for this function P, which is as yet only defined for $(x, y) \in W \times A^o$. Suppose P_1 is a function constructed similarly using sets A_1 and U_1 in some region V_j, and suppose $A^o \cap A^o_1 \neq \emptyset$. Then for $y \in A^o \cap A^o_1$ one sees readily that $P_1(\bullet, y) - P(\bullet, y)$ has a harmonic extension to a neighborhood of y, and therefore to all of W. Since $\mathbf{1}$ is superharmonic but not harmonic, there are no nonzero functions in $\Gamma(W, \mathcal{H})$; thus $P_1(\bullet, y) = P(\bullet, y)$. We may thus define P globally by allowing A and U (and V_i) to range over all possible choices of sets that satisfy the specifications made above.

It remains to verify that P is everywhere positive. Fix $y \in W$ and observe that $P(\bullet, y)$ is superharmonic in W and thus takes a minimum α. If $\alpha \leqslant 0$ then $-\alpha$ is superharmonic on W and $P(\bullet, y) - \alpha$ takes the minimum 0 on W. However, that would contradict the minimum principle on W (cf. the proof of [21, Thm. (3.4)]), Q.E.D.

Since $\mathcal{H}$ or $\mathcal{H}^*$ possesses a global positive superharmonic function if and only if it possesses a continuous global positive superharmonic function [4, Prop. 11, p. 95], and since renormalizing $(\{V_i\}_{i\in I}, \{p^i_y\}_{i\in I})$ and replacing $\mathcal{H}^*$ by a multiplicatively equivalent sheaf come to the same thing, the fact that $y \to P(x, y)$ is superharmonic for $\mathcal{H}^*$ gives us the following.

COROLLARY 4.8. — *If $\mathcal{H}$ possesses a global positive continuous superharmonic nonharmonic function, then so does $\mathcal{H}^*$. Alternatively, if $\mathbf{1} \in \overline{\mathcal{H}}_W \setminus \mathcal{H}_W$, then for a suitable normalization of $\mathcal{H}$ it is also true that $\mathbf{1} \in \overline{\mathcal{H}}^*_W \setminus \mathcal{H}^*_W$.*

Another corollary is the following result, which is trivial from the point of view of sheaf theory (see (D) of § 5 below) but analytically interesting because it is explicit.

COROLLARY 4.9. — *The mapping $\Delta : \Gamma(W, \mathcal{R}) \to \Gamma(W, \mathcal{L})$ is $1 - 1$ and onto, and its inverse is given by*

$$\lambda \to \int P(\bullet, y)\, d\lambda(y).$$

Proof. — It suffices to show that if A, U, V_i and p^i_y are as in the proof of 4.7 and λ has support in A^o, then $\Delta\left[\int P(\bullet, y)\, d\lambda(y)\right] = \lambda$. By vector integration it is clear that $\int P(\bullet, y)\, d\lambda(y)$ is harmonic outside A and that $\int P(\bullet, y)\, d\lambda(y)$ and $\int p^i_y(\bullet)\, d\lambda(y)$ differ by a harmonic function inside U, and $\Delta\left[\int p^i_y(\bullet)\, d\lambda(y)\right] = \lambda$ by definition of Δ.

If $\mathcal{H}^*$ satisfies the hypothesis of proportionality and possesses a positive global superharmonic non-harmonic function we can apply the reasoning we just gave to $\mathcal{H}^*$ (or to something multiplicatively equivalent to it), thereby constructing a positive global superharmonic non-harmonic function for $\mathcal{H}^{**} = \mathcal{H}$; with no loss of generality we can then assume that that function is $\mathbf{1}$. With that assumption present, we could go through the construction we just gave and construct a normalized global kernel for $\mathcal{H}^*$. The next proposition verifies that we get nothing new.

PROPOSITION 4.10. — *Suppose W is compact, $\mathbf{1}$ is $\mathcal{H}$-superharmonic but not $\mathcal{H}$-harmonic, and the hypothesis of proportionality holds for $\mathcal{H}^*$. Then if P is the function* *constructed in* 4.7 *above, the function* $y \to P(x, y)$ *has the properties of* 4.7 *relative to the normalization*

$$(\{V_i\}_{i \in I}, \{(y \to p_y^i(x))\}_{i \in I}) \quad \textit{of} \quad \mathcal{H}^*.$$

Proof. — The properties enumerated in 4.7 are symmetric in x and y, except that one needs to know in the present context that $y \to p_y^i(x) - P(x, y)$ has an $\mathcal{H}^*$-harmonic extension to a neighborhood of x in V_i. However, if U is an inner-regular neighborhood of x with $x \in U \subseteq V_i$ as in the construction of P given in 4.7 above, we know that hypotheses (1) and (2) of 1.13 above are satisfied for $s_y(x) = P(x, y)|U \times U \setminus$diagonal. (A) of that lemma then establishes the existence of the $\mathcal{H}^*$-harmonic extension of $y \to P(x, y) - p_y^i(x)$, Q.E.D.

Let us now turn to the case where W is compact and $\mathbf{1} \in \mathcal{H}_W$. By [21, Thm. (4.4)] we know that $\dim H^1(W, \mathcal{H}) = 1$; 3.8 gives us separatedness of the duality between $\Gamma(W, \mathcal{H}^*)$ and $H^1(W, \mathcal{H})$, and we have $\dim \Gamma(W, \mathcal{H}^*) = 1$. However, we can establish that directly, and a positivity result as well. To do this requires only that we reëxamine some of the material of [21, § 4] from the standpoint of the adjoint-sheaf theory. Let $(A, U) \to \Psi_{(A, U)}$ be a flux functional for $(W, \mathcal{H})$ as in [21, Thm. (3.8) and Def. (3.9)]. If V is a small subregion of W and X an open subset of V with compact (proper) closure in V, then we can define a function $g \in \Gamma(X, \mathcal{H}^*)$ by the following process. Let (A, U) be a special Cousin pair [21, § 3] with $\overline{X} \subseteq A \subseteq U \subseteq V$ and let p_y be the normalized kernel on V. Set

$$g(y) = \Psi_{(A, U)}[p_y(\bullet)|U \setminus A] = \Psi^c_{(A, U)}[p_y(\bullet)|\overline{U \setminus A}].$$

If μ_U and ν_A are the measures of [21, Thm. (3.8)], then we have (in the notation of that theorem)

$$\Psi^c_{(A, U)}[p_y|\overline{U \setminus A}] = \frac{1}{P(A, U)}\left\{\int p_y(x)\, d\nu_A(x) - \int p_y(x)\, d\mu_U(x)\right\}$$

and the usual vector-integration argument shows that $g(y)$ is an $\mathcal{H}^*$-harmonic function of y in X. Since X was arbitrary g is defined throughout V, and if V_1 were another small subregion of W on which we had gone through the same process and constructed a function $g_1 \in \Gamma(V_1, \mathcal{H}^*)$, the fact that the difference of the normalized kernels for V and V_1, say p_y and p_y^1, has an $\mathcal{H}^*$-harmonic extension to a neighborhood of y shows that the cocycles $p_y|(V \cap V_1)\setminus\{y\}$ and $p_y^1|(V \cap V_1)\setminus\{y\}$ are cohomologous and thus that $g(y) = \Psi_{(\{y\}, V\cap V_1)}[p_y|(V \cap V_1)\setminus\{y\}] = g_1(y)$. Since p_y is a potential on V, $g(y) > 0$ (again by [21, Thm. (3.8)]), and so we have constructed a global positive $\mathcal{H}^*$-harmonic function. The minimum principle implies that all global $\mathcal{H}^*$-harmonic functions must be multiples of g.

The function g has properties, in addition to its mere existence. With X, V and so forth as they were above, let p be a continuous potential on V, let π be the measure for which $p = \int p_y(\bullet)\, d\pi(y)$ (so that in the notation of the discussion preceding 1.10 above, $\pi = Z_V[p]$), and let E be a compact subset of X. Then if T is the homomorphism of presheaves constructed in making Def. (4.2) of [21], we have

$$\begin{aligned}
T_V[p](E) &= \Psi^c_{(A,U)}[\lambda_E p|\overline{U\setminus A}] \\
&= \frac{1}{P(A, U)}\left\{\iint_E p_y(x)\, d\pi(y)\, d\nu_A(x)\right. \\
&\qquad\qquad\left. - \iint_E p_y(x)\, d\pi(y)\, d\mu_U\right\} \\
&= \frac{1}{P(A, U)}\int_E \left\{\int p_y(x)\, d\nu_A(x) - \int p_y(x)\, d\mu_U(x)\right\} d\pi(y) \\
&= \int_E g(y)\, d\pi(y) = (g.\pi)(E) = (g.Z_V[p])(E).
\end{aligned}$$

Since V, p and E were arbitrary, this shows that for any section N of the sheaf $\mathcal{Q}$ of [21] we have $\tau N = g.\zeta N$ where τ is the sheaf homomorphism of [21, Def. (4.2)] and ζ is the sheaf isomorphism of 1.10 above. In other words, under the identification of the sheaf of charge distributions $\mathcal{Q}$ with the sheaf of measures $\mathcal{L}$, the *total* charge distribution of a given charge distribution is carried into g times the measure corresponding to the given charge distribution.

It is now routine to verify (by chasing the effects of renormalization through all the computations made above) that if the given normalization of $\mathcal{H}$ is renormalized by division by g, and the adjoint sheaf $\mathcal{H}^*$ thus replaced by $g^{-1}.\mathcal{H}^*$, that the constants become $\mathcal{H}^*$-harmonic and, because g is replaced by $g^{-1}.g = \mathbf{1}$, that $\tau = \zeta$ with this normalization. This last equality says precisely that charge distributions and total charge distributions are identified when $\mathcal{H}$ is so normalized that $\mathbf{1} \in \mathcal{H}^*$. We can summarize these considerations in the following proposition.

Proposition 4.11. — *Suppose* W *is compact and* $\mathbf{1} \in \mathcal{H}_{\mathrm{W}}$. *Then* $\Gamma(\mathrm{W}, \mathcal{H})$, $\Gamma(\mathrm{W}, \mathcal{H}^*)$, $\mathrm{H}^1(\mathrm{W}, \mathcal{H}^*)$ *and* $\mathrm{H}^1(\mathrm{W}, \mathcal{H})$ *are all one-dimensional, and* $\Gamma(\mathrm{W}, \mathcal{H}^*)$ *is generated by a positive function. For a suitable renormalization of* $\mathcal{H}$, $\mathbf{1} \in \mathcal{H}^*_{\mathrm{W}}$ *and the total charge distribution of* [21, Def. (4.2)] *of a measure* λ *in* $\mathfrak{L}$ *(with* $\mathfrak{L}$ *and* $\mathfrak{Q}$ *identified) is* λ *itself. In particular,* $\lambda \in \Gamma(\mathrm{W}, \mathfrak{L})$ *is the Laplacian of an element of* $\Gamma(\mathrm{W}, \mathfrak{R})$ *if and only if* $\int d\lambda = 0$, *by* [21, Thm. (4.3)].

This proposition is independent of the hypothesis of proportionality for $\mathcal{H}^*$. If that hypothesis is satisfied, the proposition dualizes :

Proposition 4.12. — *Suppose* W *is compact and* $\mathcal{H}^*$ *satisfies the hypothesis of proportionality. Then* $\mathcal{H}_{\mathrm{W}}$ *possesses a positive element if and only if* $\mathcal{H}^*_{\mathrm{W}}$ *does; for suitable replacement of* $\mathcal{H}$ *by a multiplicatively equivalent sheaf and renormalization of* $\mathcal{H}$, *the constants belong to both* $\mathcal{H}$ *and* $\mathcal{H}^*$, *and the total charge distributions of the charge distributions corresponding to sections of* $\mathfrak{L}$ *and* $\mathfrak{L}^*$ *can be identified with those measures.*

5. Notes.

(A) Special consideration of the ideal of measures λ for which $\int p_y(\bullet)\, d|\lambda|(y)$ is continuous, as well of the use of $\mathfrak{L}$ to designate it, occurs already in [9, § 31]; we have « localized » many of the considerations made there.

(B) If one traces back through the construction of normalizations made in 1.6, one sees that the construction of the global adjoint sheaf of 1.16 and 1.17 above is precisely the patching procedure described at the end of [21, § 5, (B)].

(C) The result of 3.4 or 4.5 above covers only the case of open subsets of R^n, of course. In the classical situation that Tillmann considered in [20], for open sets $U \subseteq R^n$ the dual of $\Gamma(U, \mathcal{H})$ is identified with the space of sections of the sheaf of germs of $\mathcal{H}$ on $(R^n \cup \{\infty\}) \setminus U$, the pairing being given by $\langle f, g\rangle = \int_\Sigma \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dS$ taken over a suitable contour Σ in U. Using the facts that $\mathcal{H}^* = \mathcal{H}$ and that those functions g have C^∞ extensions to the region enclosed by Σ, together with the classical second Green's formula, one can easily verify that this pairing is essentially the same as the pairing with $H^1_K(U, \mathcal{H}^*)$ discussed above; we leave the computations to the reader. In the presence of a normal structure on W it is particularly desirable to identify $H^1_K(U, \mathcal{H})$ with the space of germs of normal-harmonic functions at ∞ (a subject we shall discuss in a subsequent paper), and by making this identification we shall be able completely to generalize the results of [20].

(D) That $\Delta : \Gamma(W, \mathcal{R}) \to \Gamma(W, \mathcal{L})$ is an isomorphism under these circumstances follows from the exact sequence $0 \to \Gamma(W, \mathcal{H}) \to \Gamma(W, \mathcal{R}) \xrightarrow{\Delta} \Gamma(W, \mathcal{L}) \xrightarrow{\delta} H^1(W, \mathcal{H}) \to \cdots$ that goes with the sheaf exact sequence $0 \to \mathcal{H} \to \mathcal{R} \xrightarrow{\Delta} \mathcal{L} \to 0$ [8, p. 176, (*d*)]; for $\Gamma(W, \mathcal{H}) = 0$ by the minimum principle and $H^1(W, \mathcal{H}) = 0$ by [21, Thm. (4.4)].

(E) The construction of the function g in the discussion preceding 4.11 above is valid whenever the hypothesis of proportionality holds for $\mathcal{H}$, even if the other hypotheses of the adjoint-sheaf theory are not satisfied (of course, one cannot conclude that g belongs to $\mathcal{H}^*$). Moreover, the construction does not require that W be compact so much as that **1** belong to $\mathcal{H}_W$ and that a flux functional Ψ determined by $\mathcal{H}$ and a normal structure $\mathcal{H}^{\mathfrak{L}}$ be available. The same proof thus gives us the following proposition :

Proposition 5.1. — *Suppose* W *is noncompact,* $\mathbf{1} \in \mathcal{H}_W$, *and* $\mathcal{H}^{\mathfrak{L}}$ *is a normal structure subordinate to* $\mathcal{H}$ *as in* [21, Def. (1.3)]. *Identify the elements of* $\Gamma_K(W, \mathcal{Q})$ *with elements of* $\Gamma(W^*, \mathcal{Q}^{\mathfrak{L}})$ *in the natural way, so that under a normalization of* $\mathcal{H}$ *on* W *elements of* $\Gamma_K(W, \mathcal{L})$ *become identified with* *elements of* $\Gamma(W^*, \mathfrak{Q}^{\mathfrak{L}})$. *Then for a suitable renormalization of* $\mathcal{H}$, *this identification identifies elements of* $\Gamma_K(W, \mathfrak{Q})$ *with their total charge distributions.*

(F) Proposition 4.10 above constitutes a proof of the assertion of [21, § 5, (D)] that τ is an injection if it is known that the hypothesis of proportionality holds for $\mathcal{H}$.

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Manuscrit reçu le 25 juin 1969.

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