ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 19, nº 2 (1969), p. 155-168
<http://www.numdam.org/item?id=AIF_1969__19_2_155_0>

Ann. Inst. Fourier, Grenoble
19, 2 (1969), 155-168.

EXTENDING REGULAR FOLIATIONS (*)

by J. Wolfgang SMITH

1. Introduction.

In this paper M shall denote an open orientable differentiable n-manifold. To fix our ideas, we take « differentiable » to mean C^∞ throughout, and we suppose M to be Hausdorff and have empty boundary (1). Let F denote a p-dimensional differentiable foliation (2) on M, i.e. a completely integrable smooth p-dimensional differential system on M with $0 < p < n$. Thus F assigns to every $x \in M$ a p-dimensional subspace of the tangent vectorspace M_x, and moreover, every point of M lies on a unique p-dimensional maximal integral manifold of F (in the sense of Chevalley [1]). These integral manifolds will be referred to as the *leaves* of F, and we let $\pi : M \to M/F$ denote the natural projection of M onto the quotient space M/F obtained by identifying points belonging to the same leaf. The foliation is called *regular* if π admits local cross-sections. For a regular foliation F the quotient M/F can be regarded as a differentiable m-manifold (with $m = n - p$), and π will then be differentiable. However, M/F will not in general be Hausdorff. The manifold M being orientable, we can define orientability for F by the condition that M/F be orientable, and this will henceforth be assumed. The tangent bundle $\tau(M/F)$ has then an Euler class (3) χ_F, whose algebraic sign depends

(*) Research supported in part by National Science Foundation Grant GP-6648.

(1) None of these suppositions is in fact crucial.

(2) For basic terminology and results relating to foliations we refer to Palais [5], Chapter I.

(3) Milnor [4]. Although Milnor takes the base space to be Hausdorff, his construction of the Euler class does not depend upon this assumption.

upon a choice of orientation. Assuming such a choice (or alternatively taking χ_F to be determined modulo algebraic sign) we shall refer to χ_F as the *Euler class* of F. It may be remarked that the notion of an Euler class applies to nonregular foliations as well, but it cannot in general be defined in terms of a bundle over the quotient space [4].

Let us suppose, for the moment, that M/F is Hausdorff. Then it is known to be triangulable, and by the classical obstruction theory [5] it will admit a nonzero vectorfield (or a direction field) if and only if χ_F is zero. Moreover, a direction field on M/F pulls back under π to a $(p+1)$ — dimensional orientable foliation $\hat{F}$ on M such that $F \subset \hat{F}$ in the obvious sense. Such a foliation $\hat{F}$ will be called an *extension* of F. Conversely one sees that an extension of F gives rise to a direction field on M/F (whose algebraic sign is determined by a choice of orientations). When M/F is not Hausdorff it is still true that the existence of direction fields on M/F is equivalent to the existence of extensions of F, and it is also true that the vanishing of χ_F constitutes a necessary condition for the existence of these structures. The question of sufficiency, however, appears to be open. The main result of this paper asserts sufficiency in the following weakened sense:

THEOREM A. — *An orientable regular foliation on* M *with vanishing Euler class extends on relatively compact subsets of* M.

Thus when $\chi_F = 0$ an extension $\hat{F}$ of F exists at least on all relatively compact subsets $D \subset M$. It may be noted that no corresponding solution of the direction field problem for arbitrary non-Hausdorff manifolds can be envisaged.

At this point one is naturally interested to find geometric conditions on M and F which imply that $\chi_F = 0$. The following leads to one such set of conditions.

LEMMA B. — *Let* F *be a regular* 1*-dimensional foliation on* M *without compact leaves. Then* $\pi : M \to M/F$ *induces an isomorphism between the respective singular homology groups.*

[4] Smith [6].
[5] Steenrod [9].

We note that the Euler class of a q-plane bundle has order 2 when q is odd ([6]). The notation $H^q(M) = Q$ will signify that the q-dimensional singular integral cohomology of M vanishes or has no torsion of order 2, depending on whether q is even or odd, respectively. Combining Theorem A with Lemma B thus gives :

THEOREM B. — *Let* F *be a regular orientable 1-dimensional foliation on* M *without compact leaves. If* $H^{n-1}(M) = Q$, *then* F *extends on relatively compact subsets of* M.

This result can be sharpened if one replaces F by a corresponding vectorfield. A nonzero differentiable vectorfield on M will be called *nonrecurrent* if the induced foliation is regular and admits no compact leaves. Using Theorem B we establish the following results in [8] :

THEOREM C. — *Let* X *be a nonrecurrent vectorfield on* M *and let* $D \subset M$ *be relatively compact. If* $H^{n-1}(M) = Q$, *there exists a vectorfield* Y *on* D *such that* X, Y *are linearly independent and commute.*

THEOREM D. — *If* $H^{n-1}(M) = Q$, *then every relatively compact subset of* M *submerges in the plane.*

The present paper is devoted to a proof of Theorem A and Lemma B. An essential ingredient in our proof of Theorem A is a triangulation theorem which may also be of independent interest. We are greatly indebted to J. R. Munkres for having contributed the Appendix to this paper, setting forth a proof of this result.

2. Equivariant vectorfields.

Let $E_0, \ldots, E_s$ denote differentiable m-manifolds. For each pair (i, j) of indices, let

$$\varphi_{ij} : (U_{ij}, A_{ij}) \to (U_{ji}, A_{ji})$$

denote a diffeomorphism, where $U_{ij} \subset E_i$ is open and $A_{ij} \subset U_{ij}$

([6]) Milnor [4], p. 41. This again does not involve the assumption of a Hausdorff base space.

is compact. We shall assume that this family of diffeomorphisms satisfies the pseudo-group conditions

$$\varphi_{ii} = \text{identity}$$
$$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij} \quad \text{(whenever the composition is defined).}$$

Let E denote the disjoint union of the manifolds E_i. A vectorfield X on E will be called *A-equivariant* provided

$$\begin{array}{ccc} TA_{ij} & \xrightarrow{d\varphi_{ij}} & TA_{ji} \\ X\uparrow & & \uparrow X \\ A_{ij} & \xrightarrow{\varphi_{ij}} & A_{ji} \end{array}$$

commutes for every index pair (i, j). Here TA_{ij} denotes the total space of the tangent bundle $\tau(E)$ restricted to A_{ij}, and $d\varphi_{ij}$ denotes the differential of φ_{ij}.

THEOREM 1. — *Let* E *be oriented and the* φ_{ij} *orientation preserving. Let* N *be an orientable differentiable m-manifold* (not necessarily Hausdorff) *with vanishing Euler class, and* $\pi : E \to N$ *an immersion such that*

$$\begin{array}{ccc} U_{ij} & \xrightarrow{\varphi_{ij}} & U_{ji} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & N & \end{array} \tag{2.1}$$

commutes for every index pair (i, j). *Then* E *admits a differentiable nonzero A-equivariant vectorfield.*

By the triangulation theorem of J. R. Munkres (see Appendix) there exist triangulations K_i of E_i and finite subcomplexes L_{ij} of K_i such that

i) $A_{ij} \subset |L_{ij}| \subset U_{ij}$

ii) φ_{ij} maps $|L_{ij}|$ simplicially onto $|L_{ji}|$.

We shall establish the existence of a differentiable nonzero vectorfield X on E such that

$$\begin{array}{ccc} T|L_{ij}| & \xrightarrow{d\varphi_{ij}} & T|L_{ji}| \\ X\uparrow & & \uparrow X \\ |L_{ij}| & \xrightarrow{\varphi_{ij}} & |L_{ji}| \end{array} \tag{2.2}$$

commutes for all (i, j). Since $A_{ij} \subset |L_{ij}|$, X will then be A-equivariant.

Let R_L denote the equivalence relation on E generated by all pairs $(x, y) \in |L_{ij}| \times |L_{ji}|$ such that $y = \varphi_{ij}(x)$. Let E denote the quotient space E/R_L and $\alpha : E \to \overline{E}$ the projection. We observe that the given triangulation K of E induces a triangulation of $\overline{E}$. Moreover, by the usual clutching construction [7] the bundle $\tau(E)$ induces an orientable m-plane bundle ξ over $\overline{E}$, together with a bundle map $h : \tau(E) \to \xi$ over α. In other words, for every pair (i, j) were are « glueing together » the bundle spaces $T|L_{ij}|$ and $T|L_{ji}|$ via $d\varphi_{ij}$. The total space $T\xi$ of ξ is thus a quotient of TE, and $h : TE \to T\xi$ a projection. Now let

$$(x, \bar{x}, w) \in E \times \overline{E} \times N$$

such that $\bar{x} = \pi(x)$ and $w = \alpha(x)$. Let $h_x : E_x \to \xi_{\bar{x}}$, $d\pi_x : E_x \to N_w$ denote the fibre isomorphism induced by the bundle maps h and $d\pi$, respectively, and let $g_{\bar{x}} = d\pi_x \circ (h_x)^{-1}$. Commutativity of the diagrams (2.1) implies that $g_{\bar{x}}$ depends only on $\bar{x}$, and one obtains thus a function $g : T\xi \to TN$ which restricts to an isomorphism on the fibres. Since $d\pi = g \circ h$ and h is a projection, g is continuous because $d\pi$ is continuous. Hence $g : \xi \to \tau(N)$ is a bundle map.

It now follows by naturality that ξ has vanishing Euler class. Moreover, since $\overline{E}$ is triangulable, one obtains thus a nonzero cross-section $\overline{X} : \overline{E} \to T\xi$ by classical obstruction theory [8] and this pulls back under h to a nonzero vectorfield X^0 on E. But our construction clearly implies commutativity of the diagrams (2.2).

However, the construction does not guarantee differentiability of X^0. We will complete the argument by showing that X^0 can be approximated by a smooth vectorfield X without losing the commutativity conditions (2.2). Let $A \subset E$ denote an open relatively compact subset containing $|L_{ij}|$ for all (i, j), and let $B = E - A$. It is not difficult to see that there exists then a nonzero differentiable vectorfield X^*

(7) Husemoller [3], p. 123.
(8) Steenrod [9], § 39.6.

on E which agrees with X^0 on every $|L_{ij}|$ and is differentiable on B. For every index k we let

$$V_k = \bigcup_{i \leqslant k} E_i, \qquad W_k = \bigcup_{i \leqslant k} |L_{ki}|\,;$$

and we let $\mathcal{X}_k$ denote the set of all nonzero differentiable vectorfields X defined on $B \cup V_k$ such that i) X agrees with X^* on B, ii) the diagram (2.2) commutes for all pairs (i, j) with $i, j \leqslant k$. We observe that every $X \in \mathcal{X}_{k-1}$ determines a nonzero differentiable vectorfield $\hat{X}$ on $B \cup W_k$ (in an obvious way). We would like to argue that if X is « sufficiently close » to X^*, $\hat{X}$ will be near enough to X^* to extend to a nonzero vectorfield on $B \cup E_k$. By a well known result [9] this would imply that $\hat{X}$ extends to a nonzero *differentiable* vectorfield on $B \cup E_k$, and consequently that X extends to a vectorfield in $\mathcal{X}_k$. To make this precise, let ρ denote a Riemannian metric on $\tau(E)$. For every $S \subset E$ and vectorfield X defined on S, let

$$\mu(X|S) = \underset{x \in S}{\text{l.u.b.}} \; \{\rho(X_x, X_x^*)\},$$

where X_x, X_x^* denote the respective tangent vectors at x. Compactness of the spaces W_k and relative compactness of A permit us to make the following observations:

1) There exists a constant $\lambda > 1$ such that

$$\mu(X|W_k) < \lambda\mu(X|V_{k-1})$$

for all $k > 0$ and $X \in \mathcal{X}_{k-1}$.

2) There exists an $\varepsilon > 0$ such that for every $k > 0$ and $X \in \mathcal{X}_{k-1}$ with $\mu(X|W_k) < \delta < \varepsilon$, X extends to a vectorfield $Y \in \mathcal{X}_k$ with $\mu(Y|E_k) < \delta$.

But this does the trick. For we can choose $X_0 \in \mathcal{X}_0$ such that

$$\mu(X_0|E_0) < \frac{\varepsilon}{\lambda^s}.$$

This implies by 1) that

$$\mu(\hat{X}_0|W_1) < \frac{\varepsilon}{\lambda^{s-1}} < \varepsilon$$

[9] Steenrod [9], § 6.7.

and hence by 2) that X_0 extends to $X_1 \in \mathscr{X}_1$ with

$$\mu(X_1 | V_1) < \frac{\varepsilon}{\lambda^{x-1}}$$

By induction one thus obtains a vectorfield $X_s \in \mathscr{X}_s$.

3. Proof of Theorem A.

For $r > 0$ and q a positive integer let J_r^q denote the open cube in R^q given by

$$\sum_{i=1}^{q} |t_i| < r,$$

where the t_i denote natural coordinates in R^p. A differentiable chart $\psi : J_r^m \times J_s^p \to M$ will be called *flat* (with respect to the foliation F) if for every $u \in J_r^m$ the points $\{\psi(u, v) | v \in J_s^p\}$ lie on a single leaf of F. A flat chart ψ is ***regular*** if every leaf of F meets $\psi(J_r^m \times J_s^p)$ in at most one connected component. Since F is regular, every point of M is covered by a flat regular chart.

Let D be a relatively compact subset of M. There exists then a finite family of flat regular charts $\psi_j : J_2^m \times J_1^p \to M$ such that $\{\psi_j(J_1^m \times J_1^p)\}$ constitutes a covering of D. Let

$$E_j = \psi_j(J_2^m \times 0), \qquad B_j = \psi_j(J_1^m \times 0)$$
$$V = \bigcup_j \psi_j(J_2^m \times J_1^p), \qquad W = \bigcup_j \psi_j(J_1^m \times J_1^p);$$

and let $\overline{B}_i$, $\overline{W}$ denote the respective closures. Thus every E_j is diffeomorphic to an open m-cube and $\overline{B}_j \subset E_j$ is compact. Moreover, W is an open subset of M containing D. For any subset $S \subset M$, let R_S denote the equivalence relation on S consisting of all pairs (x, y) such that x and y are connected by a curve in S lying in a single leaf of F. For every index pair (i, j) let U_{ij} denote the set of all $x \in E_i$ such that $(x, y) \in R_V$ for some $y \in E_j$. We note that this point y is uniquely determined by x (regularity of ψ_j), so that one obtains functions $\varphi_{ij} : U_{ij} \to U_{ji}$. Similarly, for every pair (i, j) let A_{ij} denote the set of all $x \in \overline{B}_i$ such that $(x, y) \in R_{\overline{W}}$ for some $y \in \overline{B}_j$. The following assertions are easily verified.

LEMMA 3.1. — *For every index pair* (i, j), $U_{ij} \subset E_i$ *is open and* $A_{ij} \subset U_{ij}$ *compact. Each* φ_{ij} *constitutes a diffeomorphism of* U_{ij} *onto* U_{ji} *and maps* A_{ij} *onto* A_{ji}. *The family of these diffeomorphisms satisfies the pseudogroup conditions* (as given in Theorem 1).

The disjoint union E of the spaces E_j constitutes a differentiable m-manifold, and we note that E can be oriented so as to render every φ_{ij} orientation preserving. Moreover, the natural projection $\pi : E \to M/F$ commutes with every φ_{ij}. By Lemma 3.1 and Theorem 1 one concludes that E admits a differentiable nonzero A-equivariant vectorfield X. Since $D \subset W$, it will suffice to prove:

LEMMA 3.2. — X *induces an extension of* F *on* W.

Let F_0 denote the restriction of F to W and $\beta : W \to W/F_0$ the natural projection. Thus W/F_0 constitutes a differentiable m-manifold (not necessarily Hausdorff), and β maps each B_j diffeomorphically onto a subset $V_j \subset W/F_0$. The restriction of X to B_j consequently induces a nonzero differentiable vectorfield Y_i on V_j. Moreover, A-equivariance of X implies that Y_i, Y_j agree on $V_i \cap V_j$ for every index pair (i, j). For if $\wp \in V_i \cap V_j$ and x, y denote the corresponding points in B_i and B_j, respectively, then $(x, y) \in R_W \subset R_{\overline{W}}$. Hence $y = \varphi_{ij}(x)$ and $X_y = d\varphi_{ij}(X_x)$ by A-equivariance. Since β commutes with φ_{ij}, it follows that $d\beta(X_x) = d\beta(X_y)$, as claimed. But the subsets $\{V_j\}$ cover W/F_0, and one obtains thus a nonzero differentiable vectorfield Y on W/F_0, which in turn determines a 1-dimensional foliation H. Finally, H pulls back under $\beta : W \to W/F_0$ to an orientable $(p+1)$-dimensional foliation [10] on W which extends F.

4. Proof of Lemma B.

Let F be a regular 1-dimensional foliation on M without compact leaves, and let $\pi : M \to N$ denote the natural projection, where $N = M/F$. Neither M nor F are required

[10] By Palais [5], Chapter I, Theorem XIII.

to be orientable. It will be shown that the induced map $\pi_{\#} : C_{\#}(M) \to C_{\#}(N)$ between the respective singular chain complexes constitutes a chain homotopy equivalence.

We observe that this assertion is quite trivial in case N is Hausdorff. Choosing a complete Riemannian metric on M determines (11) a bundle structure for $\pi : M \to N$ with fibre R (the real line) and structure group G consisting of all transformations of the form $t \to (\pm t + a)$, with $a \in R$. The fibre being contractible and N being a Hausdorff manifold implies (12) that there exists a cross-section $s : N \to M$. The restriction of π to $s(N)$ is then a homeomorphism, and $s(N)$ is clearly a deformation retract of M. Thus one obtains the desired conclusion. On the other hand, if N is not Hausdorff, a cross-section of π may not exist. Consider M, for example, to be a punctured plane foliated by a parallel family of straigh lines. The leaf space N is then the real line with a single point doubled, and it is clear that a cross-section $s : N \to M$ does not exist.

To prove Lemma B, we choose a complete Riemannian metric on M and an open covering $\mathcal{V}$ of N such that every $V \in \mathcal{V}$ admits a local cross-section $s_V : V \to M$. The metric, together with s_V, permits us to define a projection $p_V : \pi^{-1}(V) \to R$, and this gives a homeomorphism $\theta_V : \pi^{-1}(V) \to V \times R$ by setting $\theta_V(x) = (\pi(x), p_V(x))$.

Let $C_{\#}(N, \mathcal{V})$ denote the subcomplex of $C_{\#}(N)$ generated by singular simplexes subordinate to $\mathcal{V}$. The inclusion $C_{\#}(N, \mathcal{V}) \to C_{\#}(N)$ is then a chain homotopy equivalence (13). Similarly we let $\mathcal{W} = \pi^{-1}(\mathcal{V})$ and observe that the inclusion $C_{\#}(M, \mathcal{W}) \to C_{\#}(M)$ is likewise a chain homotopy equivalence. It will therefore clearly suffice to show that

$$\pi_{\#} : C_{\#}(M, \mathcal{W}) \to C_{\#}(N, \mathcal{V})$$

is a chain homotopy equivalence. This will be accomplished by constructing a chain map $\tau : C_{\#}(N, \mathcal{V}) \to C_{\#}(M, \mathcal{W})$ which preserves singular simplexes and satisfies $\pi_{\#} \circ \tau = 1$. In other words, instead of constructing a cross-section $s : N \to M$ (which may not exist), we construct a chain cross-section τ

(11) Smith [8].
(12) Steenrod [9], § 12.2.
(13) Eilenberg and Steenrod [2], Theorem 8.2.

for $\pi_{\#} : C_{\#}(M, \mathfrak{W}) \to C_{\#}(N, \mathfrak{V})$. But τ determines a chain homotopy $D : C_{\#}(M, \mathfrak{W}) \to C_{\#}(M, \mathfrak{W})$ by the following construction: Let $\sigma : \Delta_q \to M$ be a singular q-simplex subordinate to $\mathfrak{W}$ and let $\sigma_0 = \tau \circ \pi_{\#}(\sigma)$. For every $y \in \Delta_q$, the points $\sigma_0(y)$ and $\sigma(y)$ belong to the same leaf of F. Since every leaf of F is homeomorphic to R, the two points determine a homeomorph $[\sigma_0(y), \sigma(y)]$ of a directed line segment. We can therefore define a singular prism $P_\sigma : \Delta_q \times I \to M$ (I denotes the unit interval) by letting $P_\sigma(y, t)$ be the point in $[\sigma_0(y), \sigma(y)]$ which divides this segment in the ration $t : 1$, this being understood in terms of the distance function on $[\sigma_0(y), \sigma(y)]$ induced by our Riemannian metric. Continuity of P_σ is immediate, and by the usual process ([14]) the correspondance $\sigma \to P_\sigma$ determines a chain homotopy D. Moreover, one can verify by an easy calculation that

$$\partial D_q + D_{q-1} \partial = 1 - \tau \circ \pi_{\#},$$

where ∂ and 1 denote the boundary operator and identity map of $C_{\#}(M, \mathfrak{W})$, respectively. It remains, therefore, to establish the existence of τ.

To this end we make the inductive hypothesis that $\tau_q : C_q(N, \mathfrak{V}) \to C_q(M, \mathfrak{W})$ has been defined for all $q < r$, subject to the conditions

$$\tau_{q-1} \circ \partial_q = \partial_q \circ \tau_q. \tag{4.1}$$

More precisely, for every singular q-simplex $\sigma : \Delta_q \to N$ subordinate to $\mathfrak{V}$, $\tau_q(\sigma)$ is assumed to be a singular q-simplex $\bar{\sigma} : \Delta_q \to M$ such that $\pi \circ \bar{\sigma} = \sigma$. Now let $\sigma : \Delta_r \to V$ denote a singular r-simplex, with $V \in \mathfrak{V}$. The function τ_{r-1} determines then a map $h_\sigma : \dot{\Delta}_p \to M$ by virtue of condition (4.1), where $\dot{\Delta}_r$ denotes the boundary of Δ_r. This defines a map $p_V \circ h_\sigma : \dot{\Delta}_r \to R$, which can be extended to a map $\varphi_\sigma : \Delta_r \to R$. Let $g_\sigma : \Delta_r \to M$ be defined by setting $g_\sigma(y) = \theta_V^{-1}(\sigma(y), \varphi_\sigma(y))$. One now has a commutative diagram

$$\begin{tikzcd}
& & \pi^{-1}(V) \arrow[d, "\pi"] \\
\dot{\Delta}_r \arrow[r] \arrow[urr, "h_\sigma"] & \Delta_r \arrow[r, "\sigma"] \arrow[ur, "g_\sigma"'] & V
\end{tikzcd}$$

([14]) Eilenberg and Steenrod [2], Chapter VII, § 6.

Setting $\tau_r(\sigma) = g_\sigma$ defines τ_r on the generators of $C_r(N, \mathfrak{V})$, and we extend by linearity. It is obvious that τ_r is a simplex preserving cross-section of $\pi_{\#}$, and commutativity of (4.2) implies condition (4.1) with $q = r$. This establishes the existence of τ.

Appendix.

(This appendix was written by J. R. Munkres.)

DEFINITION. — *Let* $f_i : |K_i| \to R^m$ *be a homeomorphism where* K_i *is a finite complex and* $i = 1, \ldots, n$. *We say that* $(K_1, f_1), \ldots, (K_n, f_n)$ *intersect in subcomplexes if for each* (i, j), $f_i^{-1}(f_i(|K_i|) \cap f_j(|K_j|))$ *is the polytope of a subcomplex* L_{ij} *of* K_i *and if* $f_j^{-1}f_i$ *is a linear isomorphism of* L_{ij} *with* L_{ji}. *They are said to intersect in full subcomplexes if each* L_{ij} *is full in* K_i. (This means that a simplex of K_i belongs to L_{ij} if all its vertices are in L_{ij}.) It is easy to see that if $(K_1, f_1), \ldots, (K_n, f_n)$ intersect in subcomplexes, then $(K'_1, f_1), \ldots, (K'_n, f_n)$ intersect in full subcomplexes, where K'_i is the first barycentric subdivision of K_i.

If $(K_1, f_1), \ldots, (K_n, f_n)$ intersect in full subcomplexes, then there exists a complex K and a homeomorphism f: $K \to R^m$ such that $f(|K|) = \bigcup_j f_j(|K_j|)$ and such that $f^{-1}f_j$ is a linear isomorphism of K_j with a subcomplex of K for each j. Furthermore, (K, f) is unique up to linear isomorphism. It is called the *union* of $(K_1, f_1), \ldots, (K_n, f_n)$. (Compare 10.1 of [EDT].)

Now suppose that $(K_1, f_1), \ldots, (K_n, f_n)$ intersect in full subcomplexes and that each $f_i : K_i \to R^m$ is a smooth imbedding, in the sense of 8.3 of [EDT]. This means not only that it is a topological imbedding which is smooth on each simplex of K_i, but also that the differential is one-to-one. The union (K, f) will not be an imbedding except under additional hypotheses. (See 10.1 of [EDT].) However, one can say the following:

LEMMA 1. — *Let* M_i *be a subcomplex of* K_i *such that* $f_i(|M_i|) \subset \operatorname{Int} f_i(|K_i|)$. *Then the union of* $(M_1, f_1), \ldots, (M_n, f_n)$ *is a smooth imbedding.*

Proof. — Let (K, f) be the union of $\{(K_i, f_i)\}$; the union M of $\{(M_i, f_i)\}$ may be taken as a subcomplex of K. Let x be a point of M_i. Then $f_i : K_i \to R^m$ triangulates a neighborhood of $f_i(x)$, and so does $f : K \to R^m$, so that $f^{-1}f_i : K_i \to K$ is a homeomorphism of $\overline{St}(x, K_i)$ with $\overline{St}(x, K)$. Since $f_i : \overline{St}(x, K_i) \to R^m$ is an imbedding, $d(f_i)_x$ is $1-1$, and hence so is df_x.

LEMMA 2. — *Let* A *be a closed subset of the differentiable manifold* M. *Let* $f : K \to M$ *be a smooth imbedding such that* $A \subset \operatorname{Int} f(|K|)$. *If there is a subcomplex* K_0 *of* K *such that* $f|K_0$ *triangulates* A, *then* $f|K_0$ *may be extended to a triangulation of* M.

This lemma is problem 10.7 of [EDT]. It can be proved by straightforward application of the triangulation techniques of J. H. C. Whitehead expounded there.

THEOREM. — *Let* $E_0, \ldots, E_n$ *be differentiable m-manifolds. Suppose that for each pair* (i, j) *of indices, we are given a diffeomorphism*

$$\varphi_{ij} : (U_{ij}, A_{ij}) \to (U_{ji}, A_{ji}),$$

where U_{ij} *is an open subset of* E_i *and* A_{ij} *is a compact subset of* U_{ij}. *Furthermore,* φ_{ii} *is the identity and* $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ *whenever the composition is defined.* (This implies $U_{ik} \supset$ domain $(\varphi_{jk} \circ \varphi_{ij}.)$

Then there are smooth triangulations K_i *of* E_i, *and finite subcomplexes* L_{ij} *of* K_i, *such that*

i) $A_{ij} \subset |L_{ij}| \subset U_{ij}$

ii) φ_{ij} *maps* $|L_{ij}|$ *simplicially onto* $|L_{ji}|$.

Proof. — We proceed by induction on n. The theorem is trivial for $n = 0$. Suppose it is true for $n - 1$.

Choose compact sets B_{0j} $(j = 1, \ldots, n)$ such that

$$A_{0j} \subset \operatorname{Int} B_{0j} \qquad \text{and} \qquad B_{0j} \subset U_{0j}.$$

Then for $1 \leqslant i < j \leqslant n$, choose a compact set $B_{ij} \subset U_{ij}$ such that

$$A_{ij} \subset B_{ij} \qquad \text{and} \qquad \varphi_{0i}(B_{0i} \cap B_{0j}) \subset B_{ij}.$$

This makes sense because

$$B_{0i} \cap B_{0j} \subset U_{0i} \cap U_{0j} = \varphi_{i0} \text{ (domain } (\varphi_{0j} \circ \varphi_{i0})) \subset \varphi_{i0}(U_{ij}),$$

so that $\varphi_{0i}(B_{0i} \cap B_{0j}) \subset U_{ij}$. Finally, for $0 \leqslant i < j \leqslant n$, set $B_{ji} = \varphi_{ij}(B_{ij})$.

Now apply the induction hypothesis to the manifolds $E_1, \ldots, E_n$, using

$$\varphi_{ij} : (U_{ij}, B_{ij}) \to (U_{ji}, B_{ji})$$

as the diffeomorphisms. We then have complexes $K^?$ smoothly triangulating E_i, and subcomplexes L_{ij} of K_i $(1 \leqslant i, j \leqslant n)$ such that $B_{ij} \subset |L_{ij}| \subset U_{ij}$ and φ_{ij} is a linear isomorphism of L_{ij} with L_{ji}.

We then proceed to triangulate E_0. First, we may assume that *mesh* K_i is less than one-third the distance from A_{i0} to $E_i - B_{i0}$, for $i = 1, \ldots, n$. (For this situation may be obtained by choosing a very large p and replacing each K_i and L_{ij} by its *pth* barycentric subdivision.) This means that for $i = 1, \ldots, n$, we may choose subcomplexes L_{i0}, M_{i0}, and N_{i0} of K_i such that

$$A_{i0} \subset |L_{i0}| \subset \operatorname{Int} |M_{i0}| \qquad \text{and} \qquad |M_{i0}| \subset \operatorname{Int} |N_{i0}| \subset B_{i0}.$$

Consider the maps $\varphi_{i0} : N_{i0} \to E_0$. Because φ_{i0} is a diffeomorphism on U_{i0} and N_{i0} is a smoothly imbedded complex in E_i, this map is a smooth imbedding of N_{i0} in E_0. We claim also these maps intersect in subcomplexes: For $\varphi_{i0}(N_{i0}) \cap \varphi_{j0}(N_{j0}) \subset B_{0i} \cap B_{0j}$, so that $\varphi_{i0}^{-1}(\varphi_{i0}(N_{i0}) \cap \varphi_{j0}(N_{j0}))$ is contained in $B_{ij} \subset L_{ij}$. This implies that

$$\varphi_{i0}^{-1}(\varphi_{i0}(N_{i0}) \cap \varphi_{j0}(N_{j0})) = N_{i0} \cap \varphi_{ji}(N_{j0} \cap L_{ji}),$$

which is clearly a subcomplex of K_i (since φ_{ji} is by assumption simplicial on L_{ji}). Futhermore, the map $\varphi_{j0}^{-1}\varphi_{i0}$ is a linear isomorphism of this subcomplex of K_i with a subcomplex of K_j, since the subcomplex is contained in L_{ij} and the map equals φ_{ij} there.

Without change of notation, let us replace each K_i, L_{i0}, M_{i0}, N_{i0}, and L_{ij} $(1 \leqslant i, j \leqslant n)$ by its first barycentric subdivision. The maps $\varphi_{i0} : N_{i0} \to E_0$ are still smooth imbeddings but now they intersect in full subcomplexes.

By Lemma 1, the union

$$\varphi_0 : M_0 \to E_0 \quad \text{of} \quad (M_{10}, \varphi_{10}), \ldots, (M_{n0}, \varphi_{n0})$$

is now an imbedding. The union of $(L_{10}, \varphi_{10}), \ldots, (L_{n0}, \varphi_{n0})$ may be considered as a subcomplex L_0 of M_0, and $\varphi_0(|L_0|)$ lies in the interior of $\varphi_0(|M_0|)$. By Lemma 2, $\varphi_0 : L_0 \to E_0$ may be extended to a smooth triangulation of E_0. Said differently, there is a complex K_0 smoothly triangulating E_0 such that φ_0 is a linear isomorphism of L_0 with a subcomplex of K_0. Then φ_{i0} is a linear isomorphism of L_{i0} with a subcomplex of K_0 which we denote by L_{0i}.

The proof of the theorem is now complete.

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Manuscrit reçu le 5 juin 1969.

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