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## CHARACTERIZATION OF SEPARABILITY FOR $LF$ -SPACES

by Giovanni VIDOSSICH

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This note characterizes the separability of  $LF$ -spaces by five equivalent conditions, one being that all members of a given defining sequence (*= suite de définition* according to [2]) are separable. These conditions imply that the dual space must be hereditarily separable and Lindelöf for the weak\* topology (*= topology  $\sigma(X', X)$  of [2]*).

Concerning uniform spaces, we shall employ the terminology (and results) of the first two chapters of [4]. We shall denote by

**F**

the scalar field, which is  $\mathbf{R}$  or  $\mathbf{C}$ ; and we shall say

*weak topology induced by  $H \subseteq F(X, Y)$*

the less fine topology on  $X$  making continuous all members of  $H$  (caution that this is a purely topological definition). Finally, an  $\mathfrak{n}_0$ -space is — according to [5] — a regular space  $X$  where there exists a countable *pseudobase*  $\mathfrak{P}$ , i.e. a countable  $\mathfrak{P} \subseteq \mathfrak{P}(X)$  such that for every compact  $K \subseteq X$  and every open  $U \subseteq X$  which contains  $K$  it follows  $K \subseteq P \subseteq U$  for a suitable  $P \in \mathfrak{P}$ .

**THEOREM.** — *Let  $X$  be an  $LF$ -space and  $(E_n)_{n=1}^\infty$  a defining sequence of  $X$ . The following statements are pairwise equivalent :*

- (1)  *$X$  is separable.*
- (2)  *$X$  is weakly separable.*
- (3) *Every weakly\* compact subset of  $X'$  is weakly\* metrizable.*

(4) *Every equicontinuous subset of  $X'$  has a countable base for the weak\* topology.*

(5) *Every  $E_n$  is separable.*

(6)  *$X$  is an  $\aleph_0$ -space.*

*Proof.* — (1)  $\rightarrow$  (2): Clear.

(2)  $\rightarrow$  (3): Let  $e: X \rightarrow X''$  be the canonical map  $x \mapsto (f(x))_{f \in X'}$  and  $K$  a weakly\* compact subset of  $X'$ . Then  $e': x \mapsto e(x)|_K$  is a continuous map from  $X$  equipped with the weak topology into the topological subspace  $e'(X)$  of  $F_p(K, \mathbf{F})$  (= product space of  $\text{Card}(K)$  copies of  $\mathbf{F}$ ). By (2),  $e'(X)$  is separable: let  $H$  be a countable dense subset of it. The weak\* topology of  $K$  is exactly the weak topology induced by  $e'(X) \subseteq F(K, \mathbf{F})$ : by [3, p. 175, Footnote], this topology equals the weak topology induced by  $H \subseteq F(K, \mathbf{F})$  and therefore it is metrizable.

(3)  $\rightarrow$  (4): Because the weak\* closure of an equicontinuous set is weakly\* compact by [2, Th. 3].

(4)  $\rightarrow$  (5): By [2, Cor. to Th. 3], there is a linear homeomorphism  $e$  from  $X$  onto a subspace  $e(X)$  of  $L_{\mathfrak{C}}(X', \mathbf{F})$ ,  $\mathfrak{C}$  being a suitable cover of  $X'$  consisting of weakly\* compact subsets of  $X'$  and  $L_{\mathfrak{C}}(X', \mathbf{F})$  the space of weakly\* continuous linear functionals on  $X'$  with the topology of uniform convergence on members of  $\mathfrak{C}$ . By [2, Th. 3], the members of  $\mathfrak{C}$  are equicontinuous and hence weakly\* metrizable by (4). By a well known theorem contained in [5, (J) and (D)],  $C_u(K, \mathbf{F})$  (= uniform space made of all weakly\* continuous maps  $K \rightarrow \mathbf{F}$  and the uniformity of uniform convergence) is separable for all  $K \in \mathfrak{C}$  and therefore the uniformity of  $C_u(K, \mathbf{F})$  has a basis of *countable* uniform coverings (as follows easily, if you want, from [4, ii. 33 and ii. 9]). Consequently the uniformity  $\pi$  of  $\prod_{K \in \mathfrak{C}} C_u(K, \mathbf{F})$  — and hence the trace of  $\pi$  on every subset — has a basis of *countable* uniform covers as it follows directly from the definition of product uniformity [4, Exercise ii. 2] (alternatively, this result may be deduced from [1, Prop. 3] and [4, ii. 33 and ii. 9]). It is well known that there is a uniform embedding  $e^*: L_{\mathfrak{C}}(X', \mathbf{F}) \rightarrow \prod_{K \in \mathfrak{C}} C_u(K, \mathbf{F})$ , the last space being equipped with the product uniformity  $\pi$ .

By what has been proved, the uniformity induced by  $\pi$  on  $e^*(e(X))$  has a basis consisting of countable uniform coverings: consequently — because a linear homeomorphism is a uniform isomorphism for the (canonical) uniformities of linear topological spaces — the (canonical) uniformity of the linear topological space  $X$  has a basis of countable uniform covers, as well as its trace on every  $E_n$ . But this trace coincides with the (canonical) uniformity of the linear topological space  $E_n (n \in \mathbf{Z}^+)$ , hence it is metrizable and consequently separable (if  $\{(U_{m,n})_{m=1}^\infty | n \in \mathbf{Z}^+\}$  is a countable base of countable uniform covers for the uniformity of  $E_n$  and if  $x_{m,n}$  is an element of  $U_{m,n}$  whenever this set is not empty, then  $\{x_{m,n} | m, n \in \mathbf{Z}^+\}$  is dense in  $E_n$ ).

(5)  $\rightarrow$  (6): By [2, Prop. 4], every compact subset of  $X$  is contained in some  $E_n$ . This, together with the fact that  $X$  induces the original topology on each  $E_n$ , implies that  $\bigcup_{n=1}^\infty \mathfrak{B}_n$  is a countable pseudobase for  $X$  whenever  $\mathfrak{B}_n$  is for  $E_n (n \in \mathbf{Z}^+)$ .

(6)  $\rightarrow$  (1): By [5, (D) and (E)]. ■

We remark that the idea of countable uniform covers may be used to show directly that every metrizable subgroup of a separable topological group must be separable.

The above theorem points out some important examples of non-metrizable  $\mathfrak{N}_0$ -spaces. [5, (J) and 10,4] imply some results on spaces of mappings between separable  $LF$ -spaces, of which we note only the following one.

**COROLLARY.** — *If an  $LF$ -space  $X$  is separable, then  $X'$  is weakly\* hereditarily Lindelöf and separable.*

*Proof.* — By « (1)  $\leftrightarrow$  (6) » of the above theorem and [5, (J) and (D), (E)]. ■

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