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## ON THE ANALYTICITY OF GENERALIZED EIGENFUNCTIONS (CASE OF REAL VARIABLES)

by Eberhard GERLACH

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The present note is a direct continuation of Chapter III in our paper [1]; its purpose is to extend the results on analyticity of the generalized eigenfunctions to the case of proper functional Hilbert spaces consisting of functions which are (real-) analytic in a domain in Euclidean space. We continue to use the notation and numbering from Chap. III in [1].

Our basic tool will be the following.

**PROPOSITION 4.** — *Let  $G$  be a domain in Euclidean space  $\mathbb{R}^n$ , and  $\mathcal{B}$  a class of functions defined everywhere in  $G$  and analytic there, and suppose that these form a proper functional Banach space  $\{\mathcal{B}, G\}$ . Then there exists a common domain  $\tilde{G}$  in complex space  $\mathbb{C}^n$ , containing  $G$ , to which all  $f \in \mathcal{B}$  can be extended analytically.*

*Proof.* — Since  $\{\mathcal{B}, G\}$  is a p.f. Banach space, to every  $x \in G$  there is an  $L(x) \in \mathcal{B}'$  ( $\mathcal{B}'$  is the continuous dual of  $\mathcal{B}$ ) such that  $f(x) = \langle f, L(x) \rangle$ . This defines a function  $L$  from  $G$  into  $\mathcal{B}'$  which is weakly-\* real-analytic. It is well-known that Banachspace-valued functions defined on a complex domain which are weakly or weakly-\* analytic are complex-analytic also in the strong topology. We shall show that  $L$  is strongly (real-) analytic; then it can be extended to a strongly analytic function  $\tilde{L}$  (still into  $\mathcal{B}'$ ) in some complex domain  $\tilde{G}$  containing  $G$ . Finally each  $f \in \mathcal{B}$  will be extended to an analytic function  $\tilde{f}$  on  $\tilde{G}$  by setting  $f(z) = \langle f, \tilde{L}(z) \rangle$  for  $z \in \tilde{G}$ .

Recall (cf. for instance [2]) that for any function  $g$  which is analytic in the fixed domain  $D \subset \mathbf{C}^1$  and for any compact  $K \subset D$ , there exists a finite number  $M(g; K)$  such that for any choice of  $\zeta, \zeta + \alpha, \zeta + \beta$  in  $K$ :

$$(6) \quad \left\| \frac{1}{\alpha - \beta} \left\{ \frac{1}{\alpha} [g(\zeta + \alpha) - g(\zeta)] - \frac{1}{\beta} [g(\zeta + \beta) - g(\zeta)] \right\} \right\| \leq M(g; K).$$

The same is true if instead of  $D$  one has a fixed open interval  $I \subset \mathbf{R}^1$ .

We shall establish existence of the strong derivatives  $\frac{\partial}{\partial x_i} L(x)$  in  $\mathcal{B}'$ . These derivatives exist in the weak-\* topology since for each  $f \in B$

$$\begin{aligned} \frac{\partial}{\partial x_i} f(x) &= \lim_{h \rightarrow 0} \frac{1}{h} (f(x + \varepsilon_i h) - f(x)) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \langle f, L(x + \varepsilon_i h) - L(x) \rangle. \end{aligned}$$

Let  $N$  be a compact neighborhood of  $x$ ; then there are numbers  $M(f; N)$  so that for all sufficiently small  $h$  and  $k$

$$\left\| \left\langle f, \frac{1}{h - k} \left\{ \frac{1}{h} [L(x + \varepsilon_i h) - L(x)] - \frac{1}{k} [L(x + \varepsilon_i k) - L(x)] \right\} \right\rangle \right\| \leq M(f; N).$$

Then by the uniform boundedness theorem, there is a constant  $M(N)$  such that  $\left\| \frac{1}{h - k} \{ \dots \} \right\| \leq M(N)$ . Letting  $h$  and  $k$  tend to zero, one now obtains existence of the strong derivative  $\frac{\partial}{\partial x_i} L(x)$ . Since all derivatives of the  $f \in \mathcal{B}$  are analytic, the preceding procedure can be repeated; thus  $L$  possesses strong derivatives of all orders. It is easy to check that  $L$  and all its derivatives are strongly continuous.

The Taylor series for  $L$  will converge strongly to the values of  $L$  if  $\|(\alpha!)^{-1} D_\alpha L(x)\|^{1/|\alpha|}$  is uniformly bounded on compacts  $K \subset G$ , with a bound independent of  $\alpha$ . (Here the  $\alpha_i$  are non-negative integers,  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ ,

$|\alpha| = \sum \alpha_i$ ,  $D_\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ ,  $\alpha! = \alpha_1! \dots \alpha_n!$ . Since all  $f \in \mathfrak{B}$  are analytic,

$$|(\alpha!)^{-1} D_\alpha f(x)|^{\frac{1}{|\alpha|}} = |\langle f, (\alpha!)^{-1} D_\alpha L(x) \rangle|^{\frac{1}{|\alpha|}}$$

(for fixed  $f$ ) is uniformly bounded on compacts  $K \subset G$ , independent of  $\alpha$ . But for variable  $f$ , this expression is a sub-additive continuous functional on  $\mathfrak{B}$ . By the uniform boundedness theorem then

$$\begin{aligned} \sup_{\|f\| \leq 1} |\langle f, (\alpha!)^{-1} D_\alpha L(x) \rangle|^{\frac{1}{|\alpha|}} &= \left( \sup_{\|f\|=1} |\langle \dots \rangle| \right)^{\frac{1}{|\alpha|}} \\ &= \|(\alpha!)^{-1} D_\alpha L(x)\|^{\frac{1}{|\alpha|}} \end{aligned}$$

is uniformly bounded on compacts  $K \subset G$ , independent of  $\alpha$ . Thus  $L$  has a strongly convergent power series expansion in some neighborhood of any point  $x \in G$ .

For each  $x \in G$ , let  $S(x)$  be the largest open ball in  $\mathbf{C}^n$ , centered at  $x$ , in which the Taylor series for  $L$  about  $x$  converges and set

$$\tilde{G} = \bigcup_{x \in G} S(x).$$

Then the series expansions yield an analytic continuation  $\tilde{L}$  of  $L$  from  $G$  to  $\tilde{G}$ . Finally, for  $f \in \mathfrak{B}$ , define

$$\tilde{f}(z) = \langle f, \tilde{L}(z) \rangle \quad \text{for } z \in \tilde{G} \quad \text{and} \quad \|\tilde{f}\| = \|f\|;$$

this gives us a p.f. Banach space  $\{\tilde{\mathfrak{B}}, \tilde{G}\}$  which is isometrically isomorphic to  $\{\mathfrak{B}, G\}$ . The proof of Proposition 4 is complete.

From now on,  $\{\mathfrak{F}, G\}$  will denote a p.f. Hilbert space consisting of analytic functions on a domain  $G \subset \mathbf{R}^n$ . Our aim is to extend the results of Corollary 2. III and Theorem 3. III in [1] to such spaces.

The anti-space  $\overline{\mathfrak{F}}$  of the Hilbert space  $\mathfrak{F}$  is identified with the dual  $\mathfrak{F}'$ , and  $\mathfrak{F}$  itself with its continuous anti-dual  $\mathfrak{F}^*$  ( $= \overline{\mathfrak{F}'} = \overline{\mathfrak{F}'}$ )<sup>(1)</sup> by means of the canonical mappings  $J$  and  $\theta$ :

$$\mathfrak{F}' = J\mathfrak{F} \quad \text{where } J \text{ is the anti-isomorphism } f \rightarrow Jf = (\cdot, f)$$

<sup>(1)</sup> For these notations, cf. L. Schwartz [3].

and

$\mathcal{F}^* = \theta\mathcal{F}$  where  $\theta$  is the isomorphism  $f \rightarrow \theta f = (f, \cdot)$ .

If  $K$  is the reproducing kernel of  $\mathcal{F}$  then for  $f \in \mathcal{F}$

$$f(x) = (f, K(\cdot, x)) = \langle f, L(x) \rangle \text{ for every } x \in G$$

where  $\langle \cdot, \cdot \rangle$  denotes the pairing of  $\mathcal{F}$  and  $\mathcal{F}'$ . Thus  $L(x) = JK_x$  and

$$K(x, y) = (K_y, K_x) = \langle K_y, JK_x \rangle = \langle J^{-1}L(y), L(x) \rangle.$$

By Proposition 4,  $L$  and  $F$  extend analytically to a complex domain  $\tilde{G}$ ; we obtain the p.f. Hilbert space  $\{\tilde{\mathcal{F}}, \tilde{G}\}$  with r.k.  $\tilde{K}$ :

$$\tilde{f}(z) = \langle f, \tilde{L}(z) \rangle = (f, J^{-1}\tilde{L}(z))$$

and

$$\tilde{K}(z, \omega) = \langle J^{-1}\tilde{L}(\omega), \tilde{L}(z) \rangle = (\tilde{K}_\omega, \tilde{K}_z).$$

Since the function  $\tilde{L}$  is strongly analytic from  $\tilde{G}$  into  $\mathcal{F}'$  and  $\tilde{K}_z = J^{-1}\tilde{L}(z)$ , we note that  $\tilde{K}(\cdot, z)$  is strongly anti-analytic for  $z \in \tilde{G}$  (i.e. the function  $\bar{z} \rightarrow \tilde{K}(\cdot, z)$  is strongly analytic from  $\bar{G} = \{z | \bar{z} \in \tilde{G}\}$  into  $\mathcal{F}'$ ). Let  $U$  denote the extension isomorphism  $U: \mathcal{F} \rightarrow \tilde{\mathcal{F}}$  constructed by Proposition 4. If  $\{g_k\}$  is a complete orthonormal system in  $\mathcal{F}$ , then so is  $\{\tilde{g}_k = Ug_k\}$  in  $\tilde{\mathcal{F}}$  and  $\tilde{K}(z, \omega) = \sum_{k=1}^{\infty} \tilde{g}_k(z)\overline{\tilde{g}_k(\omega)}$  for  $z, \omega \in \tilde{G}$ , i.e.,  $\tilde{K}$  is also a « direct » continuation of  $K$ .

**COROLLARY 2'.** — *Let  $G$  be an arbitrary domain in  $\mathbf{R}^n$  and  $\{\mathcal{F}, G\}$  any p.f. Hilbert space of functions (real-) analytic in  $G$ . Then  $\{\mathcal{F}, G\}$  is Hilbert-Schmidt expandible.*

*Proof.* — By Corollary 2, there is an H.S. operator  $T$  in  $\tilde{\mathcal{F}}$  such that  $\tilde{K}_\zeta \in T\tilde{\mathcal{F}}$  for all  $\zeta \in \tilde{G}$ . Now  $S = U^{-1}TU$  is H.S. in  $\mathcal{F}$ , and  $K_\xi \in S\mathcal{F}$  for all  $\xi \in G$ .

Now let  $A$  be a selfadjoint operator in  $\mathcal{F}$  with resolution of identity  $E(\cdot)$  and spectral measure  $\mu$ . Then  $(f, g) = (Uf, Ug)^\sim$  for all  $f, g \in \mathcal{F}$ . The operator  $\tilde{A} = UAU^{-1}$  is selfadjoint in  $\tilde{\mathcal{F}}$  and unitarily equivalent to  $A$ ; its resolution of identity is  $\tilde{E}(\cdot) = UE(\cdot)U^{-1}$ , and  $\mu$  is also a spectral measure for  $\tilde{A}$ .

Both  $\mathfrak{F}$  and  $\tilde{\mathfrak{F}}$  are H.S.-expandible. Let  $\tilde{\Lambda}_{\tilde{G}}$  denote the complement in  $\mathbf{R}^1$  of the set of all  $\lambda$  for which

$$\frac{d(\tilde{E}(\lambda)\tilde{K}_w, E(\lambda)\tilde{K}_z)}{d\mu(\lambda)} = \tilde{K}(z, w; \lambda) \text{ exists and is finite for all } z, w \in \tilde{G}$$

(similar definition for  $\Lambda_G$ , without tildas). Then  $\tilde{\Lambda}_{\tilde{G}} \supset \Lambda_G$  and  $\mu(\tilde{\Lambda}_{\tilde{G}}) = 0$ . Let  $\tilde{\mathfrak{H}}_{\tilde{G}}^{(\Omega)}$  ( $\mathfrak{H}_G^{(\Omega)}$ ) be the p.f. Hilbert space on  $\tilde{G}$  ( $G$ ) defined by the r.k.  $\tilde{K}(\cdot, \cdot; \lambda)$  ( $K(\cdot, \cdot; \lambda)$ ). For  $\tilde{f} \in \tilde{\mathfrak{F}}$ , let  $\tilde{\Lambda}_{\tilde{f}, (\tilde{G})}$  be the smallest set containing  $\tilde{\Lambda}_{\tilde{G}}$  such that for all  $\lambda \notin \tilde{\Lambda}_{\tilde{f}, (\tilde{G})}$ :

$$\left\{ \begin{array}{l} \frac{d(\tilde{E}(\lambda)\tilde{f}, \tilde{E}(\lambda)\tilde{K}_z)}{d\mu(\lambda)} = \tilde{f}(z; \lambda) \text{ exists, is finite} \\ \text{and } = 0 \text{ whenever } \tilde{K}(z, z; \lambda) = 0, \text{ for all } z \in \tilde{G} \end{array} \right.$$

and

$$\tilde{f}(\cdot, \lambda) \in \tilde{\mathfrak{H}}_{\tilde{G}}^{(\Omega)}, \frac{d\|\tilde{E}(\lambda)\tilde{f}\|^2}{d\mu(\lambda)} \text{ exists and equals } \|\tilde{f}(\cdot; \lambda)\|_{\tilde{\mathfrak{H}}_{\tilde{G}}^{(\Omega)}}^2$$

(similar definition for  $\Lambda_{f, (G)}$ , without tildas). The correspondence  $\tilde{f} \rightarrow \tilde{f}(\cdot; \lambda)$  defines  $\tilde{\mathfrak{P}}_{\tilde{G}}^{(\Omega)}$  with domain  $\tilde{\mathfrak{D}}_{\tilde{G}}^{(\Omega)} = \{\tilde{f} | \lambda \notin \tilde{\Lambda}_{\tilde{f}, (\tilde{G})}\}$ . For  $\tilde{f} \in \tilde{\mathfrak{D}}_{\tilde{G}}^{(\Omega)}$ ,  $\tilde{f}(\cdot; \lambda)$  is just the restriction of  $\tilde{f}(\cdot; \lambda)$  to the domain  $\tilde{G}$ .

**THEOREM 3'.** — *Let  $A$  be an arbitrary selfadjoint operator in  $\{\mathfrak{F}, G\}$  with spectral measure  $\mu$ . Then there is a set  $\Lambda$  on the real line,  $\mu(\Lambda) = 0$ , which is determined by Theorem 3 (and also Corollary 2, Theorem 11. I, and the above considerations) such that the generalized eigenfunctions*

$$\frac{dE(\lambda)f(x)}{d\mu(\lambda)} = f(x; \lambda) \in \mathfrak{H}_G^{(\Omega)} \text{ for } \lambda \notin \Lambda \text{ and } \tilde{f} \in \tilde{\mathfrak{D}}_{\tilde{G}}^{(\Omega)}$$

are real-analytic in the whole domain  $G$ .

*Proof.* — According to the preceding preparations, set  $Uf = \tilde{f}$ . If  $\lambda \notin \Lambda$  and  $\tilde{f} \in \tilde{\mathfrak{D}}_{\tilde{G}}^{(\Omega)}$  then  $\tilde{f}(\cdot; \lambda)$  is analytic in  $\tilde{G}$  by Theorem 3, and consequently its restriction  $f(\cdot; \lambda)$  is (real-) analytic in  $G$ .

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