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CORNELIU CONSTANTINESCU

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SOME PROPERTIES OF THE BALAYAGE OF MEASURES ON A HARMONIC SPACE

by Corneliu CONSTANTINESCU

Let X be a harmonic space with a countable basis satisfying the axioms H_0, H_1, H_2, H_3 [2] and such that there exists a locally bounded positive potential on X. Let us denote by $\mathfrak{T}_0$ the set of finite continuous potentials on X, which are harmonic outside a compact set of X, and by Λ the set of (positive Radon) measures on X for which the potentials from $\mathfrak{T}_0$ are integrable. For any $\mu \in \Lambda$ and any $A \subset X$ there exists a uniquely determined measure $\mu^A \in \Lambda$ such that

$$\int p \, d\mu^A = \int \hat{R}_p^A \, d\mu$$

for any $p \in \mathfrak{T}_0$ ([2] Lemma 4.4 (1)).

It will be proved in this paper that for any $\mu \in \Lambda$ and any subsets A, B of X we have

$$\mu^{A \cup B} \leqslant \mu^A \vee \mu^B,$$

where $\mu^A \vee \mu^B$ denotes the smallest measure on X greater then μ^A and μ^B. This will allow us to give direct proofs for some theorems about the balayage of measures, which are known only by the devious way of Markov processes. These theorems read as follows :

1° $\varepsilon_x^A \neq \varepsilon_x$ characterises the thinness of A at x, where ε_x denotes the unit mass at the point $x \in X$;

2° there exists a finite continuous potential p on X such that for any $A \subset X$ the set $\{ x \in X \mid \hat{R}_p^A(x) < p(x) \}$ is exactly the set of points of X where A is thin;

(1) In reality in this paper the relation was proved only for measures for which any finite continuous potential is integrable but the proof holds good for the measures of Λ, using the lemma 1.1. of the present paper instead of [2] lemma 4.3.

3° the fine interior of X-A is of inner μ^A-measure zero;

4° if A, B are thin at $x \in X$ then $A \cup B$ is thin at x;

5° if $\mu \in \Lambda$ and if $(A_n)_{n \in N}$, $(B_n)_{n \in N}$ are two sequences of subsets of X such that $\mu^{A_n} = \mu^{B_n}$ for any $n \in N$, then $\mu^A = \mu^B$ where

$$A = \bigcup_{n \in N} A_n, \qquad B = \bigcup_{n \in N} B_n;$$

6° let μ be a measure on X such that any compact set which is thin at any point of X is of μ-measure zero. With the aid of 2° we show first that any fine open set is μ-measurable and then that μ possesses a fine carrier (i.e. a smallest fine closed set with complement of μ-measure zero) which is a G_δ set not thin at any of its points. This last result has important consequences in the study of Markov processes on a harmonic space.

This paper is a continuation of [2]. It is supposed that the reader has read it or [1].

1. LEMMA 1.1. (2). — *Let f be a function of $\mathcal{K}_+$ (3). For any neighbourhood* U *of its support and for any positive number* ε *there exists two potentials p, q of* $\mathfrak{T}_0$ *such that* $p - q$ *is nonnegative on* X *equal to* 0 *on* X — U *and*

$$|f - (p - q)| < \varepsilon.$$

Any nonnegative hyperharmonic function on X *is the limit of an increasing sequence from* $\mathfrak{T}_0$.

By [2] lemma 4.3 there exists two finite continuous potentials p', q' such that $p' - q'$ is nonnegative on X, its support K lies in U and

$$|f - (p - q)| < \varepsilon.$$

Let g be a function of $\mathcal{K}_+$ whose support lies in U, smaller than 1 on X and equal to 1 on K. We set

$$p = R_{gp'}, \qquad q = R_{gq'}.$$

(2) This lemma was proved by N. Boboc, A. Cornea and the author in a paper which has not yet appeared. We introduce this lemma only for the purpose of extending the results to the more general set of measures Λ. The reader which is interested only in the balayage of measures with compact carriers may omit this lemma.

(3) We denote by $\mathcal{K}_+$ the set of nonnegative continuous real functions on X with compact support.

By [2] remark of proposition 3.3 p and q are finite continuous potentials on X and by [2] proposition 3.2 they are harmonic outside the support of g. Hence $p, q \in \mathcal{P}_0$. Moreover

$$p = p', \qquad\qquad q = q$$

on K. The function on X equal to p on K and equal to q on $X - K$ is hyperharmonic ([1] Satz. 1.3.10) and greater than gp'. Hence $q \geqslant p$ on $X - K$. Since $p \geqslant q$ on X, p, q possess the required properties.

The second assertion follows immediately from the remark that for any $f \in \mathcal{K}_+$ R_f belongs to $\mathcal{P}_0$ ([2] proposition 3.2 and remark of proposition 3.3).

THEOREM 1.1. — *Let* A, B *be two subsets of* X. *For any measure* $\mu \in \Lambda$, $\mu^{A \cup B} \leqslant \mu^A + \mu^B$. *For any nonnegative hyperharmonic function s on* X *we have*

$$\hat{R}_s^{A \cup B} \prec \hat{R}_s^A + \hat{R}_s^B\,.$$

Suppose first s finite on $A \cup B$ and let $\mathfrak{U}$ (resp. $\mathfrak{V}$) be the set of fine open neighbourhoods of A (resp. B). For any $U \in \mathfrak{U}$ and $V \in \mathfrak{V}$ we set

$$s(U, V) = R_s^U \wedge R_s^V\,.$$

Then it is easy to verify that

$$s + s(U, V) = R_s^U + R_s^V$$

on $U \cup V$. Hence ([2] theorem 3.2)

$$R_s^{U \cup V} + R_{s(U,V)}^{U \cup V} = R_{s+s(U,V)}^{U \cup V} = R_{R_s^U + R_s^V}^{U \cup V} = R_s^U + R_s^V.$$

$\mathfrak{U}$ and $\mathfrak{V}$ being lower directed we get further

$$\inf R_s^{U \cup V} + \inf R_{s(U,V)}^{U \cup V} = \inf R_s^U + \inf R_s^V.$$

Hence ([2] lemma 3.1 and [1], page 48)

$$R_s^{A \cup B} + \inf R_{s(U,V)}^{U \cup V} = R_s^A + R_s^B,$$

$$\hat{R}_s^{A \cup B} + (s; A, B) = \hat{R}_s^A + \hat{R}_s^B,$$

(4) $\prec$ is the specific order introduced by M. Brelot and R.-M. Hervé; it means: there exists a nonnegative hyperharmonic function t on X such that

$$R_s^{A \cup B} + t = R_s^A + R_s^B$$

where we have set

$$(s; \mathrm{A}, \mathrm{B}) = \inf \; \mathrm{R}^{\mathrm{U} \cup \mathrm{V}}_{s\,(\mathrm{U}, \mathrm{V})} \, .$$

We remark that if s' is another nonnegative hyperharmonic function on X smaller than s then

$$(s'; \mathrm{A}, \mathrm{B}) \leqslant (s; \mathrm{A}, \mathrm{B}).$$

Let p, q be two potentials of $\mathfrak{T}_0$, $q \leqslant p$. Then

$$\int (p - q)\, d\, (\mu^{\mathrm{A}} + \mu^{\mathrm{B}} - \mu^{\mathrm{A} \cup \mathrm{B}})$$
$$= \int (\hat{\mathrm{R}}_p^{\mathrm{A}} + \hat{\mathrm{R}}_p^{\mathrm{B}} - \hat{\mathrm{R}}_p^{\mathrm{A} \cup \mathrm{B}})\, d\mu - \int (\hat{\mathrm{R}}_q^{\mathrm{A}} + \hat{\mathrm{R}}_q^{\mathrm{B}} - \hat{\mathrm{R}}_q^{\mathrm{A} \cup \mathrm{B}})\, d\mu$$
$$= \int (p; \mathrm{A}, \mathrm{B})\, d\mu - \int (q; \mathrm{A}, \mathrm{B})\, d\mu \geqslant 0.$$

By lemma 1.1 it follows that $\mu^{\mathrm{A}} + \mu^{\mathrm{B}} - \mu^{\mathrm{A} \cup \mathrm{B}}$ is a nonnegative measure on X.

Let now s be arbitrary and let us denote

$$\mathrm{A}' = \{x \in \mathrm{A} \,|\, s(x) < \infty\}, \qquad \mathrm{B}' = \{x \in \mathrm{B} \,|\, s(x) < \infty\}.$$

Let x be a point of X. If

$$\hat{\mathrm{R}}_s^{\mathrm{A} \cup \mathrm{B}}\,(x) = \infty$$

then the equality

$$\hat{\mathrm{R}}_s^{\mathrm{A} \cup \mathrm{B}}\,(x) + (s; \mathrm{A}', \mathrm{B}')\,(x) = \hat{\mathrm{R}}_s^{\mathrm{A}}\,(x) + \hat{\mathrm{R}}_s^{\mathrm{B}}\,(x)$$

is trivial. Otherwise

$$\hat{\mathrm{R}}_s^{(\mathrm{A}-\mathrm{B}) \cup (\mathrm{A}-\mathrm{B})}\,(x) = \hat{\mathrm{R}}_s^{\mathrm{A}-\mathrm{A}'}\,(x) = \hat{\mathrm{R}}_s^{\mathrm{B}-\mathrm{B}'}\,(x) = 0$$

and therefore

$$\hat{\mathrm{R}}_s^{\mathrm{A} \cup \mathrm{B}}\,(x) + (s; \mathrm{A}', \mathrm{B}')\,(x) = \hat{\mathrm{R}}_s^{\mathrm{A}' \cup \mathrm{B}'}\,(x) + (s; \mathrm{A}', \mathrm{B}')\,(x) =$$
$$= \hat{\mathrm{R}}_s^{\mathrm{A}'}\,(x) + \hat{\mathrm{R}}_s^{\mathrm{B}'}\,(x) = \hat{\mathrm{R}}_s^{\mathrm{A}}\,(x) + \hat{\mathrm{R}}_s^{\mathrm{B}}\,(x).$$

Remark. — The relation

$$\mu^{\mathrm{A} \cap \mathrm{B}} + \mu^{\mathrm{A} \cup \mathrm{B}} \leqslant \mu^{\mathrm{A}} + \mu^{\mathrm{B}}$$

is not true for any A and B.

We remember that a set A *is called thin at a point* $x \in X$ *if there exists a neighbourhood* U *of* x *such that*

$$\hat{R}_1^{A \cap U}(x) < 1.$$

COROLLARY 1.1. (5). — *Let* A *be a subset of* X, x *be a point of* X, *and* ε_x *be the unit mass at the point* x. *The following conditions are equivalent :*

a) *A is thin at* x ;

b) $\varepsilon_x^A \neq \varepsilon_x$;

c) *there exists a potential* $p \in \mathfrak{T}_0$ *such that* $\hat{R}_p^A(x) < p(x)$;

d) *there exists a nonnegative hyperharmonic function* s *on* X *such that* $\hat{R}_s^A(x) < s(x)$.

a $\Rightarrow$ b. By lemma 1.1. there exists a potential $p \in \mathfrak{T}_0$ positive at x. Let U be a neighbourhood of x such that

$$\hat{R}_p^{A \cap U}(x) < p(x).$$

Then

$$\int p \, d\varepsilon_x^{A \cap U} = \hat{R}_p^{A \cap U}(x) < p(x)$$

and therefore

$$\varepsilon_x^{A \cap U}(\{x\}) < 1.$$

On the other hand

$$\varepsilon_x^{A - U}(\{x\}) = 0$$

and therefore

$$\varepsilon_x(\{x\}) \leqslant \varepsilon_x^{A \cap U}(\{x\}) + \varepsilon_x^{A-U}(\{x\}) < 1,$$

$$\varepsilon_x^A \neq \varepsilon_x.$$

b $\Rightarrow$ c follows from lemma 1.1. c $\Rightarrow$ d is trivial. In order to prove d $\Rightarrow$ a we take a real number α such that

$$\hat{R}_s^A(x) < \alpha < s(x).$$

(5) This corollary answers the problem formulated by H. Bauer ([1] page 113) which asks if the thinness of A at a point x can be characterized by the existence of a nonnegative hyperharmonic function s on X such that $R_s(x) < s(x)$.

Let U be a neighbourhood of x such that $s > \alpha$ on U. Then

$$\hat{R}_1^{A \cap U}(x) = \frac{1}{\alpha} \hat{R}_\alpha^{A \cap U}(x) \leqslant \frac{1}{\alpha} \hat{R}_s^A(x) < 1.$$

Remark. — In the proof of theorem 1.1 and corollary 1.1 it was not used the fact that X has a countable basis.

A potential p on X *is called* STRICT *if any two measures* μ, ν *on* X *coincide if*

$$\int p\, d\mu = \int p\, d\nu < \infty$$

and

$$\int^* s\, d\mu \leqslant \int^* s\, d\nu$$

for any nonnegative hyperharmonic function s on X.

The existence of a finite continuous strict potential on X can be proved like in [1] Satz 2.7.6.

COROLLARY 1.2. (6). — *For any finite strict potential p on* X *and for any set* $A \subset X$ *the set*

$$\{x \in X \mid \hat{R}_p^A(x) < p(x)\}$$

is exactly the set of points of X *where* A *is thin.*

The assertion follows immediately from corollary 1.1.

COROLLARY 1.3. — *The set of points where an arbitrary set is not thin, and the fine closure of a set which is not thin at any of its points* (7) *are sets of type* G_δ. *The fine closure of a Borel set is a Borel set.*

COROLLARY 1.4. — *For any fine Borel set* A *there exists a Borel set* B *such that* $(A - B) \cup (B - A)$ *is semipolar* (8).

Let us denote by $\mathfrak{B}$ the set of subsets of X possessing this property. Let F be a fine closed set and let F_0 be the set of points of X where F

(6) This corollary gives the answer to a problem formulated by H. Bauer ([1], page 136).

(7) Brelot called these sets subbasic ; the fine open sets are subbasic.

(8) A semipolar set is a set contained in the union of a countable family of sets which are thin at any point of X.

is not thin. By the preceding corollary F_0 is a Borel set and by [4] théorème 1 or [1] Satz 3.3.7. $F - F_0$ is semipolar. Since $F_0 \subset F$ we deduce $F \in \mathfrak{B}$. It is easy to see that $\mathfrak{B}$ is a σ-field. Hence $\mathfrak{B}$ contains all fine Borel sets.

COROLLARY 1.5 (Doob). — *Any family of fine open sets contains a countable subfamily whose union differs from the union of the whole family by a semipolar set.*

Let $(A_\iota)_{\iota \in I}$ be a family of fine open sets and let p be a finite strict potential on X. By Choquet's lemma ([3] lemma 3, page 3) there exists a sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that

$$\widehat{\inf_{n \in \mathbf{N}} \hat{R}_p^{\complement A_{\iota_n}}} \leqslant \hat{R}_p^{\complement A_\iota}$$

for any $\iota \in I$. We set .

$$A = \{x \in X \mid \widehat{\inf_{n \in \mathbf{N}} \hat{R}_p^{\complement A_{\iota_n}}}(x) < \inf_{n \in \mathbf{N}} R_p^{\complement A_\iota}(x)\}$$

By [4] théorème 1 or [1] Satz 3.3.7. A is semipolar. Let

$$x \in \bigcup_{\iota \in I} A_\iota - \bigcup_{n \in \mathbf{N}} A_{\iota_n}.$$

Then

$$\inf_{n \in \mathbf{N}} R_p^{\complement A_{\iota_n}}(x) = p(x),$$

while (corollary 1.2)

$$R_p^{\complement A_\iota}(x) < p(x)$$

for an $\iota \in I$. Hence $x \in A$ and $\bigcup_{\iota \in I} - \bigcup_{n \in \mathbf{N}} A_\iota$ is a semipolar set.

Remark. — This corollary is contained in Doob's theorem ([5] theorem 8.1) since it is possible to show that the condition (L) is satisfied by the fine topology, but the proof given here does not use the Markov processes. On the other hand it is possible to deduce this corollary from the proof of [5] theorem 2.3. replacing " ' " by " + ".

LEMMA 1.2. — *For any subset* A *of* X *there exists a fine closed set* A′ *of type* G_δ *containing* A *and such that*

$$\mu^A = \mu^{A'}$$

for any $\mu \in \Lambda$.

Let p be a finite continuous strict potential on X. By Choquet's lemma ([3] lemma 3, page 3) there exists a decreasing sequence $(p_n)_{n \in \mathbf{N}}$ of potentials on X, $p_0 = p$, such that p_n is equal to p on A and

$$\widehat{\lim_{n \to \infty} p_n} = \hat{R}_p^A .$$

We set

$$A'' = \bigcap_{n \in \mathbf{N}} \{x \in X \mid \frac{n}{n+1} p(x) < p_n(x)\}.$$

It is obvious that A″ is of type G_δ contains A and

$$R_p^{A''} \leqslant \frac{n+1}{n} p_n$$

for any $n \in \mathbf{N}$. Hence

$$\hat{R}_p^A \leqslant \hat{R}_p^{A''} \leqslant \widehat{\lim_{n \to \infty} p_n} = \hat{R}_p^A .$$

For any $x \in X$ we get

$$\int p \, d\varepsilon_x^A = \hat{R}_p^A(x) = \hat{R}_p^{A''}(x) = \int p \, d\varepsilon_x^{A''},$$

$$\varepsilon_x^A = \varepsilon_x^{A''} .$$

Hence

$$\mu^A = \int \varepsilon_x^A \, d\mu(x) = \int \varepsilon_x^{A''} \, d\mu(x) = \mu^{A''}.$$

By corollary 1.3 the fine closure A′ of A″ is a set of type G_δ. The equality $\mu^{A'} = \mu^{A''}$ is trivial.

COROLLARY 1.6. — *Any total thin set* (9) *is contained in a total thin set of type* G_δ.

The assertion follows from lemma 1.2. and corollary 1.1.

COROLLARY 1.7. (Getoor-Doob). — *Let* μ *be a measure on* X, *such that any compact total thin set is* μ*-negligible. Then any semipolar set is* μ*-negligible, any fine Borel set is* μ*-measurable and there exists a fine closed set of type* G_δ *not thin at any of its points which is the smallest fine closed set with* μ*-negligible complement.*

(9) i.e. a set which is thin at any point of X.

From the preceding corollary it follows immediately that any total thin set is μ-negligible. A semipolar set is then also μ-negligible, since it is by definition the union of a countable family of total thin sets. By corollary 1.4 any fine Borel set is μ-measurable.

By corollary 1.5 the set of μ-negligible fine open sets possesses a largest set. We denote by F its complement.

Let x be a point of F. If F is thin at x, then $\{x\}$ is semipolar and therefore μ-negligible and F-$\{x\}$ is fine closed. This contradicts the maximality of X — F. Hence F is not thin at any of its points and it is therefore a set of type G_δ (corollary 1.3).

COROLLARY 1.8. — *Let μ be a measure on* X *for which the semipolar sets are μ-negligible and let $(s_\iota)_{\iota \in I}$ be a family of nonnegative hyperharmonic functions on* X. *Then*

$$\int^* \inf_{\iota \in I} s_\iota \, d\mu = \int^* \bigwedge_{\iota \in I} s_\iota \, d\mu .$$

If the family is lower directed and at least one fuction s_ι is μ-integrable then $\inf_{\iota \in I} s_\iota$ is μ-integrable and

$$\inf_{\iota \in I} \int^* s_\iota \, d\mu = \int \inf_{\iota \in I} s_\iota \, d\mu .$$

The first assertion follows from the fact that $\inf\limits_{\iota \in I} s_\iota$ and $\bigwedge\limits_{\iota \in I} s_\iota$ differ only on a semipolar and therefore μ-negligible set ([4] théorème 1 or [1] Satz 3.3.7).

In order to prove the second assertion we remark first that $\inf\limits_{\iota \in I} s_\iota$ is fine upper semicontinuous and therefore it is a fine Borel function. We deduce further by corollary 1.7 that it is μ-measurable and therefore μ-integrable. By Choquet's lemma ([3] lemma 3, page 3) there exists a sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that $(s_{\iota_n})_{n \in \mathbf{N}}$ is decreasing and

$$\bigwedge_{n \in \mathbf{N}} s_{\iota_n} = \bigwedge_{\iota \in I} s_\iota .$$

On the other hand $\bigwedge\limits_{n \in \mathbf{N}} s_{\iota_n}$ differs from $\lim\limits_{n \to \infty} s_{\iota_n}$ on a semipolar set ([4] théorème 1 or [1] Satz 3.3.7) and therefore on a μ-negligible set. We deduce

$$\int \bigwedge_{\iota \in I} s_\iota \, d\mu = \int \bigwedge_{n \in \mathbf{N}} s_{\iota_n} \, d\mu = \int \lim_{n \to \infty} s_{\iota_n} \, d\mu$$

$$= \lim_{n \to \infty} \int s_{\iota_n} \, d\mu \geqslant \inf_{\iota \in I} \int^* s_\iota \, d\mu \geqslant \int \bigwedge_{\iota \in I} s_\iota \, d\mu.$$

2. Lemma 2.1. — *Let* μ *be a measure on* X *with compact carrier* K. *For any set* $A \subset X - K$ *and any nonnegative hyperharmonic function* s *on* X *we have*

$$\int^* \hat{R}_s^A \, d\mu = \inf_{t \in \mathfrak{S}(A, s)} \int^* t \, d\mu,$$

where $\mathfrak{S}(A, s)$ *denotes the set of nonnegative hyperharmonic functions on* X *greater than* s *on* A.

We set

$$C(A, s) = \inf_{t \in \mathfrak{S}(A, s)} \int^* t \, d\mu.$$

Obviously

$$\int^* \hat{R}_s^A \, d\mu \leqslant C(A, s)$$

with equality if A is fine open. If A is compact and does not meet K and if s is bounded on A then there exists a decreasing sequence of functions of $\mathfrak{S}(A, s)$ greater than s on a fine neighbourhood of A and converging uniformly on K to $\hat{R}_s^A$. Hence in this case we have

$$C(A, s) = \inf C(B, s) = \int \hat{R}_s^A \, d\mu < \infty,$$

where B runs through the set of fine neighbourhoods of A.

Suppose now that s is finite on A. Let $(A_n)_{n \in \mathbf{N}}$ be an increasing sequence of relatively compact subset of X-K such that s is bounded on every A_n and having A as its union, and let ε be a positive number. We shall construct by induction an increasing sequence $(B_n)_{n \in \mathbf{N}}$ of fine open sets such that

$$A_n \subset B_n \subset X - K, \qquad C(B_n, s) \leqslant C(A_n, s) + \sum_{i \leqslant n} \frac{\varepsilon}{2^i}.$$

Suppose that B_i are already constructed for $i < n$. Let B be a fine open set such that

$$A_n \subset B \subset X - K, \qquad C(B, s) \leqslant C(A_n, s) + \frac{\varepsilon}{2_n}.$$

We set $B_n = B_{n-1} \cup B$. Then ([2], Theorem 3.3)

$$C(A_{n-1}, s) + C(B_n, s) \leqslant C(B_{n-1} \cap B, s) + C(B_{n-1} \cup B, s) =$$

$$= \int (R_s^{B_{n-1} \cap B} + R_s^{B_{n-1} \cup B}) \, d\mu \leqslant \int R_s^{B_{n-1}} \, d\mu + \int R_s^B \, d\mu =$$

$$= C(B_{n-1}, s) + C(B, s) \leqslant C(A_{n-1}, s) + \sum_{i<n} \frac{\varepsilon}{2^i} + C(A_n, s) + \frac{\varepsilon}{2^n}.$$

$$C(B_n, s) \leqslant C(A_n, s) + \sum_{i \leqslant n} \frac{\varepsilon}{2^i}.$$

We get now ([2] proposition 3.4),

$$C(A, s) \leqslant C(\bigcup_{n \in \mathbf{N}} B_n, s) = \lim_{n \to \infty} C(B_n, s) \leqslant$$

$$\leqslant \lim_{n \to \infty} \int \hat{R}_s^{A_n} \, d\mu + 2\varepsilon = \int \hat{R}_s^A \, d\mu + 2\varepsilon.$$

ε being arbitrary we get

$$C(A, s) = \lim_{n \to \infty} C(A_n, s) = \int \hat{R}_s^A \, d\mu.$$

Let now A be arbitrary, but s equal to ∞ on A and

$$\int \hat{R}_s^A \, d\mu < \infty.$$

Let $(A_n)_{n \in \mathbf{N}}$ be an increasing sequence of relatively compact sets of of X⌐K having A as its union and let ε be a positive number. For any $n \in \mathbf{N}$ we can find a non negative hyperharmonic function s_n on X greater than n on A_n and such that

$$\int s_n \, d\mu < \frac{\varepsilon}{2^n}.$$

Then $\sum_{n \in \mathbf{N}} s_n$ belongs to $\mathfrak{S}(A, s)$ and

$$\int \Big(\sum_{n \in \mathbf{N}} s_n \Big) \, d\mu < 2\varepsilon.$$

ε being arbitrary we get

$$C(A, s) = 0.$$

We study now the general case. Let A′ be the subset of A where s is finite. If

$$\int \hat{R}_s^A \, d\mu = \infty$$

the required equality is proved. Otherwise it follows from the preceding considerations

$$C(A - A', s) = 0, \qquad C(A', s) = \int \hat{R}_s^{A'} \, d\mu.$$

Hence

$$C(A, s) \leqslant C(A', s) + C(A - A', s) = \int \hat{R}_s^{A'} \, d\mu \leqslant \int \hat{R}_s^A \, d\mu$$

and the lemma is proved.

Lemma 2.2. — *Let* A, B *be two subsets of* X *such that* B *is not thin at any point of* A. *Then for any* $\mu \in \Lambda$ *and any non negative hyperharmonic function* s *on* X *we have*

$$\mu^A(\{ x \in X \mid \hat{R}_s^B(x) < s(x) \}) = 0.$$

Let p be a potential of $\mathfrak{P}_0$. Since B is not thin at any point of A we have

$$p = \hat{R}_p^B$$

on A and therefore

$$\hat{R}_p^A = \hat{R}_{\hat{R}_p^B}^A,$$

$$\int (p - \hat{R}_p^B) \, d\mu^A = \int (\hat{R}_p^A - R_{\hat{R}_p^B}^A) \, d\mu = 0,$$

$$\mu^A(\{ x \in X \mid \hat{R}_p^B(x) < p(x) \}) = 0.$$

Let $(p_n)_{n \in \mathbf{N}}$ be an increasing sequence of potentials of $\mathfrak{P}_0$ converging to s (lemma 1.1). By [2] theorem 3.4

$$\{ x \in X \mid \hat{R}_s^B(x) < s(x) \} \subset \bigcup_{n \in \mathbf{N}} \{ x \in X \mid \hat{R}_{p_n}^B(x) < p_n(x) \}.$$

Hence

$$\mu^A(\{ x \in X \mid \hat{R}_s^B(x) < s(x) \}) = 0.$$

THEOREM 2.1 (10). — *For any set* $A \subset X$ *and any* $\mu \in \Lambda$ *the fine interior of* $X - A$ *is of inner* μ^A*-measure zero.*

Let K be a compact set contained in the fine interior of X-A and let p be a locally bounded strict potential on X. From lemma 2.1 it follows that there exists a decreasing sequence $(p_n)_{n \in \mathbf{N}}$ of locally bounded potentials on X equal to p on A and such that

$$\lim_{n \to \infty} \int_K p_n \, d\mu^A = \int_K \hat{R}_p^A \, d\mu^A.$$

We set for any $n \in \mathbf{N}$ and any $\varepsilon > 0$

$$B_{n,\varepsilon} = \{ x \in X \mid p(x) < p_n(x) + \varepsilon \},$$

$$C_{n,\varepsilon} = \{ x \in K \mid p(x) > p_n(x) + \varepsilon \},$$

$$C = \bigcup_{\substack{n \in \mathbf{N} \\ \varepsilon > 0}} C_{n,\varepsilon}.$$

Since $B_{n,\varepsilon}$ is thin at any point of $C_{n,\varepsilon}$ we get by corollary 1.2

$$C_{n,\varepsilon} \subset \{ x \in X \mid \hat{R}_p^{B_{n,\varepsilon}}(x) < p(x) \}.$$

Since $B_{n,\varepsilon}$ is a fine neighbourhood of A we get from the preceding lemma

$$\mu^A(C_{n,\varepsilon}) \leqslant \mu^A(\{ x \in X \mid \hat{R}_p^{B_{n,\varepsilon}}(x) < p(x) \}) = 0,$$

$$\mu^A(C) = 0.$$

On $K - C$ we have $p \leqslant \lim_{n \to \infty} p_n$. Hence

$$\int_{K-C} (p - \hat{R}_p^A) \, d\mu^A \leqslant \lim_{n \to \infty} \int_K (p_n - \hat{R}_p^A) d\mu = 0.$$

Since A is thin at any point of K it follows that $p - \hat{R}_p^A$ is positive on K. We deduce therefore from this relation

$$\mu^A(K - C) = 0, \quad \mu^A(K) \leqslant \mu^A(C) + \mu^A(K - C) = 0.$$

COROLLARY 2.1. (11). — *If two subsets* A, B *of* X *are thin at a point* $x \in X$ *then* $A \cup B$ *is thin at* x.

(10) This theorem (together with corollary 1.3) gives the answer to a problem formulated by H. Bauer ([1], page 119).

(11) This problem was formulated to the author by H. Bauer and was the starting point of the present paper.

We may suppose $x \in A \cup B$. Then

$$\varepsilon_x^{A \cup B}(\{x\}) \leqslant \varepsilon_x^{\{x\}}(\{x\}) + \varepsilon_x^{A-\{x\}}(\{x\}) + \varepsilon_x^{B-\{x\}}(\{x\}) \quad \text{(theorem 1.1)},$$

$$\varepsilon_x^{\{x\}}(\{x\}) < 1 \quad \text{(corollary 1.1)},$$

$$\varepsilon_x^{A-\{x\}}(\{x\}) = \varepsilon_x^{B-\{x\}}(\{x\}) = 0 \quad \text{(theorem 2.1)}.$$

Hence

$$\varepsilon_x^{A \cup B}(\{x\}) < 1, \qquad \varepsilon_x^{A \cup B} \neq \varepsilon_x$$

and the corollary follows now from corollary 1.1.

Corollary 2.2. — *Let* $A \subset B \subset X$. *The following assertions are equivalent :*

a) $\mu^A = \mu^B$ *for any* $\mu \in \Lambda$.

b) *the fine interior of* $X - A$ *is of inner* μ^B*-measure zero for any* $\mu \in \Lambda$.

a $\Rightarrow$ b is an immediate consequence of the theorem. In order to prove b $\Rightarrow$ a we take a $\mu \in \Lambda$, a $p \in \mathfrak{T}_0$, an open neighbourhood U of the fine closure of A, and a point $x \in X$. Since p and R_p^U coincide on U and since $X - U$ is by hypothesis of ε_x^B-measure zero we have

$$\hat{R}_p^B(x) = \int p\, d\varepsilon_x^B = \int R_p^U\, d\varepsilon_x^B = \hat{R}_{R_p^U}^B(x) \leqslant R_p^U(x).$$

U and x being arbitrary and p continuous we get

$$\hat{R}_p^B \leqslant R_p^{\tilde{A}} = R_p^A,$$

where $\tilde{A}$ denotes the fine closure of A. Hence

$$\hat{R}_p^A \leqslant \hat{R}_p^B \leqslant \hat{R}_p^A, \qquad \mu^A = \mu^B.$$

3. Lemma 3.1. (12). — *Let* A, C *be two subsets of* X, $A \subset C$, *such that the fine closure* $\tilde{A}$ *of* A *is a Borel set. Then for any* $x \in X$ *the restriction of* ε_x^C *to* $\tilde{A} - \{x\}$ *is smaller than the restriction of* ε_x^A *to* $\tilde{A} - \{x\}$. *If* x *does not belong to the fine closure of* $C - A$ *or if* A *is not thin at* x *then the restriction of* ε_x^C *to* $\tilde{A}$ *is smaller than the restriction of* ε_x^A *to* $\tilde{A}$.

Let K be a compact subset of $A - \{x\}$ and let $(U_n)_{n \in N}$ be a decreasing sequence of open sets forming a fundamental system of neighbourhoods of K. We set for any $n \in N$

$$A_n = C \cap (A \cup U_n), \qquad C_n = C - U_n.$$

By theorem 1.1 we get

$$\varepsilon_x^C (K) \leqslant \varepsilon_x^{A_n} (K) + \varepsilon_x^{C_n} (K) = \varepsilon_x^{A_n} (K).$$

Let p be a potential of $\mathfrak{T}_0$. By [2] theorem 3.3 we have

$$\hat{R}_p^{A \cap U_n} + \hat{R}_p^{A_n} \leqslant R_p^A + \hat{R}_p^{U_n \cap C}.$$

Let ε be a positive number and p_0 be a finite positive potential on X. By lemma 2.1 there exists a nonnegative hyperharmonic function on X greater than p on K such that

$$s(x) < \hat{R}_p^K (x) + \varepsilon$$

Since p is continuous the set $\{y \in X \mid p(y) < s(y) + \varepsilon\, p_0(y)\}$ is a neighbourhood of K. Hence it contains an U_n for a sufficiently large n and we get

$$\hat{R}_p^{U_n \cap C}(x) \leqslant \hat{R}_p^{U_n}(x) \leqslant s(x) + \varepsilon\, p_0(x) < \hat{R}_p^K (x) + \varepsilon + \varepsilon\, p_0(x).$$

On the other hand if a nonnegative hyperharmonic functions is greater than p on $A \cap U_n$ then it is greater than p on its fine closure which contains K. Hence

$$\hat{R}_p^K \leqslant R_p^{A \cap U_n}.$$

Using the obtained inequalities we get

$$\hat{R}_p^K (x) + \hat{R}_p^{A_n} (x) \leqslant \hat{R}_p^A (x) + \hat{R}_p^K (x) + \varepsilon + \varepsilon\, p_0(x).$$

ε being arbitrary we deduce

$$\lim_{n \to \infty} \hat{R}_p^{A_n} (x) = \hat{R}_p^A (x).$$

By lemma 1.1 this means that $(\varepsilon_x^{A_n})_{n \in N}$ converges vaguely to ε_x^A. This fact together with the first inequality gives

$$\varepsilon_x^C (K) \leqslant \liminf_{n \to \infty} \varepsilon_x^{A_n} (K) \leqslant \varepsilon_x^A (K),$$

which proves the first assertion of the lemma.

(12) This lemma is a generalization of the principle of harmonic measure from the theory of functions.

Suppose now that x does not belong to the fine closure of C — A. Then by theorem 1.1 and theorem 2.1.

$$\varepsilon_x^{C}(\{x\}) \leqslant \varepsilon_x^{A}(\{x\}) + \varepsilon_x^{C-A}(\{x\}) = \varepsilon_x^{A}(\{x\}).$$

This relation and the first assertion of the lemma implies the second assertion of the lemma.

If A is not thin at x, then C is not thin at x and we get (corollary 1.1)

$$\varepsilon_x^{A} = \varepsilon_x = \varepsilon_x^{C}.$$

THEOREM 3.1. — *For any subsets* A, B *of* X *and any* $\mu \in \Lambda$ *we have*

$$\mu^{A \cup B} \leqslant \mu^{A} \vee \mu^{B}.$$

Suppose first that A, B are Borel sets and fine closed and let us denote by A_0 (resp. B_0) the set of points of X where A (resp. B) is not thin. By corollary 1.3 A_0 and B_0 are Borel sets.

Let f be a function of $\mathcal{K}_+$. We have (Bourbaki, Intégration, ch. V, § 3, théorème 1)

$$\int_{A_0} f\,d\mu^{A\cup B} = \int \Big(\int_{A_0} f\,d\varepsilon_x^{A\cup B}\Big)\,d\mu(x)$$

$$= \int_{A_0}\Big(\int_{A_0} f\,d\varepsilon_x^{A\cup B}\Big)\,d\mu(x) + \int_{X-A_0}\Big(\int_{A_0} f\,d\varepsilon_x^{A\cup B}\Big)\,d\mu(x).$$

For any $x \in A_0$ we have $\varepsilon_x^{A\cup B} = \varepsilon_x = \varepsilon_x^{A}$ and therefore

$$\int_{A_0}\Big(\int_{A_0} f\,d\varepsilon_x^{A\cup B}\Big)\,d\mu(x) = \int_{A_0}\Big(\int_{A_0} f\,d\varepsilon_x^{A}\Big)\,d\mu(x).$$

By the preceding lemma for any $x \in X - A_0$ the restriction of $\varepsilon_x^{A\cup B}$ to A_0 is smaller than the restriction of ε_x^{A} to A_0. Hence

$$\int_{X-A_0}\Big(\int_{A_0} f\,d\varepsilon_x^{A\cup B}\Big)\,d\mu(x) \leqslant \int_{X-A_0}\Big(\int_{A_0} f\,d\varepsilon_x^{A}\Big)\,d\mu(x).$$

We deduce

$$\int_{A_0} f\,d\mu^{A\cup B} \leqslant \int_{A_0}\Big(\int_{A_0} f\,d\varepsilon_x^{A}\Big)\,d\mu(x) + \int_{X-A_0}\Big(\int_{A_0} f\,d\varepsilon_x^{A}\Big)\,d\mu(x)$$

$$= \int \Big(\int_{A_0} f\,d\varepsilon_x^{A}\Big)\,d\mu(x) = \int_{A_0} f\,d\mu^{A} \leqslant \int_{A_0} f\,d(\mu^{A} \vee \mu^{B}).$$

Similarly we prove

$$\int_{B_0-A_0} f\,d\mu^{A\cup B} \leqslant \int_{B_0-A_0} f\,d(\mu^A \vee \mu^B).$$

We have

$$\int_{A-(A_0\cup B_0)} f\,d\mu^{A\cup B} = \int \Big(\int_{A-(A_0\cup B_0)} f\,d\varepsilon_x^{A\cup B}\Big)\,d\mu(x)$$

$$= \int_{A-(A_0\cup B_0)} \Big(\int_{A-(A_0\cup B_0)} f\,d\varepsilon_x^{A\cup B}\Big)\,d\mu(x)$$

$$+ \int_{X-(A-(A_0\cup B_0))} \Big(\int_{A-(A_0\cup B_0)} f\,d\varepsilon_x^{A\cup B}\Big)\,d\mu(x).$$

If $x \in A - (A_0 \cup B_0)$ the set $(A \cup B) - A$ is thin at x and therefore, by the preceding lemma, the restriction of $\varepsilon_x^{A\cup B}$ to A is smaller than the restriction of ε_x^A to A. Hence

$$\int_{A-(A_0\cup B_0)} \Big(\int_{A-(A_0\cup B_0)} f\,d\varepsilon_x^{A\cup B}\Big)\,d\mu(x)$$

$$\leqslant \int_{A-(A_0\cup B_0)} \Big(\int_{A-(A_0\cup B_0)} f\,d\varepsilon_x^{A}\Big)\,d\mu(x).$$

If $x \in X - (A_0 \cup B_0))$ then, by the preceding lemma, the restriction of $\varepsilon_x^{A\cup B}$ to $A - (A_0 \cup B_0)$ is smaller than the restriction of ε_x^A to $A - (A_0 \cup B_0)$ and we get

$$\int_{X-(A-(A_0\cup B_0))} \Big(\int_{A-(A_0\cup B_0)} f\,d\varepsilon_x^{A\cup B}\Big)\,d\mu(x)$$

$$\leqslant \int_{X-(A-(A_0\cup B_0))} \Big(\int_{A-(A_0\cup B_0)} f\,d\varepsilon_x^{A}\Big)\,d\mu(x).$$

Putting together the obtained inequalities we deduce

$$\int_{A-(A_0\cup B_0)} f\,d\mu^{A\cup B} \leqslant \int_{A-(A_0\cup B_0)} \Big(\int_{A-(A_0\cup B_0)} f\,d\varepsilon_x^{A}\Big)\,d\mu(x)$$

$$+ \int_{X-(A-(A_0\cup B_0))} \Big(\int_{A-(A_0\cup B_0)} f\,d\varepsilon_x^{A}\Big)\,d\mu(x)$$

$$= \int \Big(\int_{A-(A_0\cup B_0)} f\,d\varepsilon_x^{A}\Big)\,d\mu(x)$$

$$= \int_{A-(A_0\cup B_0)} f\,d\mu^{A} \leqslant \int_{A-(A_0\cup B_0)} f\,d(\mu^A \vee \mu^B).$$

Similarly we prove

$$\int_{B-(A\cup A_0\cup B_0)} f\,d\,\mu^{A\cup B} \leqslant \int_{B-(A\cup A_0\cup B_0)} f\,d\,(\mu^A \vee \mu^B).$$

If we add now the four inequalities we get by theorem 2.1.

$$\begin{aligned}\int f\,d\,\mu^{A\cup B} &= \int_{A_0} f\,d\,\mu^{A\cup B} + \int_{B_0-A_0} f\,d\,\mu^{A\cup B} + \int_{A-(A_0\cup B_0)} f\,d\,\mu^{A\cup B} \\ &\quad + \int_{B-(A\cup A_0\cup B_0)} f\,d\,\mu^{A\cup B} \\ &\leqslant \int_{A_0} f\,d\,(\mu^A \vee \mu^B) + \int_{B_0-A_0} f\,d\,(\mu^A \vee \mu^B) \\ &\quad + \int_{A-(A_0\cup B_0)} f\,d\,(\mu^A \vee \mu^B) + \int_{B-(A\cup A_0\cup B_0)} f\,d\,(\mu^A \vee \mu^B) \\ &= \int f\,d\,(\mu^A \vee \mu^B).\end{aligned}$$

Let now A, B be arbitrary subsets of X. By lemma 1.2. there exists two fine closed Borel sets A′, B′ of X, $A \subset A'$, $B \subset B'$, such that

$$\mu^A = \mu^{A'}, \qquad \mu^B = \mu^{B'}, \qquad \mu^{A\cup B} = \mu^{A'\cup B'}$$

for any $\mu \in \Lambda$. We deduce

$$\mu^{A\cup B} = \mu^{A'\cup B'} \leqslant \mu^{A'} \vee \mu^{B'} = \mu^A \vee \mu^B.$$

COROLLARY 3.1. — *Let* $(A_n)_{n\in \mathbf{N}}$ *be a sequence of subsets of* X *and let* μ *be a measure of* Λ. *If there exists a measure on* X *greater than* μ^{A_n} *for any* $n \in \mathbf{N}$ *then*

$$\mu^A \leqslant \bigvee_{n\in \mathbf{N}} \mu^{A_n},$$

where A *denotes the union of* $(A_n)_{n\in \mathbf{N}}$.

We set for any $m \in \mathbf{N}$

$$B_m = \bigcup_{n\leqslant m} A_n.$$

By induction we can show using the theorem that

$$\mu^{B_m} \leqslant \bigvee_{n\leqslant m} \mu^{A_n} \leqslant \bigvee_{n\in \mathbf{N}} \mu^{A_n}.$$

From [2] theorem 3.4 it follows that μ^{B_n} converges vaguely to μ^A. Hence

$$\mu^A \leqslant \bigvee_{n\in \mathbf{N}} \mu^{A_n}.$$

COROLLARY 3.2. — *Let $\mu \in \Lambda$ and let $(A_n)_{n \in \mathbf{N}}$, $(B_n)_{n \in \mathbf{N}}$ be two sequences of subsets of* X *such that for any $n \in \mathbf{N}$.*

$$\mu^{A_n} = \mu^{B_n}.$$

Then

$$\mu^{A} = \mu^{B},$$

where

$$A = \bigcup_{n \in \mathbf{N}} A_n, \qquad B = \bigcup_{n \in \mathbf{N}} B_n.$$

We set for any $n \in \mathbf{N}$

$$C_n = A_n \cup B_n, \qquad C = \bigcup_{n \in \mathbf{N}} C_n.$$

For any $p \in \mathfrak{T}_0$ we have by theorem 3.1

$$\int p\, d\mu^{A_n} = \int \hat{R}_p^{A_n} d\mu \leqslant \int \hat{R}_p^{C_n} d\mu = \int p\, d\mu^{C_n} \leqslant \int p\, d\,(\mu^{A_n} \vee \mu^{B_n})$$
$$= \int p\, d\mu^{A_n}.$$

Hence

$$\mu^{A_n} = \mu^{C_n}.$$

We set for any $n \in \mathbf{N}$

$$D_n = \{\, x \in X \mid \varepsilon_x^{A_n} = \varepsilon_x^{C_n} \}.$$

For any $p \in \mathfrak{T}_0$ and any $n \in \mathbf{N}$ we have

$$\int (\hat{R}_p^{C_n} - \hat{R}_p^{A_n})\, d\mu = \int p\, d\mu^{C_n} - \int p\, d\mu^{A_n} = 0.$$

Hence

$$\mu\,(\{\, x \in X \mid \hat{R}_p^{C_n}(x) > \hat{R}_p^{A_n}(x)\,\}) = 0.$$

From this relation we get immediately (lemma 1.1 and corollary 1.1)

$$\mu\,(X - D_n) = 0, \qquad \mu\,(X - \bigcap_{n \in \mathbf{N}} D_n) = 0.$$

We set for any $n \in \mathbf{N}$

$$A'_n = \bigcup_{m \leqslant n} A_m, \qquad C'_n = \bigcup_{m \leqslant n} C_m,$$

and denote by A''_n a fine closed Borel set containing A'_n and such that

$$\nu^{A'_n} = \nu^{A''_n}$$

for any $\nu \in \Lambda$ (lemma 1.2). Let n be a natural number and x be a point of $\bigcap_{n \in \mathbf{N}} D_n$. From the preceding corollary and theorem 2.1 we get

$$\varepsilon_x^{C'_n}(X - A''_n) \leqslant \sup_{m \leqslant n} \varepsilon_x^{C_m}(X - A''_n) = \sup_{m \leqslant n} \varepsilon_x^{C'_m}(X - A''_n) = 0,$$

$$\varepsilon_x^{A''_n \cup C'_n}(X - A''_n) \leqslant \sup(\varepsilon_x^{A''_n}(X - A''_n), \varepsilon_x^{C'_n}(X - A''_n)) = 0.$$

If $x \notin A''_n$ or if A''_n is not thin at x we get from lemma 3.1

$$\varepsilon_x^{A''_n \cup C'_n} \leqslant \varepsilon_x^{A''_n} = \varepsilon_x^{A'_n}.$$

If A_n is thin at x and x belongs to A''_n then any A_m is thin at x for any $m \leqslant n$. Since $x \in \bigcap_{n \in \mathbf{N}} D_n$ we see by corollary 1.1 that C_n is thin at x for any $m \leqslant n$. Hence by corollary 2.1 C'_n is thin at x and again by lemma 3.1 we get

$$\varepsilon_x^{A''_n \cup C'_n} \leqslant \varepsilon_x^{A'_n}.$$

For any $p \in \mathfrak{T}_0$ we deduce

$$\hat{R}_p^{A'_n}(x) \leqslant \hat{R}_p^{C'_n}(x) \leqslant \hat{R}_p^{A''_n \cup C'_n}(x) = \int p \, d\varepsilon_x^{A''_n \cup C'_n}$$

$$\leqslant \int p \, d\varepsilon_x^{A'_n} = \hat{R}_p^{A'_n}(x),$$

$$\hat{R}_p^{A'_n}(x) = \hat{R}_p^{C'_n}(x).$$

From this equality we get

$$\varepsilon_x^{A'_n} = \varepsilon_x^{C'_n}.$$

We deduce further

$$\mu^{C'_n} = \int \varepsilon_x^{C'_m} d\mu(x) = \int_{\bigcap_{n \in \mathbf{N}} D_n} \varepsilon_x^{C'_n} d\mu(x) = \int_{\bigcap_{n \in \mathbf{N}} D_n} \varepsilon_x^{A'_n} d\mu(x)$$

$$= \int \varepsilon_x^{A'_n} d\mu(x) = \mu^{A'_n}.$$

By [2] theorem 3.4 $(\mu^{A'_n})_{n \in \mathbf{N}}$ (resp. $(\mu^{C'_n})_{n \in \mathbf{N}}$) converges vaguely to μ^A (resp. μ^C) and therefore

$$\mu^A = \mu^C.$$

We prove similarly

$$\mu^B = \mu^C.$$

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Corneliu CONSTANTINESCU,
Institut de Mathématiques
de l'Académie Roumaine
Str. M. Eminescu 47
Bucarest 9 (Roumanie)