ANNALES DE L'INSTITUT FOURIER

NICOLAS TH. VAROPOULOS

Sets of multiplicity in locally compact abelian groups

Annales de l'institut Fourier, tome 16, nº 2 (1966), p. 123-158
<http://www.numdam.org/item?id=AIF_1966__16_2_123_0>

Ann. Inst. Fourier, Grenoble
16, 2 (1966), 123-158.

SETS OF MULTIPLICITY IN LOCALLY COMPACT ABELIAN GROUPS

by N. Th. VAROPOULOS

Introduction, notations and remarks.

Let G be a locally compact additive abelian group. In what follows we shall use freely standardized and well-established notations and terminology. We shall for instance always denote by $0 = 0_G$ the zero element of G. We shall denote by $\mathbf{Z}$ and $\mathbf{R}^n$ the group of the integers and the Euclidean n-space respectively.

$M(G) \rhd M_0(G)$ denotes the complex Banach algebra of all bounded complex Radon measures on G, and the closed ideal of those, whose Fourier transform vanishes at the infinity of $\hat{G}$, the character group of G. $M^+(G) \subset M(G)$ will denote the cone of positive measure and $M_0^+(G) = M^+(G) \cap M_0(G)$. $M(G)$ has a natural involution $\mu \to \tilde{\mu} = \overline{\mu(-x)}$. For $\mu \in M(G)$, $s(\mu)$ will denote the support of μ i.e. the smallest closed set whose complement is a μ-null set. We shall denote by h_G the Haar measure of G which is unique up to multiplicative constant; when G is compact h_G will always be normalised by $h_G(G) = 1$. We shall denote by $L_1(G)$ the algebra of elements of $M(G)$ which are absolutely continuous with respect to h_G.

For P, $Q \subset G$ subsets of G and $n \in \mathbf{Z}$ we denote:
ξ_P the characteristic function of P i.e. $\xi_P(x) = 1$ if $x \in P$; and $\xi_P(x) = 0$ if $x \notin P$.

$$P + Q = \{x + y; \quad x \in P, \quad y \in Q\} \subset G$$
$$nP = \left\{ sgn(n) \sum_{j=1}^{|n|} x_j; \quad x_j \in P, \quad 1 \leqslant j \leqslant |n| \right\} \subset G$$
$$Gp(P) = Gp\{x; x \in P\}.$$

τ_n: $G \to G$ the continuous endomorphism: $\tau_n(g) = ng$ for $g \in G$. For any set X, $|X|$ will denote the cardinal number of X. For any $x \in \mathbf{R}$, $x \geqslant 0$; $[x] = \sup\{n \in \mathbf{Z}; n \leqslant x\}$ will denote the integral part of x.

We shall also use the letter C, possibly with suffixes, to denote absolute positive constants, appearing in various formulae (not necessarily the same constant everywhere).

Finally we shall follow N. Bourbaki [1] for measure theory and Loève's book [4] for probability theory.

We now make a number of definitions:

Definition I. — *A subset* $P \subset G$ *will be called strongly independent if, for all* N, *positive integer, any family of* N *distinct points of* P, $(p_j \in P)_{j=1}^N$, *and any family of* N *integers,* $(n_j \in \mathbf{Z})_{j=1}^N$ *such that* $\sum_{j=1}^N n_j p_j = 0_G$, *we must have* $\tau_{n_j}(P) = 0_G$ *for all* $1 \leqslant j \leqslant N$.

Definition S. — *A positive Radon measure on* G,

$$0 \neq \mu \in M^+(G)$$

will be called an S-*measure (Salem) if:*

(i) $\mu \in M_0(G)$

(ii) $s(\mu)$ *is compact and* $h_G[Gp(s(\mu))] = 0.$

Definition S*. — *A positive Radon measure on* G,

$$0 \neq \mu \in M_0^+(G),$$

will be called an S*-*measure if it is an* S-*measure and if:*

(iii) $m \in \mathbf{Z}, \quad \tau_m(G) \neq 0_G \Longrightarrow \overline{h}_G[g \in G; \quad mg \in Gp(s(\mu))] = 0.$

Definition R. — *A subset* $P \subset G$ *will be called an* R-*set (Rudin) of* G, *if it is perfect, strongly independent and if there exists* $\mu \in M_0^+(G)$, $\mu \neq 0$, *such that* $s(\mu) \subset P$.

The two main theorems of this paper can be stated:

Theorem S. — *Every non discrete, locally compact, abelian group has* S-*measures.*

Theorem R. — *Every non discrete, metrisable, locally compact, abelian group has* R-*sets.*

We point out that the only point of introducing the concept of S*-measures is, that it is through them that R-sets will be constracted. In many important cases every S-measure is automatically an S*-measure. We have:

LEMMA 0.1. — *Let* G *be a denumerable at infinity, locally compact, abelian group, then we can afirm that every* S*-measure on* G *is an* S**-measure, provided that the following hypothesis* (H) *is satisfied for* G.

(H) *For every open subgroup* $\Omega \subset G$ *and every* $m \in \mathbf{Z}$ *we have:*

$$\tau_m(G) \neq O_G \Longrightarrow h_G[\tau_m(\Omega)] \neq 0.$$

Proof. — Observe that for an S-measure μ on G and every $m \in \mathbf{Z}$, the set $[g \in G;\ mg \in Gp(s(\mu))]$ is a Borel subgroup of G; thus if it has a positive h_G measure it must be an open subgroup; and that is impossible if $\tau_m(G) \neq O_G$ by the hypothesis (H).

LEMMA 0.2. — *The following groups satisfy hypothesis* (H).

(α) $\mathbf{R}$ *and* $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ *the one dimensional torus.*

(β) $\mathbf{U}(p) = (\mathbf{Z}(p^\infty))^\wedge$ *the additive group of p-adic integers for some prime p.*

(γ) $G = \prod_{n=1}^{\infty} \mathbf{Z}(p_n)$ *for prime numbers* $p_n (n \geqslant 1)$ *such that* $p_n \xrightarrow[n\to\infty]{} \infty$.

Proof. — (α) Immediate.

(β) By dualising a well-known property of $\mathbf{Z}(p^\infty)$ (cf. [3] 2 (9)) we see that every non zero closed subgroup of $\mathbf{U}(p)$ has finite index in $\mathbf{U}(p)$ and thus positive Haar measure. It suffices then to observe for any $\Omega \subset \mathbf{U}(p)$ open subgroup, and $m \neq 0$, $\tau_m(\Omega)$ is a non zero compact subgroup.

(γ) The open subgroups $\Omega_N = \prod_{n \geqslant N} \mathbf{Z}(p_n)$ ($N \in \mathbf{Z}$, $N \geqslant 1$) form a neighbourhood basis of O_G. And to see property (H) it suffices to observe that, since $p_n \xrightarrow[n\to\infty]{} \infty$, for all $m \in \mathbf{Z}$, $m \neq 0$, there exists $N_0 = N_0(m)$ such that $\tau_m(\Omega_N) = \Omega_N$ for all $N \geqslant N_0$.

The material of this paper is divided:

§ 1 The main tools and specialized notations for our contractions are introduced.

§ 2 We treat the case $G = \prod_{n=1}^{\infty} \mathbf{Z}(p_n)$ where p_n $(n \geqslant 1)$ are prime numbers increasing very rapidly (in a sense to be specified).

§ 3 We treat the case $G = \mathbf{U}(p) = (\mathbf{Z}(p^\infty))^\wedge$ the group of p-adic integers, for some fixed prime p.

§ 4 We treat the case $G = \prod_{n=1}^{\infty} G_n$ where $G_n \cong \mathbf{Z}(p^N)$, for some fixed prime p and some fixed $N \in Z$. $N \geqslant 1$, and all $n \geqslant 1$.

§ 5 We treat the case $G = \prod_{n=1}^{\infty} \mathbf{Z}(p^n)$ for some fixed prime p.

§ 6 We prove Theorem S.

§ 7 We prove Theorem R.

It might be worth observing that § 2, § 3, § 4 do not depend on each other and that § 5 depends only on § 3.

The names of Salem and Rudin, we use, are justified by [7] and [5].

1. The main tools and notations for the constructions.

We start by introducing some notations that will prove useful.

For, r, $m \in \mathbf{Z}$; $r \geqslant 1$, $m \geqslant 1$ we set:

$$\sigma(r;\, m) = \frac{1}{m} \sum_{j=0}^{m-1} \left|\cos \frac{2\pi j}{m}\right|^r$$

and:

$$\sigma(r) = \sigma(r;\, \infty) = \int_0^{2\pi} |\cos \theta|^r \, d\theta$$

we have by the mean value theorem:

$$|\sigma(r, m) - \sigma(r)| \leqslant \frac{C_1 r}{m} \tag{1.1}$$

also elementary considerations give [7] p. 537:

$$\sigma(r) \leqslant \frac{C_2}{\sqrt{r}}. \tag{1.2}$$

Let now for the rest of this paragraph denote by G a metrisable, compact, abelian group and let:

$$\Sigma \qquad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq \cdots \supseteq \bigcap_{n=1}^{\infty} G_n = \{O_G\}$$

be a series of compact open subgroups.

We shall associate with G and Σ two probability spaces:

$$(\Omega, \mathcal{B}, P) \qquad \text{and} \qquad (\Omega', \mathcal{B}', P')$$

as follows:

$$\Omega = \prod_{n=1}^{\infty} G_n \qquad \text{and} \qquad \Omega' = \prod_{n=1}^{\infty} L_n$$

where $L_n = G_n \times G_n$; $\mathcal{B}$ and $\mathcal{B}'$ are the topological Borel fields of Ω and Ω' respectively (for the Tychonov topology of course);

$$P = h_\Omega = \bigotimes_{n=1}^{\infty} h_{G_n} \qquad \text{and} \qquad P' = h_{\Omega'} = \bigotimes_{n=1}^{\infty} h_{L_n}.$$

We shall now define on Ω and Ω' two random Radon measures of G as follows:

DEFINITION Ω. — *With each*

$$\omega \in \Omega[\omega = (g_1, g_2, \ldots, g_n, \ldots);\ g_n \in G_n]$$

we associate a sequence of Radon measures on G $\{\mu_{j,\omega}\}_{j=1}^{\infty}$ *defined by:*

$$s(\mu_{j,\omega}) = \{g_j, g_j^{-1}\} \qquad \textit{and} \qquad \mu_{j,\omega}(\{g_j\}) = \mu_{j,\omega}(\{g_j^{-1}\}) = \frac{1}{2};$$

for $j \geqslant 1$ *we then define*:

$$\mu = \mu_\omega = \mathop{\bigstar}_{j=1}^{\infty} \mu_{j,\omega} = \lim_{N} \mathop{\bigstar}_{j=1}^{N} \mu_{j,\omega} \in M^+(G).$$

We shall call $\mu = \mu_\omega$ *the* Ω $(= \Omega[G, \Sigma])$*-random measure associated with* Σ, *and* $\mu_j = \mu_{j,\omega}$ *its* j^{th} *component.*

DEFINITION Ω' (only needed for § 4). — *With each*

$$\omega' \in \Omega'[\omega' = (l_1, l_2, \ldots, l_n, \ldots);\ l_n = (g_n, g_n') \in L_n = G_n \times G_n]$$

we associate a sequence of Radon measures on G $\{\mu'_{j,\omega'}\}_{j=1}^{\infty}$

defined by:

$$s(\mu'_{j,\omega'}) = \{g_j, \ g'_j\} \qquad \text{and} \qquad \mu'_{j,\omega'}(\{g_j\}) = \mu'_{j,\omega'}(\{g'_j\}) = \frac{1}{2};$$

for $j \geqslant 1$ *we then define:*

$$\mu' = \mu'_{\omega'} = \mathop{\bigstar}_{j=1}^{\infty} \mu'_{j,\omega'} = \lim_{\mathrm{N}} \mathop{\bigstar}_{j=1}^{\mathrm{N}} \mu'_{j,\omega'} \in \mathrm{M}^+(\mathrm{G}).$$

We shall call $\mu' = \mu'_{\omega'}$ *the* Ω' $(= \Omega'[\mathrm{G}, \Sigma])$*-random measure associated with* Σ, *and* $\mu'_j = \mu'_{j,\omega'}$ *its* j^{th} *component.*

We make at once a number of important remarks:

Remarks. — (1.i.) The convergence of the infinite, convolution product appearing in both definitions Ω and Ω' is taken for the vague topology of measures, and is assured by the fact that $\bigcap_{n=1}^{\infty} \mathrm{G}_n = \{0_{\mathrm{G}}\}$ and $s(\mu_{j,\omega}) \subset \mathrm{G}_j$ and $s(\mu'_{j,\omega'}) \subset \mathrm{G}_j$ for all $j \geqslant 1$; $\omega \in \Omega$, $\omega' \in \Omega'$.

(1.ii) For all $\omega \in \Omega$ and $\omega' \in \Omega'$ we have:

$$\mu_\omega \geqslant 0, \qquad \mu'_{\omega'} \geqslant 0; \qquad \|\mu_\omega\| = \|\mu'_{\omega'}\| = 1.$$

(1.iii) For each $\chi \in \hat{\mathrm{G}}$, $\{\hat{\mu}_j(\chi)\}_{j=1}^{\infty}$ (resp. $\{\hat{\mu}'_j(\chi)\}_{j=1}^{\infty}$) is a sequence of (complex) independent random variables defined on the probability space Ω (resp Ω') and we have:

$$\hat{\mu}(\chi) = \prod_{j=1}^{\infty} \hat{\mu}_j(\chi) \qquad \text{and} \qquad \hat{\mu}'(\chi) = \prod_{j=1}^{\infty} \hat{\mu}'_j(\chi);$$

from that it follows at once that for all $r \in \mathbf{Z}$, $r \geqslant 1$:

$$\mathbf{E}|\hat{\mu}(\chi)|^r = \prod_{j=1}^{\infty} \mathbf{E}|\hat{\mu}_j(\chi)|^r \qquad \text{and} \qquad \mathbf{E}|\hat{\mu}'(\chi)| = \prod_{j=1}^{\infty} \mathbf{E}|\hat{\mu}'_j(\chi)|.$$

1. iv) Let us assume that $\chi \in \hat{\mathrm{G}}$ is an element of finite order or equivalently that $m = |\mathrm{G}/\ker\chi| < +\infty$ then we see at once that if $m_j = |\mathrm{G}_j/\mathrm{G}_j \cap \ker\chi|$ then:

$$\mathbf{E}|\hat{\mu}_j(\chi)|^r = \sigma(r; m_j).$$

(1.v) In the particular case when for some $\mathrm{N} \in \mathbf{Z}$, $\mathrm{N} \geqslant 1$

$$\mathrm{G} = \prod_{n=1}^{\infty} \mathrm{G}^{(n)} \qquad \text{with} \qquad \mathrm{G}^{(n)} \cong \mathbf{Z}(2^{\mathrm{N}}) \quad (n \geqslant 1),$$

and

$$G_j = \prod_{n \geqslant A_j} G^{(n)} \qquad \text{for} \qquad j \geqslant 1 \qquad \text{and} \qquad 1 \leqslant A_1 \leqslant A_2 \leqslant \cdots.$$

We have for every $\chi \in \hat{G}$

$$\mathbf{E}|\hat{\mu}_j'(\chi)| = \begin{cases} 1 & \text{if} \quad \chi|G_j \equiv 1 \\ \alpha_j(\chi) < 1 & \text{if} \quad \chi|G_j \not\equiv 1 \end{cases}$$

and observe that just as above

$$\alpha_j(\chi) = \alpha(m_j) = \frac{1}{2m_j^2} \sum_{0 \leqslant \alpha, \beta \leqslant m_j - 1} \left| \exp \frac{2\pi\alpha i}{m_j} + \exp \frac{2\pi\beta i}{m_j} \right|$$

depends only on $m_j = |G_j/G_j \cap \ker\chi|$ which can take only the values $2, 4, \ldots, 2^N$ if $\chi|G_j \not\equiv 1$. So there exists $\alpha = \sup_{1 \leqslant k \leqslant N} \alpha(2^k)$ independent of χ and j such that

$$\chi|G_j \not\equiv 1 \Longrightarrow \alpha_j(\chi) \leqslant \alpha < 1.$$

We now prove some lemmas.

LEMMA 1.1 (only needed for § 4). — *Let* $X \subset \hat{G}$ *be a set of characters of* G *and assume that*

$$\sum_{\chi \in X} \mathbf{E}|\hat{\mu}(\chi)| < +\infty \quad \left(\text{resp.} \sum_{\chi \in X} \mathbf{E}|\hat{\mu}'(\chi)| < +\infty\right)$$

then almost surely $\hat{\mu}(\chi) \xrightarrow[\chi \in X;\, \chi \to \infty]{} 0$ (*resp. almost surely* $\hat{\mu}'(\chi) \xrightarrow[\chi \in X;\, \chi \to \infty]{} 0$).

Proof. — We prove the result for the Ω-random measures, the one for Ω'-random measures is proved identically.

Using the elementary properties of mathematical expectation we see that (observe $|X| \leqslant |\hat{G}| \leqslant \aleph_0$):

$$\mathbf{E}\left\{\sum_{\chi \in X} |\hat{\mu}(\chi)|\right\} = \sum_{\chi \in X} \mathbf{E}|\hat{\mu}(\chi)| \quad < +\infty$$

thus almost surely:

$$\sum_{\chi \in X} |\hat{\mu}(\chi)| < +\infty \Longrightarrow \text{almost surely } \hat{\mu}(\chi) \xrightarrow[\chi \in X;\, \chi \to \infty]{} 0.$$

LEMMA 1.2. — *Suppose that there exists a family of positive integers* $\{\rho_\chi\}_{\chi\in\hat{G}}$ *such that for all values of* $\lambda \geqslant 0$

$$\sum_{\chi\in\hat{G}} \lambda^{\rho_\chi}\, \mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} < +\infty.$$

Then almost surely:

$$\hat{\mu}(\chi) \xrightarrow[\chi\to\infty]{} 0.$$

Proof. — Just as in the proof of Lemma 1.1 we have for all $\lambda \geqslant 0$

$$\mathbf{E}\left\{\sum_{\chi\in\hat{G}} \lambda^{\rho_\chi}|\hat{\mu}(\chi)|^{\rho_\chi}\right\} = \sum_{\chi\in\hat{G}} \lambda^{\rho_\chi}\, \mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} < +\infty$$

and almost surely:

$$\sum_{\chi\in\hat{G}} \lambda^{\rho_\chi}|\hat{\mu}(\chi)|^{\rho_\chi} < +\infty.$$

From that we conclude that almost surely:

$$\overline{\lim_{\chi\to\infty}}\, |\hat{\mu}(\chi)| \leqslant \frac{1}{\lambda}$$

and λ being arbitrary it suffices to take a sequence $\lambda_n \xrightarrow[n\to\infty]{} \infty$ to obtain the required result.

LEMMA 1.3. — *Suppose that* Σ *in* G *is such that:*

$$\underline{\lim}_{n}\, \frac{\log h_G(G_n)}{n} = -\infty$$

then for all $\omega \in \Omega$ *and all positive integer* r *we have:*

$$h_G[rs(\mu_\omega)] = 0$$

and thus since $s(\mu_\omega) = -s(\mu_\omega)$ *we have:*

$$h_G[Gp(s(\mu_\omega))] = 0.$$

Proof. — We have in general for all $N \in \mathbf{Z}$, $N \geqslant 1$ and $\omega \in \Omega$:

$$rs(\mu_\omega) \subseteq \sum_{j=1}^{N} rs(\mu_{j,\omega}) + G_N$$

and thus:

$$h_G[rs(\mu_\omega)] \leqslant 2^{rN} h_G(G_N) = \exp\{r(\log 2)N + \log h_G(G_N)\} = K_N;$$

and by the hypothesis we have $\underline{\lim}_N K_N = 0$. That proves the result.

Now let G be of the form $G = \prod_{n=1}^{\infty} G_n$ for G_n compact abelian groups. And let $\mathcal{G} = \{g_n \in G_n \subset G\}_{n=1}^{\infty}$ and

$$\mathcal{K} = \{K_n \in \mathbf{Z}\}_{n=1}^{\infty} \qquad \text{such that} \qquad K_1 = 1;$$

$K_{n+1} > K_n \quad n \geqslant 1.$

Definition Λ' (only needed for § 4). — *With* $\mathcal{G}$ *and* $\mathcal{K}$ *we associate a sequence of measures of* G $\{\lambda_n\}_{n=1}^{\infty}$ *defined by:*

$$\lambda_n \geqslant 0, \qquad \|\lambda_n\| = 1, \qquad s(\lambda_n) = \bigcup_{K_n \leqslant j < K_{n+1}} (g_j \cup g_j^{-1}),$$

$$x \in s(\lambda_n) \Longrightarrow \lambda_n(\{x\}) = \frac{1}{|s(\lambda_n)|}; \qquad \text{for} \qquad n \geqslant 1$$

we also define:

$$\lambda = \bigotimes_{n=1}^{\infty} \lambda_n = \mathop{\bigstar}_{n=1}^{\infty} \lambda_n \in M^+(G)$$

and we call it the Λ $(= \Lambda[G;\ \mathcal{G},\ \mathcal{K}])$*-measure associated to* $\mathcal{G}$ *and* $\mathcal{K}$*; and* λ_j *we call its* j^{th} *component.*

Definition Λ (only needed for § 4). — *With* $\mathcal{G}$ *and* $\mathcal{K}$ *we associate a sequence of measures of* G $\{\lambda'_n\}_{n=1}^{\infty}$ *defined by:*

$$\lambda'_n \geqslant 0, \qquad \|\lambda'_n\| = 1, \qquad s(\lambda'_n) = \bigcup_{K_n \leqslant j < K_{n+1}} g_j \cup O_G,$$

$$x \in s(\lambda'_n) \Longrightarrow \lambda'_n(\{x\}) = \frac{1}{|s(\lambda'_n)|}; \qquad \textit{for} \qquad n \geqslant 1$$

we also define:

$$\lambda' = \bigotimes_{n=1}^{\infty} \lambda'_n = \mathop{\bigstar}_{n=1}^{\infty} \lambda'_n \in M^+(G)$$

and we call it the Λ' $(= \Lambda'[G;\ \mathcal{G},\ \mathcal{K}])$*-measures associated to* $\mathcal{G}$ *and* $\mathcal{K}$*; and* λ'_j *we call its* j^{th} *component.*

Remarks. — (1.vi) we have:

$$\lambda \geqslant 0, \qquad \lambda' \geqslant 0; \qquad \|\lambda\| = \|\lambda'\| = 1$$

(1.vii) for all $\chi \in \hat{G}$ we have:

$$\hat{\lambda}(\chi) = \prod_{j=1}^{\infty} \hat{\lambda}_j(\chi) \qquad \text{and} \qquad \hat{\lambda}'(\chi) = \prod_{j=1}^{\infty} \hat{\lambda}'_j(\chi)$$

we finish this paragraph with a very technical:

LEMMA 1.4 (only needed for § 4 and § 5). — *Suppose* $G = \prod_{n=1}^{\infty} G_n$ *with* $G_n = \mathbf{Z}(p^{N_n})$ *for some fixed prime* p, *and* $1 \leqslant N_1 \leqslant N_2 \leqslant \cdots$ *positive integers.*

Suppose $\mathfrak{L} = \{L_n\}_{n=1}^{\infty}$ *is a non decreasing sequence of positive integers such that:*

$$(\alpha) \qquad \lim_{n\to\infty} \frac{L_n}{n \log\log n} = +\infty$$

Suppose further that $\{\varphi_n^k \in M(G)\}_{n=1}^{\infty}$, $1 \leqslant k \leqslant r$, *are* r *families of Radon measures on* G *such that:*

$$(\beta) \qquad \varphi_n^k \geqslant 0 \quad \textit{and} \quad \|\varphi_n^k\| = 1 \quad \textit{for} \quad n \geqslant 1; 1 \leqslant k \leqslant r$$

$$(\gamma) \qquad s(\varphi_n^k) \subset \prod_{j \geqslant L_n} G_j \quad \textit{for} \quad n \geqslant 1, 1 \leqslant k \leqslant r$$

$$(\delta) \quad |s(\varphi_n^k)| = O(\log n) \quad \text{as} \quad n \to \infty \quad \textit{for} \quad 1 \leqslant k \leqslant r$$

under the above conditions if we denote by:

$$\varphi^k = \mathop{\bigstar}_{n=1}^{\infty} \varphi_n^k \qquad \text{and by} \qquad \varphi = \mathop{\bigstar}_{k=1}^{r} \varphi^k.$$

Then for all $m \in \mathbf{Z}$ *we have:*

$$\tau_m(G) \neq 0_G \Longrightarrow h_G[g \in G; \qquad mg \in Gp(s(\varphi))] = 0.$$

Proof. — The convergence for the vague topology of the $\mathop{\bigstar}_{n=1}^{\infty} \varphi_n^k$ $(1 \leqslant k \leqslant r)$ is assured by (α) and (γ).

Let us now denote for all $R \in \mathbf{Z}$, $R \geqslant 1$;

$$\Gamma_R = \sum_{k=1}^{r} Rs(\varphi^k) - \sum_{k=1}^{r} Rs(\varphi^k)$$

and observe that:

(1.3) $$Gp(s(\varphi)) = \bigcup_{R \geqslant 1} \Gamma_R$$

also for all $n \in \mathbf{Z}$ $n \geqslant 1$ and $1 \leqslant k \leqslant r$ we have:

$$s(\varphi^k) \subset \sum_{j=1}^{n} s(\varphi_j^k) + \prod_{j \geqslant L_n} G_j$$

thus:

(1.4) $$\Gamma_R \subset D_R^n + \prod_{j \geqslant L_n} G_j$$

where:

$$D_R^n = \sum_{k=1}^{r} \sum_{j=1}^{n} Rs(\varphi_j^k) - \sum_{k=1}^{r} \sum_{j=1}^{n} Rs(\varphi_j^k)$$

and using condition (δ) we see at once that:

(1.5) $$\log |D_R^n| = 0\,(n \log \log n).$$

Let us now fix $\zeta \in \mathbf{Z}$ $\zeta \geqslant 0$ such that $\tau_{p^\zeta}(G) \neq O_G$ then we have for all $t \in G$:

(1.6) $$h_{\tau_{p^\zeta}(G)} \Big[\Big(t + \prod_{j \geqslant L_n} G_j \Big) \cap \tau_{p^\zeta}(G) \Big] \leqslant h_{\tau_{p^\zeta}(G)} \Big[\prod_{j \geqslant L_n} \tau_{p^\zeta}(G_j) \Big] = \Big\{ \prod_{j=1}^{L_n - 1} |\tau_{p^\zeta}(G_j)| \Big\}^{-1} \leqslant p^{-L_n + C_\zeta}$$

where C_ζ is a constant (depending on ζ).

Now putting (1.4), (1.5) and (1.6) together we see that:

$$h_{\tau_{p^\zeta}(G)}[\Gamma_R \cap \tau_{p^\zeta}(G)] \leqslant C \exp\,(C_1\, n \log \log n - C_2 L_n) \xrightarrow[n \to \infty]{} 0$$

by condition (α). And using (1.3) we see that:

(1.7) $$h_G[g \in G;\ p^\zeta g \in Gp(s(\varphi))] \leqslant h_{\tau_{p^\zeta}(G)}[Gp(s(\varphi)) \cap \tau_{p^\zeta}(G)] = 0.$$

Now if $q \not\equiv 0 \pmod p$, τ_q, is an automorphism of G (It is (1-1) continuous and $\tau_q(G) \supset \sum_{n=1}^{\infty} G_n$) and thus preserves the measure of sets; that observation combined with (1.7) completes the proof of the Lemma.

2. The case $G = \prod_{n=1}^{\infty} \mathbf{Z}(p_n)$ for a « very rapidly incrasing » sequence of primes $\{p_n\}_{n=1}^{\infty}$.

Definition. — *A sequence of primes $\{p_n\}_{n=1}^{\infty}$ will be called very rapidly increasing if the following conditions are satisfied.*

(α) $\qquad p_{n+1} > p_1 p_2 \dots p_n \qquad$ for $\qquad n \geqslant 1.$

(β) If $q_n = \left[\dfrac{\log p_n}{n}\right]$ *the integral part of* $\dfrac{\log p_n}{n}$ *then* $q_{n+1} > q_n$ *for* $n \geqslant 1$.

(γ) $\qquad p_n \geqslant \exp(e^{n^2}) \qquad$ *for* $\qquad n \geqslant 1.$

Remarks (2.i). — Observe that given an arbitrary sequence of primes $\{r_n\}_{n=1}^{\infty}$ such that $\sup\limits_n r_n = +\infty$ we can extract a subsequence $\{r_{n_j}\}_{j=1}^{\infty}$ which is very rapidly increasing (with j). We can now state:

Theorem Π_1. — *If $\{p_n\}_{n=1}^{\infty}$ is a very rapidly increasing sequence of primes and if* $G = \prod_{n=1}^{\infty} G^{(n)}$ *with* $G^{(n)} = \mathbf{Z}(p_n)$ *then* G *has* S*-measures.*

Proof. — Let us define for all $n \in \mathbf{Z}$, $n \geqslant 1$:

$$K_n = \inf\left\{ r \in \mathbf{Z}\,;\; r \geqslant 1,\; \frac{\log p_r}{r} \geqslant n \right\}$$

and observe at once that:

$$K_1 = 1 \qquad \text{and} \qquad K_n \uparrow \infty.$$

Using condition (β) of the definition, an preserving the notation $q_n = \left[\dfrac{\log p_n}{n}\right]$, we see that for all $n \in \mathbf{Z}$, $n \geqslant 1$:

$$(2.1) \quad \{r \in \mathbf{Z}\,;\; K_r = n+1\} = \{r \in \mathbf{Z}\,;\; q_n < r \leqslant q_{n+1}\} \neq \varnothing;$$

and from that we deduce at once that for all $n \in \mathbf{Z}$, $n \geqslant 1$:

$$(2.2) \quad |\{j \in \mathbf{Z}\,;\; j \geqslant 1,\; K_j \leqslant n+1\}| = \sup_{K_j \leqslant n+1} j = \sup_{K_j = n+1} j = q_{n+1}.$$

Let us now define a series of subgroups of G:

$$\Sigma \qquad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq \cdots \supseteq \bigcap_{n=1}^{\infty} G_n = \{0_G\}$$

by

$$G_n = \prod_{j \geqslant K_n} G^{(j)} \qquad \text{for all} \qquad n \geqslant 1.$$

We shall prove the theorem Π_1 by proving the following fact:

« The Ω-random measure $\mu = \mu_\omega$ associated to the series Σ is almost surely an S-measure of G. »

Towards that taking (2.1) and condition (γ) of the definition into account we see that

$$\overline{\lim_n} \{(\log p_{K_n-1}) n^{-1}\} \geqslant \lim_m \{(\log p_m)(q_m + 1)^{-1}\} = +\infty$$

and that together with $h_G(G_n) = (p_1 p_2 \ldots p_{K_n-1})^{-1}$ implies that $\underline{\lim}_n \dfrac{\log h_G(G_n)}{n} = -\infty$ and thus taking Lemma 1.3 into acount we see that for all $\omega \in \Omega$:

$$(2.3) \qquad h_G[Gp(s(\mu_\omega))] = 0.$$

Let us now define for all $\chi \in \hat{G}$:

$$\nu_\chi = \sup\Big\{ r \in \mathbf{Z}; \quad \chi | \prod_{j \geqslant r} G^{(j)} \not\equiv 1 \Big\} \quad \text{if} \quad \chi \neq 0_{\hat{G}}; \qquad \nu_{0_{\hat{G}}} = 0$$

and:

$$\rho_\chi = e^{\nu_\chi^2}$$

and let us observe at once that for $j \geqslant 1$:

$$(2.4) \qquad K_j \leqslant \nu_\chi \Longrightarrow \infty > |G_j / G_j \cap \mathrm{Ker}\chi| \geqslant p_{\nu_\chi};$$

and

$$(2.5) \qquad |\{\chi \in \hat{G};\ \nu_\chi = j\}| = p_1 p_2 \ldots p_{j-1}(p_j - 1).$$

Now in general using (1. iii) and (1.iv) we see that:

$$(2.6) \quad \mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} = \prod_{j=1}^{\infty} \mathrm{E}|\hat{\mu}_j(\chi)|^{\rho_\chi} = \prod_{K_j \leqslant \nu_\chi} \mathrm{E}|\hat{\mu}_j(\chi)|^{\rho_\chi} = \prod_{K_j \leqslant \nu_\chi} \sigma(\rho_\chi; m_j)$$

where $m_j = |G_j/G_j \cap \ker\chi|$ and where an empty product is interpreted as 1.

Now using (1.1) and (1.2) we see that for all N, $m \in \mathbf{Z}$, $N \geqslant 1$, $m \geqslant p_N$

$$(2.7) \quad \sigma(e^{N^2}; m) \leqslant \sigma(e^{N^2}) + \frac{C_1 e^{N^2}}{m} \leqslant C_2 e^{-\frac{N^2}{2}} + \frac{C_1 e^{N^2}}{p_N} \leqslant C e^{-\frac{N^2}{2}}$$

putting (2.4) (2.6) and (2.7) together and using (2.2) we see that: for $\nu_\chi = N \geqslant 2$:

$$\mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} \leqslant \exp\left\{\left(\log C - \frac{N^2}{2}\right)|\{j \in \mathbf{Z};\ j \geqslant 1,\ K_j \leqslant N\}|\right\}$$
$$= \exp\left\{\left(\log C - \frac{N^2}{2}\right) q_N\right\}$$

from that and (2.5) and conditions (α) (γ) of the definition we deduce that for $N \geqslant 2$

$$(2.8) \quad \sum_{\nu_\chi = N} \mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} \leqslant p_1 p_2 \ldots p_N \exp\left\{\left(\log C - \frac{N^2}{2}\right) q_N\right\}$$
$$\leqslant p_N^2 \exp\left\{\left(\log C - \frac{N^2}{2}\right) q_N\right\} = \exp\left\{2 \log p_N + \left(\log C - \frac{N^2}{3}\right) q_N\right\}$$
$$= 0\left\{\exp\left(-\frac{N}{3} e^{N^2}\right)\right\}$$

and from that it follows that for all $\lambda \geqslant 0$

$$\sum_{\chi \in \hat{G}} \lambda^{\rho_\chi} \mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} \leqslant \sum_N \lambda^{e^{N^2}} \sum_{\nu_\chi = N} \mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} < +\infty.$$

Therefore the conditions of Lemma 1.2 are satisfied and so the Ω-random measure $\mu = \mu_\omega \in M_0(G)$ almost surely. If we combine that with (2.3) and with remark (1.ii) we have the required result that almost surely μ is an S-measure of G.

3. The case G — U(p) the p-adic integers.

THEOREM U. — *For any prime p the compact additive group of p-adic integers* $\mathbf{U}(p)$ *has* S-*measures.*

Proof. — We fix the prime p once and for all, and write $G = \mathbf{U}(p)$ for the group of p-adic integers. Observe then that:

$$G = \mathbf{U}(p) = \varprojlim_n \mathbf{Z}(p^n) \qquad \text{and that} \qquad \hat{G} = \mathbf{Z}(p^\infty) = \varinjlim_n \mathbf{Z}(p^n)$$

more explicitly to the ascending chain of $\hat{G}$:

$$\{O_G\} \subseteq \mathbf{Z}(p) \subseteq \mathbf{Z}(p^2) \subseteq \cdots \subseteq \mathbf{Z}(p^n) \subseteq \cdots \subseteq \mathbf{Z}(p^\infty) = \hat{G}$$

corresponds the polar descending chain of G:

$$G \supseteq (\mathbf{Z}(p))^0 \supseteq (\mathbf{Z}(p^2))^0 \supseteq \cdots \supseteq (\mathbf{Z}(p^n))^0 \supseteq \cdots \supseteq \{O_G\} = (\hat{G})^0$$

and of course for $n < m$

$$(\mathbf{Z}(p^n))^0/(\mathbf{Z}(p^m))^0 \cong \mathbf{Z}(p^{m-n}).$$

Let us also observe that for positive integers M and N:

$$\text{(3.1)} \quad \chi \in (\mathbf{Z}(p^M))\diagdown(\mathbf{Z}(p^{M-1})) \subset \hat{G} \Longrightarrow |(\mathbf{Z}(p^N))^0/(\mathbf{Z}(p^N))^0 \cap \operatorname{Ker}\chi| = p^{\sup\{M-N,0\}}.$$

We shall denote now for all n positive integer:

$$K_n = [n(\log n)^{1/4}] = \text{the integral part of } n(\log n)^{1/4}$$

and let us observe that:

$$K_1 = 0 \qquad \text{and} \qquad K_n \uparrow \infty.$$

Also for arbitrary $\alpha \geqslant 0$ we verify that there exists $n_0 = n_0(\alpha)$ s.t. for $n \geqslant n_0$ we have:

$$\text{(3.2)} \quad |\{j \in \mathbf{Z};\ j \geqslant 1,\ K_j \leqslant n - \alpha \log n\}| = \sup\{j \in \mathbf{Z};\ K_j \leqslant n - \alpha \log n\} \geqslant \frac{n}{(\log n)^{1/2}}.$$

Let us now define a series of subgroups:

$$\Sigma \quad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq \cdots \supseteq \bigcap_{n=1}^{\infty} G_n = \{O_G\}$$

by

$$G_n = (\mathbf{Z}(p^{K_n}))^0 \qquad \text{for} \qquad n \geqslant 1.$$

We shall prove Theorem U by proving that:

« The Ω-random measure $\mu = \mu_\omega$ associated with Σ is almost surely an S-measure. »

Towards that we observe that since

$$\log h_G(G_n) = -(\log p)K_n \qquad \text{and since} \qquad \frac{K_n}{n} \xrightarrow[n\to\infty]{} \infty$$

the conditions of Lemma 1.3 are satisfied and so for all $\omega \in \Omega$ we have:

$$(3.3) \qquad h_G[Gp(s(\mu_\omega))] = 0.$$

Let us now define for all $\chi \in \hat{G}$:

$$\rho_\chi = \inf\{r \in \mathbf{Z}; \qquad \chi \in \mathbf{Z}(p^r) \subseteq \hat{G}\}$$

we have for: $N \in \mathbf{Z},\ N \geqslant 1$:

$$(3.4) \quad |\{\chi \in \hat{G};\ \rho_\chi = N\}| = |\mathbf{Z}(p^N) \setminus \mathbf{Z}(p^{N-1})| = (p-1)p^{N-1}.$$

Now using remarks (1.iii) and (1.iv) and (3.1) we see that: for $\chi \neq 0_{\hat{G}}$:

$$(3.5) \quad \mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} = \prod_{K_j \leqslant \rho_\chi} \sigma(\rho_\chi;\ p^{\rho_\chi - K_j}) \leqslant \prod_{K_j \leqslant \rho_\chi - \alpha \log \rho_\chi} \sigma(\rho_\chi;\ p^{\rho_\chi - K_j})$$

where $\alpha = \frac{3}{2}(\log p)^{-1}$ and where empty products are interpreted as 1. But with that choice of α using (1.1) and (1.2) we see that: for

$$N \geqslant 1, \qquad K_j \leqslant N - \alpha \log N \Longrightarrow \frac{N}{p^{N-K_j}} \leqslant \frac{1}{\sqrt{N}}$$
$$\Longrightarrow \sigma(N;\ p^{N-K_j}) \leqslant \frac{C}{\sqrt{N}}$$

which together with (3.2) and (3.5) implies that there exists. N_0 such that for $N \geqslant N_0$ and $\rho_\chi = N$:

$$\mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} \leqslant \left(\frac{C}{\sqrt{N}}\right)^{|\{j \in \mathbf{Z};\ j \geqslant 1,\ K_j \leqslant N - \alpha \operatorname{Log} N\}|} \leqslant \left(\frac{C}{\sqrt{N}}\right)^{\frac{N}{(\operatorname{Log} N)^{1/2}}}$$

and from that we deduce using (3.4) that:

$$\sum_{\rho_\chi = N} \mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} \leqslant p^N \left(\frac{C}{\sqrt{N}}\right)^{\frac{N}{(\operatorname{Log} N)^{1/2}}}$$
$$\leqslant \exp\left\{(\log p)N + \left(\log C - \frac{1}{2}\log N\right)\frac{N}{(\log N)^{1/2}}\right\}$$
$$= O\left\{\exp\left(-\frac{1}{3}N(\log N)^{1/2}\right)\right\}.$$

And that implies that for all $\lambda \geqslant 0$:

$$\sum_{\chi \in \hat{G}} \lambda^{\rho_\chi} \mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} = \sum_{N} \lambda^{N} \sum_{\rho_\chi = N} \mathbf{E}|\hat{\mu}(\chi)|^{\rho_\chi} < +\infty.$$

Therefore the conditions of Lemma 1.2. are satisfied and thus almost surely $\mu = \mu_\omega \in M_0(G)$. That fact combined with (3.3) and remark (1.ii) implies that almost surely the Ω-random measure μ is an S-measure of G.

4. The case $G = \prod_{j=1}^{\infty} G^{(j)}$ for $G^{(j)} = Z(p^N)$ for some fixed prime and some fixed $N \in Z$, $N \geqslant 1$.

THEOREM Π_2. — *If* $G = \prod_{j=1}^{\infty} G^{(j)}$ *where* $G^{(j)} = \mathbf{Z}(p^N)$ *for some fixed prime p and some fixed* $N \in \mathbf{Z}$, $N \geqslant 1$; *then* G *has* S*-*measures.*

Proof. — Let us fix once and for all: $\mathcal{K} = \{K_j \in \mathbf{Z}\}_{j=1}^{\infty}$ satisfying the following conditions:

$$\begin{aligned} &1 = K_1 < K_2 < \cdots < K_n < \cdots, \qquad K_{n+1} - K_n \uparrow \\ (4.1) \qquad &K_n \sim n \log n \quad (n \to \infty), \\ &(K_{n+1} - K_n - \log n) = 0\,(1) \quad (n \to \infty) \end{aligned}$$

but arbitrary otherwise,

$$(\text{e.g. set } K_{j+1} - K_j = [\log i] + 1 \quad (j \geqslant 1)).$$

Observe now that:

$$\hat{G} = \sum_{j=1}^{\infty} \hat{G}^{(j)} \qquad \text{and} \qquad \hat{G}^{(j)} \cong \mathbf{Z}(p^N);$$

for $\chi \in \hat{G}$ let us introduce:

$$\nu_\chi = \sup \left\{ r \in \mathbf{Z};\ \chi \Big| \prod_{j=r}^{\infty} G^{(j)} \not\equiv 1 \right\} \quad \text{if} \quad \chi \neq O_{\hat{G}}, \quad \text{and} \quad \nu_{O_{\hat{G}}} = 0.$$

$$\varphi_\chi = \sup_{K_r \leqslant \nu_\chi} r$$

$$\delta_\chi = \left| \left\{ r \in \mathbf{Z};\ r \geqslant 1,\ \chi \Big|_{\prod_{K_r \leqslant j < K_{r+1}} G^{(j)}} \not\equiv 1 \right\} \right|$$

let us also define for $j \geqslant 1$:

$$X_j = \{\chi \in \hat{G}; \quad \varphi_\chi = j, \quad \delta_\chi \leqslant (\log i)^2\} \subseteq \hat{G}$$

and:

$$X = \bigcup_{j=1}^{\infty} X_j \subseteq \hat{G}$$

and let us observe that using (4.1) we have:

$$(4.2) \quad |X_j| \leqslant j^{(\log j)^2} p^{N(K_{j+1}-K_j)(\log j)^2} \leqslant \exp\{C(\log j)^3\}.$$

Let now $\mathcal{G} = \{g_n \in G\}_{n=1}^{\infty}$ be a family that satisfies:

$$(4.3) \qquad g_n \in G^{(n)} \setminus \tau_p(G^{(n)}).$$

And let us define:

$$\lambda[G; \mathcal{G}, \mathcal{K}] = \lambda = \bigotimes_{j=1}^{\infty} \lambda_j$$

and
$$\lambda'[G; \mathcal{G}, \mathcal{K}] = \lambda' = \bigotimes_{j=1}^{\infty} \lambda'_j;$$

the Λ and Λ' measures associated to $\mathcal{G}$ and $\mathcal{K} = \{K_j\}_{j=1}^{\infty}$, and let us observe that when $p \neq 2$ we have the following two facts:

a) If $x \in s(\lambda_n)$ then we have $\lambda_n(\{x\}) = \dfrac{1}{2(K_{n+1} - K_n)}$.

b) $s(\lambda_n) = - s(\lambda_n)$ and by (4.3) $Gp[s(\lambda_n)] = \prod_{K_n \leqslant j < K_{n+1}} G^{(j)}$ for $\chi \in \hat{G}$ if $\chi\Big|_{\prod_{K_n \leqslant j < K_{n+1}} G^{(j)}} \not\equiv 1$, then χ cannot be constant on $s(\lambda_n)$ and so there exist two points $x_1, x_2 \in s(\lambda_n)$ with

$$\chi(x_j) = \exp\left(\frac{2\pi r_j i}{p^N}\right) \quad j = 1,2 \qquad \text{and} \qquad r_1 \not\equiv r_2 \pmod{p^N}.$$

From observations *a*) and *b*) using (4.1) we deduce at once:

$$(4.4) \quad \chi\Big|_{\prod_{K_n \leqslant j < K_{n+1}} G^{(j)}} \not\equiv 1 \Longrightarrow |\hat{\lambda}_n(\chi)| \leqslant 1 - \frac{\beta}{K_{n+1} - K_n} \leqslant 1 - \frac{\beta}{\log \varphi_\chi + C_1}$$

for some $\beta = \beta(p, N) > 0$ and C_1 constants (independent of n and χ).

Then from (4.4) and remarks (1.vii) we see that if $p \neq 2$:

$$(4.5) \qquad |\hat{\lambda}(\chi)| \leqslant \left(1 - \frac{\beta}{\log \varphi_\chi + C_1}\right)^{\delta_\chi}.$$

Also we have analogously for $p = 2$

$$\chi\Big|_{\prod_{K_n \leqslant j < K_{n+1}} G^{(j)}} \not\equiv 1 \Longrightarrow |\hat{\lambda}'(\chi)| \leqslant 1 - \frac{\gamma}{K_{n+1} - K_n + 1} \leqslant 1 - \frac{\gamma}{\log \varphi_\chi + C_2}$$

for some $\gamma = \gamma(N) > 0$.

And from that it follows just as above that for $p = 2$

$$(4.6) \qquad |\hat{\lambda}'(\chi)| \leqslant \left(1 - \frac{\gamma}{\log \varphi_\chi + C_2}\right)^{\delta_\chi}.$$

Let us now define the series of subgroups of G:

$$\Sigma = \Sigma[G, \mathcal{K}],$$

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq \cdots \supseteq \bigcap_{n=1}^{\infty} G_n = \{O_G\}$$

by

$$G_n = \prod_{j \geqslant K_n} G^{(j)} \qquad \text{for} \qquad \text{all} \qquad n \geqslant 1;$$

and let us consider $\mu = \mu_\omega$ and $\mu' = \mu'_{\omega'}$ the Ω and Ω' random measures associated to the series.

Now using remark (1.iii) we obtain the following estimates

$$\mathbf{E}|\hat{\mu}(\chi)| = \prod_{j=1}^{\infty} \mathbf{E}|\hat{\mu}_j(\chi)| = \prod_{j \leqslant \varphi_\chi} \mathbf{E}|\hat{\mu}_j(\chi)|$$

with the usual interpretation of the empty product as 1.

Now since for $j \leqslant \varphi_\chi$ we have $\chi|G_j \not\equiv 1$, we see, using remark (1.iv), that for $p \neq 2$:

$$j \leqslant \varphi_\chi \Longrightarrow \mathbf{E}|\hat{\mu}(\chi)| \leqslant \sup_{1 \leqslant n \leqslant N} \{\sigma(1, p^n)\} = \delta < 1;$$

so finally we conclude:

$$(4.7) \qquad \mathbf{E}|\hat{\mu}(\chi)| \leqslant \delta^{\varphi_\chi}.$$

Analogously for $p = 2$ we see, using remark (1.iii) that

$$\mathbf{E}|\hat{\mu}'(\chi)| = \prod_{j=1}^{\infty} \mathbf{E}|\hat{\mu}_j'(\chi)| = \prod_{j \leqslant \varphi_\chi} \mathbf{E}|\hat{\mu}_j'(\chi)|;$$

and just as above using (1.v) we see that for $p = 2$:

$$j \leqslant \varphi_\chi \Longrightarrow \mathrm{E}|\hat{\mu}_j'(\chi)|. \leqslant \alpha = \alpha(\mathrm{N}) < 1;$$

so finally:

$$\mathbf{E}|\hat{\mu}'(\chi)| \leqslant \alpha^{\varphi_\chi}. \tag{4.8}$$

Now using (4.2) and (4.7) we see that

$$\sum_{\chi \in \mathbf{X}} \mathbf{E}|\hat{\mu}(\chi)| = \sum_{\mathrm{M}} \sum_{\chi \in \mathbf{X}_\mathrm{M}} \mathbf{E}|\hat{\mu}(\chi)| \leqslant \sum_{\mathrm{M}} |\mathrm{X}_\mathrm{M}| \sup_{\chi \in \mathbf{X}_\mathrm{M}} (\mathrm{E}|\hat{\mu}(\chi)|)$$
$$\leqslant \sum_{\mathrm{M}} \exp\{\mathrm{C}(\log \mathrm{M})^3 + (\log \delta)\mathrm{M}\} < +\infty$$

and thus the conditions of Lemma 1.1 are satisfied and we conclude that for $p \neq 2$:

$$\hat{\mu}(\chi) \xrightarrow[\chi \in \mathbf{X};\, \chi \to \infty]{} 0 \quad \text{almost surely.} \tag{4.9}$$

In an entirely analogously fashion we see that for $p = 2$:

$$\hat{\mu}'(\chi) \xrightarrow[\chi \in \mathbf{X};\, \chi \to \infty]{} 0 \quad \text{almost surely.} \tag{4.10}$$

Let us now observe that for all $\omega \in \Omega$ ($\omega' \in \Omega'$ respectively) the sequences of measures $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_{n,\omega}\}_{n=1}^{\infty}$

$$(\{\lambda_n'\}_{n=1}^{\infty} \quad \text{and} \quad \{\mu_{n,\omega'}'\}_{n=1}^{\infty} \quad \text{respectively})$$

satisfy the conditions of Lemma 1.4 if we take $\mathfrak{L} = \mathfrak{K}$. So we deduce that for all $\omega \in \Omega$ and all $\omega' \in \Omega'$ and all $m \in \mathbf{Z}$ with $\tau_m(\mathrm{G}) \neq \mathrm{O}_\mathrm{G}$ we have:

$$\begin{aligned} h_\mathrm{G}[g \in \mathrm{G};\ mg \in \mathrm{G}p(s(\lambda * \mu_\omega))] \\ = h_\mathrm{G}[g \in \mathrm{G};\ \mathrm{m}g \in \mathrm{G}p(s(\lambda' * \mu_{\omega'}'))] = 0. \end{aligned} \tag{4.11}$$

Thus finaly taking (4.11) into account and remarks (1.ii) and (1.vi) we see that Theorem Π_2 will follow if we prove:

(P) (i) For $p \neq 2$ almost surely $\lambda * \mu_\omega \in \mathrm{M}_0(\mathrm{G})$
(ii) For $p = 2$ almost surely $\lambda' * \mu_{\omega'}' \in \mathrm{M}_0(\mathrm{G})$

more explicitly:

P(i) $\Longrightarrow \lambda * \mu$ is an S*-measure of G almost surely when $p \neq 2$.

P(ii) $\Longrightarrow \lambda' * \mu'$ is an S*-measure of G almost surely when $p = 2$.

To prove P(i) it suffices, taking (4.9) into account, to prove that, if $\omega \in \Omega$ is such that $\hat{\mu}(\chi) \xrightarrow[\chi \in X;\, \chi \to \infty]{} 0$ then:

$$\hat{\lambda}(\chi)\hat{\mu}_\omega(\chi) \xrightarrow[\chi \to \infty]{} 0. \tag{4.12}$$

This we do; towards that let for $j \geqslant 1$;

$$X_j \subseteq T_j = \{\chi \in \hat{G};\ \ \varphi_\chi = j\} \subset \hat{G};\ \ j_1 \neq j_2 \Longrightarrow T_{j_1} \cap T_{j_2} = \varnothing$$

then:

$$|T_j| < +\infty, \qquad \bigcup_{j=1}^{\infty} T_j \cup O_{\hat{G}} = \hat{G}; \tag{4.13}$$
$$\chi \in T_j \setminus X_j \Longrightarrow \delta_\chi > (\log j)^2.$$

From that using remarks (1.ii) and (1.vi) and (4.5) and the choice of μ_ω we see that:

$$s_j = \sup_{\chi \in X_j} |\hat{\mu}_\omega(\chi)\ \hat{\lambda}(\chi)| \leqslant \sup_{\chi \in X_j} |\hat{\mu}_\omega(\chi)| \xrightarrow[j \to \infty]{} 0$$

and:

$$\sigma_j = \sup_{\chi \in T_j \setminus X_j} |\hat{\mu}_\omega(\chi)\ \hat{\lambda}(\chi)| \leqslant \sup_{\chi \in T_j \setminus X_j} |\hat{\lambda}(\chi)|$$
$$\leqslant \left(1 - \frac{\beta}{\log j + C_1}\right)^{(\log j)^2} \xrightarrow[j \to \infty]{} 0$$

so: $$\sup_{\chi \in T_j} |\hat{\mu}_\omega(\chi)\ \hat{\lambda}(\chi)| \leqslant \max\{s_j;\ \sigma_j\} \xrightarrow[j \to \infty]{} 0$$

and that together with (4.13) implies (4.12).

The proof of P(ii) follows in an entirely analogous line when we use (4.10) and (4.6).

5. The case $G = \prod_{j=1}^{\infty} Z(p^j)$ for some fixed prime p.

Let us in this paragraph readopt some of the notations of § 3; in particular let:

$K_j = [j(\log j)^{1/4}]$ be the integral part of $j(\log j)^{1/4}$ for $j \geqslant 1$.

Let us also fix $\{H_j\}_{j=1}^{\infty}$ a sequence of integers such that:

$$(5.1) \quad H_1 = 1, \qquad H_j \uparrow \infty, \qquad \frac{H_{j+1}}{\sum_{r=1}^{r=j} H_r} \geqslant (j+1)^2 \text{ for } j \geqslant 1.$$

Let us also introduce for all $n \geqslant 1$;

$$\begin{array}{llll} \alpha(n) = \inf\{j \in \mathbf{Z}; & j \geqslant 1, & H_j \geqslant K_n\} & \alpha(n) \uparrow \infty \\ \beta(n) = \sup\{j \in \mathbf{Z}; & j \geqslant 1, & K_j \leqslant n\} & \beta(n) \uparrow \infty \end{array}$$

and we observe that:

$$(5.2) \quad [n \in \mathbf{Z};\ n \geqslant 1,\ \alpha(n) \leqslant m] = [n \in \mathbf{Z}, n \geqslant 1, K_n \leqslant H_m] \Longrightarrow \sup_{\alpha(n) \leqslant m} n = \beta(H_m)$$

and that:

$$(5.3) \qquad \beta(n) = O\left(\frac{n}{(\log n)^{1/8}}\right) \qquad \text{as} \qquad n \to \infty$$

we can then state:

Theorem Π_3. — *If* $G = \prod_{j=1}^{\infty} G_j$ *with* $G_j = \mathbf{Z}(p^{H_j})$ $(j \geqslant 1)$ *and* $\{H_j\}_{j=1}^{\infty}$ *satisfying* (5.1); *then* G *has* S*-*measures*.

Proof. — Observe that $\hat{G} = \sum_{j=1}^{\infty} \hat{G}_j$ where $\hat{G}_j \cong \mathbf{Z}(p^{H_j})$ and that the canonical injections $\hat{G}_j \to \mathbf{Z}(p^{\infty})$ $(j \geqslant 1)$ induce a projection:

$$q: \hat{G} \to \mathbf{Z}(p^{\infty})$$

which by polarity induces a canonical injection:

$$q^0 = i: \mathbf{U}(p) = (\mathbf{Z}(p^{\infty}))^{\wedge} \to G.$$

Let now $\nu = \bigstar_{j=1}^{\infty} \nu_j$ be an S-measure of $\mathbf{U}(p)$ as constructed in § 3, and using the injection i let us identify ν_j $(j \geqslant 1)$ and ν with elementary M(G), going back then to § 3, we see that for that identification, for all $n \geqslant 1$

$$(5.4) \qquad s(\nu_n) \subseteq i\{(\mathbf{Z}(p^{K_n}))^0\} \subseteq \prod_{j \geqslant \alpha(n)} G_j;$$

and

$$|s(\nu_n)| = 2.$$

Let us also fix once and for all:

$$\{\theta_j \in M^+(G_j)\}_{j=1}^{\infty}$$

satisfying the following conditions: $\theta_1 = \delta_0$ and for $j \geqslant 2$ we have

$$||\theta_j|| = 1; \quad |s(\theta_j)| = p^{H_{j-1}}; \quad 0 \not\equiv \chi \in \mathbf{Z}(p^{H_{j-1}}) \subset \hat{G}_j \Longrightarrow \hat{\theta}_j(\chi) = 0$$

such a sequence of measures exists always, e.g. consider $[G_j : \mathbf{Z}(p^{H_j - H_{j-1}})]$ a section of $G_j \to \mathbf{Z}(p^{H_{j-1}})$ in G_j and give equal masses to all of its points.

Using that sequence let us define $\theta = \bigotimes_{j=1}^{\infty} \theta_j \in M(G)$ and let us also define $\varphi = \nu * \theta$ we have of course $\varphi \geqslant 0$ and $||\varphi|| = 1$ and we shall prove that φ is an S*-measure of G.

It is trivial to see that $\varphi \in M_0(G)$. Indeed by the definition of θ and the fact that for all $\chi \in \hat{G}$ $\hat{\theta}(\chi) = \prod_{j=1}^{\infty} \hat{\theta}_j(\chi)$, we see that if $\{\chi_\alpha \in \hat{G}\}_{\alpha \in A}$ is a net in $\hat{G}$ such that $\chi_\alpha \in \operatorname{supp} \hat{\varphi} \subset \operatorname{supp} \hat{\theta}$ for all $\alpha \in A$, and $\chi_\alpha \xrightarrow[\alpha \in A]{} \infty$ we have $q(\chi_\alpha) \xrightarrow[\alpha \in A]{} \infty$ in $\mathbf{Z}(p^\infty)$; thus $\hat{\nu}(\chi_\alpha) \xrightarrow[\alpha \in A]{} 0 \Longrightarrow \hat{\varphi}(\chi_\alpha) \xrightarrow[\alpha \in A]{} 0$.

We proceed to prove that the condition on the support of φ is satisfied. Towards that using the argument developed the end of the proof of Lemma 1.4 we see that it suffices to prove that for all $\zeta \in \mathbf{Z}$ $\zeta \geqslant 0$ we have:

$$h_{\tau_{p^\zeta}(G)}[Gp(s(\varphi)) \cap \tau_{p^\zeta}(G)] = 0. \tag{5.5}$$

To show that, let us observe that for all $N \in \mathbf{Z}$, $N \geqslant 1$ we have, using (5.4) and (5.2):

$$s(\nu) \subseteq \sum_{j=1}^{\beta(H_N)} s(\nu_j) + \prod_{j>N} G_j$$

and thus:

$$s(\varphi) \subseteq \sum_{j=1}^{\beta(H_N)} s(\nu_j) + \sum_{j=1}^{N} s(\theta_j) + \prod_{j>N} G_j.$$

Therefore using the arguments of the proof of Lemma 1.4 and in particular (1.6) and what follows: we have for all fixed $R \in \mathbf{Z}$ and $N > |\zeta|$

$$h_{\tau_p \zeta(G)}[(Rs(\varphi) - Rs(\varphi)) \cap \tau_{p^\zeta}(G)] \leqslant 2^{2R\beta(H_N)} p^{2R \sum_{j=1}^{N-1} H_j} p^{-\left(\sum_{j=1}^{N} H_j - \zeta N\right)}$$
$$= \exp\left(- C_1 H_N + C_2 \sum_{j=1}^{N-1} H_j + C_3 N + C_4 \beta(H_N)\right) \xrightarrow[n \to \infty]{} 0$$

by (5.1) (5.3) ($C_j > 0$, $1 \leqslant j \leqslant 4$ are constants). This together with the fac that $Gp(s(\varphi)) = \bigcup_{R=1}^{\infty} (Rs(\varphi) - Rs(\varphi))$ prove (5.5) and completes the proof.

6. Proof of Theorem S and application.

We start with a lemma on locally compact abelian groups:

LEMMA 6.1. — (i) *Let* G *be a compact group and* $H \subset G$ *be a closed subgroup then:*

α) *If* G/H *has* S-*measures so has* G.

β) *If* G/H *has* S*-*measures and is in addition a group unbounded order* ($m \neq 0 \Longrightarrow \tau_m(G/H) \neq O_{G/H}$) *then* G *has* S*-*measures also.*

(ii) *If* G *and* H *are locally compact groups and if they both have* S*-*measures* (*respectively* S-*measures*) *then* $G \times H = K$ *has* S*-*measures* (*respectively* S-*measures*).

(iii) *If* G *is a locally compact group and* $\Omega \subset G$ *is an open subgroup of* G *then if* Ω *has* S-*measures so has* G.

Proof. — (i) Observe that if $p: G \to G/H$ is the canonical projection, there exists a canonical injection

$$\bar{p}: M(G/H) \to M(G)$$

(cf. [1] ch .7) such that:

$$(6.1) \quad p^{-1}(M_0(G/H)) \subset M_0(G), \ \bar{p}(M^+(G/H)) \subset M^+(G); \quad s(\bar{p}(\mu)) = \bar{p}^{-1}(s(\mu)).$$

From that we see at once that for all $\mu \in M(G/H)$ and $m \in \mathbf{Z}$, $m \neq 0$:

$$(6.2) \quad p[g \in G; \ mg \in Gp\{s(\bar{p}(\mu))\}] = [\dot{g} \in G/H; \ m\dot{g} \in Gp(s(\mu))]$$

putting $m = 1$ and μ an S-measure of G/H in (6.2) we see using (6.1) that $\bar{p}(\mu)$ is an S-measure of G and that proves α.

To prove β we let μ be an S*-measure of G/H, and $m \neq 0$, $m \in \mathbf{Z}$, then by the hypothesis on G/H we see that

$$h_{G/H}[\dot{g} \in G/H; \quad m\dot{g} \in Gp(s(\mu))] = 0,$$

and that, together with (6.1) and (6.2) implies that $\bar{p}(\mu)$ is an S*-measure of G.

(ii) Let λ and μ be S*-measures of G and H respectively, then if $\nu = \lambda \otimes \mu \in M(K)$, we have:

$$(6.3) \qquad 0 \neq \nu \in M_0(K), \qquad \nu \geqslant 0;$$

also if $m \in \mathbf{Z}$ is such that $\tau_m(K) \neq O_K$, then either $\tau_m(G) \neq O_G$ or $\tau_m(H) \neq O_H$ or both; suppose then that $\tau_m(G) \neq O_G$ then by the hypothesis:

$$h_G[g \in G; \quad mg \in Gp(s(\lambda))] = 0;$$

and that implies:

$$h_K[k \in G; \quad mk \in Gp(s(\nu))] = 0;$$

which together with (6.3) implies that ν is an S*-measure of K.

The result about S-measures follows similarily and is only simpler.

(iii) Is trivial.

Definition 6.1. — (i) *We shall call a compact abelian group* G *a group of type* T *if* $\hat{G}$ *has torsion free elements.*

(ii) *We shall call a compact abelian group* G *a group of type* **U** *if* $\hat{G}$ *contains a subgroup* $B \subset \hat{G}$, $B \cong \mathbf{Z}(p^\infty)$ *for some prime* p.

(iii) *We shall call a compact abelian group* G *a group of type* Π *if* $\hat{G}$ *contains a subgroup* $\Sigma \subset \hat{G}$ *such that* $\Sigma \cong \sum_{n=1}^{\infty} \mathbf{Z}(p_n^{N_n})$, *where* p_n *are primes and* $N_n \in \mathbf{Z}$, $N_n \geqslant 1$ $(n \geqslant 1)$, *and where* $p_n^{N_n} \underset{n\to\infty}{\longrightarrow} \infty$.

Observe that the above classification is not identical with the one given in [8] § 5. With our classification we can obtain:

COROLLARY 6.1. — (i) *Groups of type* T *and* **U** *have* S*-*measures.*

(ii) *Groups of type* Π *have* S*-*measures.*

Proof. — (i) It follows from a theorem by Salem [7], from theorem U, and from Lemma 0.1 and Lemma 0.2, that **T**, the one dimensional torus, and $\mathbf{U}(p)$, the additive group of p-adic integers, for any prime p, have S*-measures. Our result follows from that, Lemma 6.1 (i) β, and the simple observation that if $B \subset \hat{G}$ and $B \cong \mathbf{Z}$ or $\mathbf{Z}(p^\infty)$ then $G/B^0 \cong \mathbf{T}$ or $\mathbf{U}(p)$.

(ii) Let G be a group of type Π and let $\hat{G} \supseteq \Sigma \cong \sum_{n=1}^{\infty} \mathbf{Z}(p_n^{N_n})$ with $p_n^{N_n} \xrightarrow[n\to\infty]{} \infty$ as in the definition. We distinguish two mutually exclusive cases.

a) $\sup_n p_n < +\infty$: then there exists $\Sigma_1 \subseteq \Sigma \subseteq \hat{G}$ such that $\Sigma_1 \cong \sum_{n=1}^{\infty} \mathbf{Z}(p^{N_n})$ with $N_n \xrightarrow[n\to\infty]{} \infty$ and some fixed prime p. Then there exists $\Sigma_2 \subseteq \Sigma_1$ with $\Sigma_2 \cong \sum_{j=1}^{\infty} \mathbf{Z}(p^{H_j})$ where H_j satisfy (5.1) : then $G/\Sigma_2^0 \cong \prod_{j=1}^{\infty} \mathbf{Z}(p^{H_j})$, and thus, a simple use of Theorem Π_3 and Lemma 6.1 (*i*) β. proves the result.

b) $\sup_n p_n = +\infty$: then just as above we see that

$$\Sigma_1 \subseteq \Sigma \subseteq \hat{G}$$

where $\Sigma_1 \cong \sum_{n=1}^{\infty} \mathbf{Z}(p_n)$ with $p_n \xrightarrow[n\to\infty]{} \infty$, and by remark (2.i) we may assume that $\{p_n\}_{n=1}^{\infty}$ is a very rapidly increasing sequence of prismes. Then $G/\Sigma_1^0 \cong \prod_{n=1} \mathbf{Z}(p_n)$, and thus G has S*-measures by Theorem Π_1 and Lemma 6.1 (i) β.

We next prove:

LEMMA 6.2. — *If a compact abelian group* G *is neither a group of type* T, *type* **U** *or type* Π, *then it is a group of bounded order, and thus* ([3], 8) $G \cong \prod_{\alpha\in A} \mathbf{Z}(p_\alpha^{n_\alpha})$, *where for all* $\alpha \in A$ p_α *is a prime number and* n_α *is a positive integer such that*

$$\sup_{\alpha\in A} p_\alpha^{n_\alpha} < +\infty.$$

Proof. — Using duality we see that to prove the Lemma it suffices to prove that:

« If A ($=\hat{G}$) is a discrete abelian group such that: (α) A is a torsion group.

(β) A is not of bounded order.

(γ) A is a reduced group. $\Longleftrightarrow$ A contains no divisible subgroups. $\Longleftrightarrow$ A contains no subgroups isimorphic to any $\mathbf{Z}(p^\infty)$ for any p prime number (use (α)).

Then A contains a subgroup $\Sigma \subset A$ such that

$$\Sigma \cong \sum_{n=1}^{\infty} \mathbf{Z}(p_n^{N_n})$$

for $\{p_n^{N_n}\}_{n=1}^{\infty}$ a sequence of powers of primes with $p_n^{N_n} \xrightarrow[n\to\infty]{} \infty$.»

To prove the above we can make on A the extra assumption: (δ) « A is a p-primary group for some prime p. » Indeed, anyway $A = \bigoplus_p A_p$ is the direct sum of its primary components, if the number of those components is infinite the conclusion above follows at once, otherwise we see that one of those components must satisfy (β) (and of course also (α) and (γ)), and we can argue on that component.

Now to prove the above statement it suffices to show that, for every group A satisfying conditions (α)-(δ), and for every $N \in \mathbf{Z}$, $N \geqslant 1$, there exists a direct decomposition $A = B \oplus C$ where $\tau_{p^N}(B) \neq O_B$ and $\tau_{p^N}(C) \neq O_C$. For then, it would follow that either B or C or both satisfy conditions (α) — (δ), and thus iterating the decomposition with increasing N, we would obtain:

$$A \supseteq D \cong B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus \cdots$$

such that $\tau_{p^n}(B_n) \neq O_{B_n}$ ($n \geqslant 1$), and that clearly implies our assertion.

Thus suppose that for some A satisfying conditions (α) — (δ), and some positive integer N, such a decomposition is impossible. From that contradictory hypothesis it follows that in all direct decompositions of $A = B \oplus C$ we always have either

$$\tau_{p^N}(B) = O_B \qquad \text{or} \qquad \tau_{p^N}(C) = O_C.$$

Now let $\{A = B_\theta \oplus C_\theta\}_{\theta\in\Theta}$ be a family of direct decompositions where Θ is a simply ordered set such that

$$\theta_1 \leqslant \theta_2 \Longrightarrow B_{\theta_1} \subseteq B_{\theta_2}$$

and such that $\tau_{p^N}(B_\theta) = O_A$; $\theta \in \Theta$. Then we claim that $B = \bigcup_{\theta\in\Theta} B_\theta$ is a direct component of A, of order $\leqslant p^N$. Indeed $\tau_{p^N}(B) = O_A$ trivially, further B is a serving (pure) subgroup of A (indeed $x \in B$ and $ny = x$ some $y \in A$ and $n \in \mathbf{Z} \Longrightarrow x \in B_\theta$ some $\theta \in \Theta$ and $y = b_\theta + c_\theta$ with $b_\theta \in B_\theta$ $c_\theta \in C_\theta \Longrightarrow nb_\theta = x$), so our assertion is a consequence of a well-known theorem in abelian group theory ([3] 8-Th. 7].

From the above it follows that the direct summands of A or order $\leqslant p^N$ have maximal elements. Let K be such a maximal element and let $A = K \oplus L$. Now L satisfies conditions (α) — (δ) and thus it is a decomposable group ([3] 9 Th. 10) $L = L_1 \oplus L_2$, $L_1 \neq 0$, $L_2 \neq 0$ and thus by our contradictory hypothesis either $\tau_{p^N}(L_1) = 0$ or $\tau_{p^N}(L_2) = 0$ and in either case we contradict the maximality of K.

We now prove:

Lemma 6.3. — (i) *If* G *is a metrisable compact abelian group then it has* S*-*measures.*

(ii) *If* G *is any compact abelian group then it has* S-*measures.*

Proof. — Taking corrollary 6.1 and Lemma 6.2 into account, it suffices to prove our Lemma making, in both (i) and (ii), the extra assumption that G is a group of bounded order, then:

(i) $G = G_1 \oplus G_2 \oplus \dots \oplus G_R$ for some $R \geqslant 1$ and where $G_j \cong \prod_{n=1}^{\infty} G_n^{(j)}$ with $G_n^{(j)} \cong \mathbf{Z}(p_j^{N_j})$ for p_j prime numbers and N_j positive integers $(1 \leqslant j \leqslant R)$. Our result then follows from Theorem Π_2 and Lemma 6.1 (ii).

(ii) G contains a closed subgroup $H \subset G$ such that $G/H \cong \prod_{n=1}^{\infty} G_n$ where $G_n \cong \mathbf{Z}(p)$ for some fixed prime p and then the result follows from Theorem Π_2 and Lemma 6.1.i.α.

We shall now use the following classical structure theorem (e.g. Cf. [6] 2-4).

(Σ) « Every locally compact abelian group G contains $\Omega \subset G$ an open subgroup such that $\Omega \cong \mathbf{R}^n \times K$ for some $n \in \mathbf{Z}$, $n \geqslant 0$; and K a compact group. »

Now using Salem's result [7] which asserts that $\mathbf{R}$ has S-measures, using also Lemma 6.3(i), Lemma 6.1(ii) and Lemmas 0.1 and Lemma 0.2 our structure theorem (Σ) above we see that we have:

COROLLARY 6.2. — *Every metrisable non discrete locally compact abelian group, contains a countable at infinity open subgroup Ω which has* S*-*measures.*

Proof of theorem S. — Using again Lemma 6.3 (ii) the result of Salem [7] and Lemma 6.1 (ii) we see that every group of the from $\mathbf{R}^n \times K$ where K is a compact abelian group, has S-measures provided that either $n \geqslant 1$ or $|K| \geqslant \aleph_0$. Theorem S then follows from that, our structure theorem (Σ) above, and Lemma 6.1 (iii).

We now proceed to give an application of theorem S. Towards that we start by describing a particular case a classical decomposition of M(G) due to Raicov (Cf. [2] § 2).

For $S \subset G$ any compact subset of the locally compact abelian group G let:

$$\mathcal{R}(S) = \{g + mS - nS; \quad g \in G, \quad m \geqslant 0, \quad n \geqslant 0\}$$

and let:

$$I(S) = \{m \in M(G); \quad \forall R \in \mathcal{R}(S), \quad R \text{ is an } m\text{-null set}\}$$

$$R(S) = \left\{m \in M(G);\ \exists \{R_j \in \mathcal{R}(S)\}_{j=1}^{\infty} \text{ s.t. } G \setminus \bigcup_j R_j \text{ is } m\text{-null}\right\}.$$

Then it can be proved [2] that $x \in I(S)$ and $y \in R(S) \Longrightarrow x \perp y$ (are mutually singular) and $I(S) \lhd M(G)$ is an ideal, while R(S) is a subalgebra, and we have the decomposition

$$M(G) = I(S) \oplus R(S).$$

Let us also define:

$\rho_s\colon M(G) \to \mathbf{C}$ by $\rho_s(m) = \int dr$ where $m = i + r$; $i \in I(S)$

and $r \in R(S)$ is the orthogonal (unique) decomposition of m. It is then easy to see that ρ_s is a complex homomorphism i.e. $\rho_s \in \mathfrak{M}(M(G))$ [2] the maximal ideal space of $M(G)$. Also it is easy to verify that for all $m \in M(G)$ we have:

$$\rho_s(\tilde{m}) = \overline{\rho_s(m)}$$

in other words that ρ_s is a symmetric ideal.

Let us then apply that decomposition starting from $S = s(\mu)$, for μ an S-measure of G, normalised by $||\mu|| = 1$. We verify then at once that $L_1(G) \subseteq I(S)$, so that $\rho_s(L_1(G)) = 0$ i.e. $\rho_s \notin \hat{G}$ for the canonical identification of $\hat{G}$ into a subset $\hat{G} \subset \mathfrak{M}(M(G))$. What is more to the point $\rho_s \notin \overline{(\hat{G})}$, the topological closure (for the Gelfand topology) of $\hat{G}$ in $\mathfrak{M}(M(G))$, for we have: $\tau \in \overline{(\hat{G})} \setminus \hat{G} \Longrightarrow \hat{\mu}(\tau) = 0$, while we have $\hat{\mu}(\rho_s) = \rho_s(\mu) = 1$.

So we have proved.

Application. — If G is a non discrete locally compact abelian group then there exists σ a symmetric maximal ideal of $M(G)$ such that:

$$\sigma \supset L_1(G) \qquad \text{and} \qquad \sigma \not\supset M_0(G).$$

7. Theorem R.

We start from a lemma on locally compact abelian groups:

LEMMA 7.1. — (i) *If* G *is a metrisable, locally compact group, and if it has* S*-*measures, then it also has an* S*-*measure* λ, *such that* $O_G \notin s(\lambda)$ *and* $s(\lambda)$ *is totally disconnected.*

(ii) *If* G *is a locally compact group, and if* Ω *is an open subgroup, and if* Ω *contains* R-*sets, then* G *contains* R-*sets also.*

Proof. — (i) Indeed if ν is an S*-measure of G then it suffices to find $0 \neq \lambda \leqslant \nu$ and such that $O_G \notin s(\lambda)$ and $s(\lambda)$ is totally disconnected. This can be done using simple arguments of general topology and Radon measure theory.

(ii) It is trivial.

Remarks. — (7.i) In general if μ is an S-measure, $s(\mu)$ has no isolated points since $\mu \in M_0(G)$. Thus in the above Lemma 7.1(i) $s(\lambda)$ is a Cantor set.

Let now G for the rest of this paragraph denote an abelian, metrisable, countable at infinity, locally compact group, and let us fix on it d a translation invariant metric. Let us also fix λ an S*-measure on G which is as in Lemma 7.1(i) i.e. $O_G \notin s(\lambda) = \Lambda$ and Λ is a Cantor set, and let us normalise it by: $\|\lambda\| = 1$.

Then by the hypothesis on λ and G we have for:

$$j = 0,1,2, \ldots; \ \Lambda = \bigcup_{\alpha=1}^{2^j} \Lambda_j^\alpha$$

where $\{\Lambda_j^\alpha\}$; $j = 0,1,2, \ldots, 1 \leqslant \alpha \leqslant 2^j$ are compact sets satisfying:

(α) $\Lambda_j^\alpha \cap \Lambda_j^\beta = \emptyset$ for $\alpha \neq \beta$ and all $j \geqslant 0$

(β) $\sup\limits_{1 \leqslant \alpha \leqslant 2^j} (\operatorname{diam} \Lambda_j^\alpha) \xrightarrow[j \to \infty]{} 0$ where of course for any $E \subseteq G$ $\operatorname{diam} E = \sup\limits_{e_1, e_2 \in E} d(e_1, e_2)$

(γ) $\Lambda_j^\alpha \supseteq \Lambda_{j+1}^\alpha \cup \Lambda_{j+1}^{2^j+\alpha}$ for $j \geqslant 0$ and $1 \leqslant \alpha \leqslant 2^j$.

Let us also denote by:

$$\lambda_j^\alpha = \xi_{\Lambda_j^\alpha} \lambda \in M_0^+(G) \qquad \text{for all} \qquad j \geqslant 0; \ 1 \leqslant \alpha \leqslant 2^j.$$

Let us now denote:

$$G^j = \prod_{\alpha=1}^{2^j} G_\alpha \qquad \text{with} \qquad G_\alpha = G, \qquad 1 \leqslant \alpha \leqslant 2^j \qquad \text{for} \qquad j \geqslant 0$$

and for all $g^j = (g_1^j, g_2^j, \ldots, g_{2^j}^j) \in G^j$ and $x \in G$.

Let us define:

$$\begin{aligned} &(7.2) \quad (\alpha) \quad \Lambda_j^\alpha[x] = x + \Lambda_j^\alpha; j \geqslant 0, 1 \leqslant \alpha \leqslant 2^j \\ &\qquad\quad\ \ (\beta) \quad \lambda_j^\alpha[x] = \delta_x \bigstar \lambda_j^\alpha \in M_0^+(G); j \geqslant 0, 1 \leqslant \alpha \leqslant 2^j \\ &\qquad\quad\ \ (\gamma) \quad \lambda[g^j] = \sum_{\alpha=1}^{2^j} \lambda_j^\alpha[g_\alpha^j] \in M_0^+(G); j \geqslant 0. \end{aligned}$$

We have of course:

$$(7.3) \quad \|\lambda[g^j]\| = 1, \qquad \lambda[g^j] \geqslant 0, \qquad s(\lambda[g^j]) \subseteq \bigcup_{\alpha=1}^{2^j} \Lambda_j^\alpha[g_\alpha^j].$$

We then compute, taking (7.1) and (7.2) into account for all $j \geqslant 0$ and $\chi \in \hat{G}$:

$$(\lambda[g^j])^\wedge(\chi) - (\lambda[g^{j+1}])^\wedge(\chi) = \sum_{\alpha=1}^{2^j} \{(\lambda_{j+1}^{\alpha}[g_\alpha^j])^\wedge(\chi) - (\lambda_{j+1}^{\alpha}[g_\alpha^{j+1}])^\wedge(\chi)\}$$
$$+ \sum_{\alpha=1}^{2^j} \{(\lambda_{j+1}^{\alpha+2^j}[g_\alpha^j])^\wedge(\chi) - (\lambda_{j+1}^{\alpha+2^j}[g_{\alpha+2^j}^{j+1}])^\wedge(\chi)\}$$

and using (7.2) and (7.3) and observing that

$$(\lambda_j^\alpha[x])^\wedge(\chi) = \chi(x)\hat{\lambda}_j^\alpha(\chi) \qquad \text{for} \qquad x \in G,$$

we see that there exists $\{\varepsilon_j > 0\}_{j=0}^\infty$ such that:

$$g^j \in G^j \qquad \text{for all} \qquad j \geqslant 0; \qquad g^0 = O_G \in G^0 = G;$$

for all $j \geqslant 0$

(7.4) $$D[g^j, g^{j+1}] = \sup\{d(g_\alpha^j, g_\alpha^{j+1}) + d(g_\alpha^j, g_{\alpha+2^j}^{j+1}); \; 1 \leqslant \alpha \leqslant 2^j\} \leqslant \varepsilon_j$$

implies:

(7.5) $$\sup_\chi |(\lambda[g^j])^\wedge(\chi) - (\lambda[g^{j+1}])^\wedge(\chi)| \leqslant 2^{-j}$$

and

$$s(\lambda[g^j]) \subseteq \Lambda + K \qquad \text{for} \qquad j \geqslant 0$$

and K some compact neighbourhood of O_G.

From (7.5) it follows at once that if $\{g^j\}_{j=0}^\infty$ is a sequence satisfying (7.4) then $\lambda[g^j] \to l$ for the vague topology of measures and by (7.3).

(7.6) $$l \geqslant 0, \qquad \|l\| = 1, \qquad l \in M_0(G).$$

Let us now denote for all $j \geqslant 0$:

$$\Sigma = \{n \in \mathbf{Z}; \tau_n(G) \neq O_G\} \subseteq \mathbf{Z}$$
$$\Sigma_j = \left\{(n_\alpha \in \mathbf{Z})_{\alpha=1}^{2^j}; |n_\alpha| \leqslant j+1 \text{ for } 1 \leqslant \alpha \leqslant 2^j, \sum_{\alpha=1}^{2^j} \tau_{n_\alpha}(G) \neq O_G\right\}$$

and let us also denote for $j \geqslant 0$:

$$H_j = \bigcup_{(n_\alpha)_{\alpha=1}^{2^j} \in \Sigma_j} \left\{g^j \in G^j; \sum_{\alpha=1}^{2^j} n_\alpha g_\alpha^j = O_G\right\} \subseteq G^j$$

and for $j \geqslant 0$

$$\mathrm{K}_j = \bigcup_{(n_\alpha)_{\alpha=1}^{2^j} \in \Sigma_j} \left\{ g^j \in \mathrm{G}^j ;\quad \sum_{\alpha=1}^{2^j} n_\alpha g_\alpha^j \in \mathrm{G}p\Lambda \right\} \subseteq \mathrm{G}^j.$$

We observe at once that:

$$(7.7) \qquad h_{\mathrm{G}^j}(\mathrm{K}_j) = 0 \qquad \text{for} \qquad j \geqslant 0$$

indeed by the definition of Σ, Σ_j and K_j to prove (7.7) it suffices to prove that:

if $j \geqslant 0$, $1 \leqslant \alpha \leqslant 2^j$ and $n_{\alpha_0} \in \Sigma \subset \mathbf{Z}$ and $n_\alpha \in \mathbf{Z}$ for $\alpha \neq \alpha_0$

arbitrary, then:

$$h_{\mathrm{G}^j}[g^j \in \mathrm{G}^j ;\quad \sum_{\alpha=1}^{2^j} n_\alpha g_\alpha^j \in \mathrm{G}p\Lambda] = 0 ;$$

and that follows from Fubini's theorem and the fact that the section of the set $\left[g^j \in \mathrm{G}^j ;\ \sum_{\alpha=1}^{2^j} n_\alpha g_\alpha^j \in \mathrm{G}p\Lambda \right]$ by the « plane »

$$g_\alpha^j = x_\alpha \qquad \text{for} \qquad \alpha \neq \alpha_0 ; \qquad 1 \leqslant \alpha \leqslant 2^j \quad \text{is}$$
$$\mathrm{S}(x) \cong \left[g \in \mathrm{G} ;\quad n_{\alpha_0} g \in - \sum_{\alpha \neq \alpha_0} n_\alpha x_\alpha + \mathrm{G}p\Lambda \right]$$

and so is either $\emptyset$ or a coset of a subgroup of G of measure zero by the definition of λ an S*-measure of G, and the definition of Σ.

We then prove:

Lemma 7.2. — *For all* $j \geqslant 1$ *and* $g^{j-1} \in \mathrm{G}^{j-1}$ *and* $\varepsilon > 0$ *we can find* $g^j = \{g_\alpha^j\}_{\alpha=1}^{2^j} \in \mathrm{G}^j$ *such that:*

$$\mathrm{D}[g^{j-1}, g^j] < \varepsilon \qquad \textit{and} \qquad \left(\prod_{\alpha=1}^{2^j} \Lambda_j^\alpha[g_\alpha^j] \right) \cap \mathrm{H}_j = \emptyset.$$

Proof. — Observe that for $j \geqslant 1$:

$$(7.8) \qquad \left(\prod_{\alpha=1}^{2^j} \Lambda_j^\alpha[g_\alpha^j] \right) \cap \mathrm{H}_j \neq \emptyset \Longrightarrow g^j \in \mathrm{K}_j \subset \mathrm{G}^j ;$$

it is also trivial, since $\{g^j \in \mathrm{G}^j ;\ \mathrm{D}[g^{j-1}, g^j] < \varepsilon\} \subset \mathrm{G}^j$ is an open subset, that:

$$(7.9) \quad h_{G^j}[g^j \in G^j; \qquad D[g^{j-1}, g^j] < \varepsilon] > 0 \quad j \geqslant 1.$$

Thus the Lemma follows by comparing (7.7), (7.8) and (7.9).
Now we choose inductively three families:

$$\{g^j = (g^j_\alpha)_{\alpha=1}^{2^j} \in G^j\}_{j=0}^\infty, \qquad \{\eta_j > 0\}_{j=0}^\infty, \qquad \{\rho_j > 0\}_{j=0}^\infty$$

satisfying for all $j \geqslant 0$ the following conditions:

(i) $\qquad g^0 = O_G \quad$ and $\quad D[g^j, g^{j+1}] < \eta_j$

(ii) $\qquad \rho_j \geqslant \sum_{k=j}^{\infty} \eta_k$

(iii) $\qquad \eta_j < \varepsilon_j$

(iv) $\qquad \left(\prod_{\alpha=1}^{2^j} (\Lambda_j^\alpha[g^j_\alpha])_{\rho_j}\right) \cap H_j = \emptyset$

where for any $E \subset G$ and any

$$\rho > 0 \quad E_\rho = \{x \in G; \quad d(x, E) \leqslant \rho\}$$

(v) $\qquad \rho_j \leqslant \dfrac{1}{j+1}.$

To see how that choice is carried out, suppose that for some $j \geqslant 0$ $g^j \in G^j$ has been chosen such that:

$$\left(\prod_{\alpha=1}^{2^j} \Lambda_j^\alpha[g^j_\alpha]\right) \cap H_j = \emptyset.$$

Then we can choose $\rho_j \leqslant \dfrac{1}{j+1}$ such that (iv) above holds, and impose on the $\{\eta_k\}_{k=j}^\infty$ the condition that they are small enough so that (ii) and (iii) hold. Then using Lemma (7.2) we can choose $g^{j+1} \in G^{j+1}$ such that (i) holds and that

$$\left(\prod_{\alpha=1}^{2^{j+1}} \Lambda_{j+1}^\alpha[g^{j+1}_\alpha]\right) \cap H_{j+1} = \emptyset.$$

And that completes the indective step. The induction starts trivially since $H_0 = \{O_G\} \subseteq G$ and thus:

$$\Lambda[g^0] \cap H_0 = \Lambda \cap \{O_G\} = \emptyset.$$

Now by (i) and (iii) as well as by (7.4) and (7.5) and (7.6) for such a choice of $\{g^j\}_{j=0}^\infty$, $\lambda[g^j] \to l \in M_0(G)$ for the vague topology of measures and $||l|| = 1$. Also from (7.2) and (7.3) and (i), (ii) and (iii) above it follows that for $i \geqslant j \geqslant 0$

$$s(\lambda[g_j]) \subseteq \bigcup_{\alpha=1}^{2^j} (\Lambda_j^\alpha[g_\alpha^j])_{\rho_j} = X_j;$$

and thus also $s(l) \subset X_j$ for all $j \geqslant 0$, and so $s(l) \subseteq \bigcap_{j=0}^{\infty} X_j$. And from that we see that the relations (iv) and (v) and the conditions (7.1) imply that $s(l)$ is a strongly independent set; in fact rather more can be asserted, namely:

« For any $N \in \mathbf{Z}$, $N \geqslant 1$, and any $\{\sigma_k \in s(l)\}_{k=1}^{N}$ distinct points, if $(n_k \in \mathbf{Z})_{k=1}^{N}$ are such that $\sum_{k=1}^{N} n_k\sigma_k = O_G$, then for all $1 \leqslant k \leqslant N$ $n_k \notin \Sigma$, which implies that $\tau_{n_k}(G) = O_G$ ».

Thus we have proved the:

Lemma 7.3. — *If G is a metrisable locally compact group and if it has* S^**-measures then it has* R*-sets.*

Proof of Theorem R. — The Lemma 7.3 the corollary 6.2 put together with the Lemma 7.1(ii) imply Theorem R at once.

Remarks. — (7.ii) Observe that the condition of metrisability cannot be relaxed for the construction of R-sets. Indeed if $\mu \in M_0(G)$, for G a non metrisable compact group, then

$$|\mathrm{supp}\ \hat{\mu}| \leqslant \aleph_0,$$

and thus there exists $\nu \in G/[Gp(\mathrm{supp}\ \hat{\mu})]^0$ such that μ can be identified canonically with ν (as in [1] ch. 7). Thus in particular $s(\mu)$ must contain a coset of $[Gp(\mathrm{supp}\ \hat{\mu})]^0$, which is a non trivial group ($[Gp(\mathrm{supp}\ \hat{\mu})]^0$ is not even metrisable); and thus $s(\mu)$ cannot be independent.

(7.iii) Any metrisable R-set contains a totally disconnected R-set (cf. Lemma 7.1).

We finish up the paragraph by stating without proof what we think is the main application of Theorem R.

If for any algebra R we denote:

$$R^2 = \left\{ \sum_{j=0}^{N} \lambda_j x_j y_j \middle| \ N \geqslant 1;\ \lambda \in G;\ x_j,\ y_j \in R \right\}$$

and if we denote the algebra of continuous measures:

$$M_c(G) = \{m \in M(G);\quad \forall x \in G\{x\} \text{ is a } m\text{-null set}\}$$

which is seen at once to be a closed ideal of M(G) then we have:

Application. — For any G non discrete locally compact abelian group:

(i) $M_c/\overline{M_c^2}$ is a non separable Banach space.

(ii) $M_0/\overline{M_0^2}$ is an infinite dimensional Banach space.

(iii) If in addition G is metrisable then:

$$M_0 \not\subset \overline{M_c^2}.$$

The proof of that result will be given in a publication which will follow this one [9].

I should like to finish up by expressing my gratitude to the C.N.R.S. for supporting this research financially, and to the Département de Mathématiques de la Faculté des Sciences d'Orsay, that offered to me for the second year running its generous hospitality.

BIBLIOGRAPHY

[1] N. Bourbaki, *Livre VI Integration.*
[2] E. Hewitt, *Michigan Math. J.*, 5, (1958), 149-158.
[3] I. Kaplanski, Infinite Abelian Groups, *The University of Michigan press.*
[4] M. Loève, Probability Theory, *Van Nostrand.*
[5] W. Rudin, Fourier Stieltjes transforms of measures on independant sets, *Bull. Amer. Math. Soc.*, 66 (1960).
[6] W. Rudin, Fourier analysis on groups, *Interscience tract*, 12.
[7] R. Salem, On sets of multiplicity for trigonometric series, *Amer. Journ. fo Math.*, 64 (1942), 531-538.
[8] N. Th. Varopoulos, The functions that operate on B_0 (Γ) of a discrete group, *Bull. Soc. Math. France*, 93 (1965) (to appear).
[9] N. Th. Varopoulos, Sur les mesures de Radon d'un groupe localement compact abélien, *C.R. Acad. Sc. Paris*, t. 260, 1059-1062 (1965).

Manuscrit reçu le 7 juillet 1965.

N. Th. Varopoulos,
Trinity College,
Cambridge (Angleterre).
et Département de Mathématiques
Fac. des Sci. Orsay.