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ON SETS FILLED BY ASYMPTOTIC SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Consider an ordinary differential equation

$$x' = f(t, x)$$

 $x = (x_1, \ldots, x_n), \quad f = (f_1, \ldots, f_n).$

Assumption I. Suppose that the domain D of f(t, x) is open, f(t, x) is continuous on D and through each point

A: $t = a_0, \quad x = a = (a_1, a_2, \ldots, a_n)$

of D passes only one integral x = x(t, A) of (1).

Denote by $(\alpha(A), \beta(A))$ the maximal interval on which there exists the integral passing through A. We shall denote

X(t, A) = (t, x(t, A)) for $t \in (\alpha(A), \beta(A))$.

Let E be an open subset of D. In the following we shall deal with the set Z(E) of such points A, that $X(t, A) \in E$ for $a_0 \leq t < \infty$. Obviously set Z(E) depends on both set E and system (1). It is evident that $E \subset F$ implies $Z(E) \subset Z(F)$. Let φ be a family of subsets F of D. We shall consider the following properties of equation (1).

PROPERTY I (of equation (1) in respect to E and φ). — For every $F \in \varphi$ Z(E) \cap F is empty or consists of one point.

PROPERTY II. — For every $F \in \varphi$ Z(E) \cap F is not empty. Let I⁺(A) denote the set of all points B = X(t, A) for $t \ge a_0$. We say that the point $A \in P(G) \cap D$, where P(G) denotes the boundary of an open set G, is the point of egress from G (with respect to equation (1) and set D) if there exists such an integral x(t) of (1) and a positive number $\varepsilon > 0$ that

$$x(a_0) = a$$
 and $(t, x(t)) \in G$

for $a_0 - \varepsilon < t < a_0$ (under Assumption I, $X(t, A) \in G$ for $a_0 - \varepsilon < t < a_0$). If no point of P(G) is a point of egress from G then $A \in G$ implies $I^+(A) \subset G$. If Property I is satisfied and $B \in Z(E) \cap F$ then $(F - B) \cap Z(E) = \emptyset$, where F - B denotes the set of all points of the set F except the point B. It follows that for every $A \in F$, $A \neq B$ either $I^+(A) \sim \epsilon E$ or $\beta(A) < \infty$. Let G be such a set that $\overline{G} \cap E$ has no common point with a plane $t = c > a_0$, where \overline{G} denotes the closure of G, then $I^+(A) \subset \overline{G}$ implies $A \sim \epsilon Z(E)$.

LEMMA. — Suppose Assumption I and the following conditions. For each set $G_i(i = 1, ...)G_i \cap E$ is contained in a halfspace $t < c_i$. No point of $P(G_i)$ is a point of egress. Set F satisfies inclusion $F - O \subset \bigcup_{i=1}^{\infty} G_i$. Then $(F - O) \cap Z(E) = \emptyset$.

THEOREM 1. — Suppose Assumption I and the following conditions. The intersection E(s) of a given set E and the plane t = s satisfies the inequality diam(E(t)) < p(t), where p(t) is a positive function continuous on $(-\infty, \infty)$. No point of $P(G_i)$ is a point of egress in respect to the equation

$$x' = f(t + a_0, x + a(t)) - f(t + a_0, a(t)),$$

where a_0 is a real number and x = a(t) is such a Lipchitzian function that the right side of the equation is defined. Set F satisfies inclusion $F - O \subset \bigcup_{i=1}^{\infty} G_i$. For any *i* and *s* there exists a constant c(i, s) that dist $(G_i(t), 0) \ge p(t+s)$ for $t \ge c(i, s)$, where $G_i(s)$ is the intersection of G_i and the plane t = s.

Under these assumptions if $A \in Z(E)$, then

$$(\mathbf{F}(\mathbf{A}) - \mathbf{A}) \cap \mathbf{Z}(\mathbf{E}) = \mathbf{\emptyset},$$

where F(A) denotes set obtained from A by translation of R^{n+1} transforming O on A.

THEOREM 2. — If assumptions of Theorem 1 are satisfied and F is a plane then equation (1) possesses property I in respect to E and the family of planes parallel to F (and of the same dimension).

Suppose now that set F is a plane and in the coordinate system t, x = (u, v), $u = (u_1, \ldots, u_k)$, $v = (v_1, \ldots, v_{n-k})$ it has the equation t = 0, u = 0. Now Property I (for the family of planes $t = c_0$, $u = (c_1, \ldots, c_k)$, c_i arbitrary) is necessary and sufficient for set Z(E) to be the graph of a single-valued function v = q(t, u). Putting $g = (f_1, \ldots, f_k)$, $h = (f_{k+1}, \ldots, f_n)$ system (1) takes the form

(2)
$$u' = g(t, u, v), \quad v' = h(t, u, v).$$

The following result formulated in terms of inequalities can be obtain from Theorem 1 formulated in terms of sets (1)

THEOREM 3. — Suppose that system (2) satisfies Assumption I and that the functions g(t, u, v), h(t, u, v) for

$$(t, u, v) \in \mathbf{D}, \qquad (t, \overline{u}, \overline{v}) \in \mathbf{D}$$

satisfy inequalities

(3)
$$(g(t, u, v) - g(t, \overline{u}, \overline{v})) (u - \overline{u}) \leqslant \gamma(t) (u - \overline{u})^2$$

for $|v - \overline{v}| = |u - \overline{u}|$, where |z| denotes Euclidean distance of point z from 0,

(4)
$$(h(t, u, v) - h(t, \overline{u}, \overline{v})) (v - \overline{v}) \ge \gamma(t) (v - \overline{v})^2$$
,

for

$$|u-\overline{u}|\leqslant |v-\overline{v}|,$$

where $\gamma(t)$ is a function summable in every finite interval, and such that

$$\int_0^\infty \gamma(s) \ ds = \infty,$$

then set Z of points A lying on the integrals of (2) (remaining in D) bounded for $a_0 \leq t < \infty$ is a graph of a single-valued function v = q(t, u) defined in a certain set $S(S \in \mathbb{R}^{k+1})$ satisfying the Lipschitz condition with respect to all the variables

(1) Such kind of formulation was suggested by T. Wazewski.

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and in particular the condition

 $|q(t, u) - q(t, \overline{u})| \leq |u - \overline{u}|$

in the set S or the set Z is an empty set.

Theorem 3 is a particular case of theorem 2 in [1].

Now for illustration of Property II we present a variant of an example from [2].

Let system (2) satisfy Assumption I on a neighbourhood D of the set $H: |u| \leq 1, |v| \leq 1, -\infty < t < \infty$. Moreover suppose that g(t, u, v)u < 0 for $|u| = 1, |v| \leq 1$ and arbitrary t, h(t, u, v) > 0 for $|v| = 1, |u| \leq 1$ and arbitrary t.

Under these assumptions for every \overline{u} , $|\overline{u}| < 1$ and arbitrary \overline{t} , there exists \overline{v} , that $I^+(\overline{t}, \overline{u}, \overline{v}) \in H$.

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