ON THE AXIOMATIC OF HARMONIC FUNCTIONS II

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In this paper we shall use constantly the notations and definitions from [2].

1. The fine topology of $X$ is the least fine topology on $X$, which is finer than the given topology and with respect to which the superharmonic functions (not necessarily defined on the whole space $X$) are continuous. A set $V \ni x$ is a fine neighbourhood of $x$ if and only if either $x$ is an interior point of $V$ for the initial topology or there exists a superharmonic function $s$, defined on a neighbourhood of $x$, such that

$$s(x) < \liminf_{x \in V \ni y \ni z} s(y).$$

**Theorem 1.** — Any point of $X$ possesses a fundamental system of fine neighbourhoods which are compact and connected in the initial topology of $X$.

Since the theorem has a local character we may suppose that there exists a harmonic positive function and a positive potential on $X$. Dividing the sheaf of harmonic functions by a positive harmonic function the fine topology does not change; we may suppose therefore that the constants are harmonic functions.

Let $x \in X$ and $V$ be a fine neighbourhood of $x$. It is sufficient to suppose that $x$ is not an interior point of $V$. There exists then a superharmonic function $s$ on $X$ such that

$$s(x) < \liminf_{x \in V \ni y \ni z} s(y).$$

Let $\alpha$ be a real number

$$s(x) < \alpha < \liminf_{x \in V \ni y \ni z} s(y)$$

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and $U$ be a regular domain containing $x$ such that $s$ is greater than $\alpha$ on $\overline{U} - V$. We denote by $F$ the set

$$F = \{ y \in \overline{U} \mid s(y) \leq \alpha \}$$

and by $K$ the component of $F$ containing $x$. It is sufficient to prove that $K$ is a fine neighbourhood of $x$.

Let $C$ be a component of $F$ contained in $U$. There exists then an open set $G$, $C \subset G \subset U$, $F \cap \partial G = \emptyset$. Since $s > \alpha$ on $\partial G$ it follows $s > \alpha$ on $G$ which contradicts the inequality $s \leq \alpha$ on $C$. Consequently any component of $F$ has a non-empty intersection with $\partial U$.

Let $\beta$ and $\varepsilon$ be positive numbers such that $s + \beta > \alpha$ on $\partial U$ and $s(x) + \varepsilon \beta < \alpha$. Let $K'$ be a compact set in $\partial U - K$ such that

$$\omega_{x}^{U}(\partial U - (K \cup K')) < \varepsilon.$$ 

We denote by $u$ the function on $\overline{U}$ equal to

$$y \rightarrow \omega_{y}^{U}(\partial U - (K \cup K'))$$

on $U$ equal to 1 on $\partial U - (K \cup K')$ and equal to 0 on $K \cup K'$. $u$ is lower semicontinuous. We denote by $F_{\varepsilon}$ the set

$$F_{\varepsilon} = \{ y \in \overline{U} \mid s(y) + \beta u(y) \leq \alpha \}.$$ 

Obviously $F_{\varepsilon} \subset F$ and $x \in F_{\varepsilon}$. There exists an open set $G$, $K \subset G$, $K' \cap F \cap G = \emptyset$, $F \cap \partial G = \emptyset$. Let $y \in F_{\varepsilon} \cap G$ and $C_{y}$ be the component of $F_{\varepsilon}$ containing $y$. $C_{y}$ is contained in $G$ since $F_{\varepsilon} \cap \partial G = \emptyset$. $C_{y} \cap \partial U$ is not empty, as it was shown above; let $z \in C_{y} \cap \partial U$. If $z \in K$ then

$$s(y) + \beta u(y) = s(y) + \beta > \alpha$$

which contradicts the relation $z \in F_{\varepsilon}$. Hence $z \in K$ and $C_{y} \subset K$, $y \in K$, $F_{\varepsilon} \cap G \subset K$. Since

$$\liminf_{y \in \partial U} (s(y) + \beta u(y)) = \liminf_{y \in \partial G} (s(y) + \beta u(y))$$

$$\geq \liminf_{y \in \partial G} (s(y) + \beta u(y)) \geq \alpha > s(x) + \beta u(x).$$

$K$ is a fine neighbourhood of $x$.

2. We shall suppose in this paragraph that there exists a positive potential on $X$. 
Theorem 2. — For any non-negative superharmonic function $s$ on $X$ and any set $E \subset X$, $\hat{R}^E_x$ is equal to $s$ on the fine interior of $E$.

Let $x$ be a fine interior point of $E$. Then [3] (pag. 435)

$$\lim_{[U, U]} \int_{(x - E) \cap U} d\omega^E_x = 0,$$

where $U$ denotes the filter of sections of regular neighbourhoods of $x$. If $s$ is bounded in a neighbourhood of $x$ then

$$s(x) \geq \hat{R}^E_x(x) = \lim_{U \uparrow} \tilde{R}^E_x \omega^U_x$$

$$\geq \lim \sup_{U \uparrow} \tilde{R}^E_x \omega^U_x = \lim \sup_{U \uparrow} \frac{1}{\ell} \int_{(x - E) \cap U} s \omega^U_x$$

$$\geq \lim_{U \uparrow} \int s \omega^U_x - \lim \sup_{U \uparrow} \int_{(x - E) \cap U} s \omega^U_x = s(x).$$

In the general case let $\mathcal{E}$ be the set of continuous finite positive superharmonic functions dominated by $s$. We have

$$s(x) \geq \hat{R}^E_x(x) \geq \sup_{E \in \mathcal{E}} \hat{R}^E_x(x) = \sup_{E \in \mathcal{E}} s'(x) = s(x).$$

Corollary 1. — A polar set has no fine interior points.

Theorem 3. — Let $G$ be a fine open set and $s$ be a non-negative superharmonic function. Then

a) $\hat{R}^G_x = R^G_x$;

b) $R^G_x = R^E_x$ for any $E \subset G$;

c) $R^G_x = \sup_{s' \in \mathcal{E}} R^G_x$, where $\mathcal{E}$ is an increasingly directed set of superharmonic functions with $s = \sup_{s' \in \mathcal{E}} s'$ on $G$;

d) $\hat{R}^G_x = \sup_{K \subset G} \hat{R}^K_x$ where $K$ is compact.

Corollary 2. — For any fine open set $G$ and any measure $\mu$ with compact carrier we have

$$\int s \, d\mu^G = \int \hat{R}^G_x \, d\mu. \quad (1)$$

where $s$ is an arbitrary non-negative superharmonic function.

(1) $\mu^G$ is the balayaged measure of $\mu$ on $G$ [3] (p. 447).
This relation follows from Theorem 3 c) taking $\mathcal{H}$ as the set of all continuous finite positive superharmonic functions smaller than $s$.

**Lemma 1.** — Let $s$ be a positive superharmonic function on $X$ and $F \subseteq X$ be a closed non-empty set, non-polar if $X \notin \mathfrak{B}$. $s$ is resolutive for the normed Dirichlet problem on $X - F$ and we have

$$R^F_s = H^{X-F}_s \quad (2)$$
onumber

on $X - F$.

Since $F$ is non-polar if $X \notin \mathfrak{B}$, there exists a locally bounded positive potential on any component of $X - F$. Let $s_0$ be a positive continuous superharmonic function on $X$. We want to prove that $X - F$ is an $\text{MP}_0$-set [1]. Let $s' \in \mathcal{H}_{s_0}^{X-F}$. Then $s' + \varepsilon s_0$ is non-negative outside a compact set contained in $U$ for any $\varepsilon > 0$. From [2] (Theorem 2) it follows $s' + \varepsilon s_0 \geq 0$. $\varepsilon$ being arbitrary we get $s' \geq 0$ and $X - F$ is an $\text{MP}_0$-set. By [1] (Corollary 3) the restrictions of the functions $\min (s, n s_0)$ on $\partial U$ are resolutive. Since $\min (s, n s_0) \uparrow s$ for $n \uparrow \infty$ it follows that the restriction of $s$ on $\partial U$ is resolutive.

Let $\bar{s} \in \mathcal{H}_{s_0}^{X-F}$ and $s'$ be the function on $X$ equal to $s$ on $F$ and equal to $\min (s, \bar{s})$ on $X - F$. $s'$ is superharmonic and dominates $s$ on $F$. Hence $\bar{s} \geq R^F_s$, $H^{X-F}_s \geq R^F_s$ on $X - F$. The converse inequality is trivial.

**Theorem 4.** — Let $s$ be a non-negative superharmonic function. For any $E \subseteq X$ such that $s$ is finite on $E$

$$R^E_s = \inf_{G \supseteq E} R^G_s,$$

where $G$ is fine open.

Obviously

$$R^E_s \leq \inf_{G \supseteq E} R^G_s.$$

Let $s'$ be a non-negative superharmonic function on $X$, $s' \geq s$ on $E$ and $\theta$ be a real number, $0 < \theta < 1$. The set

$$G = \{ x \in X | s'(x) > \theta s(x) \}$$

[*] The normed Dirichlet problem and the associated notions were introduced in [1].
is fine open and contains \(E\). We have \(s' \geq s\) on \(G\) and therefore

\[
\frac{s'}{G} \geq R^G_s \geq \inf_{G \supseteq E} R^G_s.
\]

\(s', \theta\) being arbitrary we get

\[
R^E_s \geq \inf_{G \supseteq E} R^G_s.
\]

**Theorem 5.**

Let \(s_1, s_2\) be non-negative superharmonic functions and \(E\) be an arbitrary set. Then

\[
R^E_{s_1 + s_2} = R^E_{s_1} + R^E_{s_2}, \quad R^E_{s_1 + s_2} = R^E_{s_1} + R^E_{s_2}.
\]

If \(E\) is compact the relation follows from lemma 1. By theorem 3 \(d)\) it can be extended to fine open sets and by theorem 4 to arbitrary \(E\) subjected to the condition that \(s_1 + s_2\) is finite on \(E\). In the general case we have

\[
R^E_{s_1 + s_2} \leq R^E_{s_1} + R^E_{s_2}
\]

and

\[
R^E_{s_1 + s_2} = R^E_{s_1} + R^E_{s_2}
\]
on \(E\). We denote by \(E'\) the set

\[
E' = \{y \in E | s_1(y) + s_2(y) < \infty\}.
\]

Let \(x \in X - E\). If \(R^E_{s_1 + s_2}(x) = \infty\) the required equality holds at \(x\). On the contrary case there exists a non-negative superharmonic function \(s_0\) on \(X\) finite at \(x\) and \(s_0 \geq s_1 + s_2\) on \(E\). For any non-negative superharmonic function \(s\) on \(X\) and any \(\varepsilon > 0\) we have

\[
R^E_s \leq R^E_{s'} \leq R^E_s + \varepsilon s_0.
\]

Hence

\[
R^E_s(x) = R^E_{s'}(x).
\]

We have therefore

\[
R^E_{s_1 + s_2}(x) = R^E_{s_1 + s_2}(x) = R^E_s(x) + R^E_{s_2}(x) = R^E_{s_1}(x) + R^E_{s_2}(x).
\]

The second equality follows immediately from the first one.

\(^{(*)}\) This theorem was proved by R.-M. Hervé under the supplementary hypothesis that \(X\) has a countable basis, and either \(s_1, s_2\) are continuous or \(E\) is closed or \(E\) is open or the axiom D is fulfilled \([3]\).
BIBLIOGRAPHY

