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ON THE AXIOMATIC OF HARMONIC FUNCTIONS I

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The aim of the present paper is to present some remarks on Brelot's axiomatic of harmonic functions [2] and to show that any space, which locally has a countable basis and on which there exists a positive superharmonic function, possesses a countable basis.

1. Let X be a locally compact connected space and $\mathcal{H}$ a sheaf on X, of real vector spaces of continuous functions (1) called harmonic functions. An open set $U \subset X$ is called regular if it is non-empty, relatively compact and, if for any continuous function f on the boundary ∂U of U, there exists a unique function on $\overline{U}$ equal to f on ∂U and harmonic on U, non-negative if f is non-negative. The restriction of this function on U will be denoted by H_f^U. For any $x \in U$ the functional

$$f \to H_f^U(x)$$

is linear and non-negative on the real vector space of continuous functions on ∂U. There exists therefore a measure $\omega_x^U = \omega_x$ on ∂U, called harmonic measure, such that

$$H_f^U(x) = \int f \, d\omega_x^U$$

for any continuous function f on ∂U.

We assume that $\mathcal{H}$ satisfies the following axioms.

A_1. *The regular domains form a basis of* X.

A_2. *The limit of any increasing sequence of harmonic functions on a domain is either harmonic or identically infinite.*

(1) The term « function » means, in this paper, « real finite function ».

If u is a non-negative harmonic function on a domain U, then it follows from $\mathbf{A}_2$, considering the sequence $\{nu\}$, that u is either positive or identically zero.

THEOREM 1. — *Let* $\mathfrak{U}$ *be an increasingly directed set of harmonic functions on a domain* U; *the least upper bound of* $\mathfrak{U}$ *is either harmonic or identically infinite.*

We shall prove the assertion using an idea from R. NEVANLINNA (*Uniformisierung*, Springer Verlag, 1953).

Let us suppose the least upper bound of $\mathfrak{U}$ is not identically infinite and let x be a point of U at which it is finite. There exists an increasing sequence $\{u_n\}$ of $\mathfrak{U}$ such that

$$\lim_{n\to\infty} u_n(x) = \sup_{u\in\mathfrak{U}} u(x).$$

We denote

$$u_0 = \lim_{n\to\infty} u_n.$$

Then u_0 is harmonic by $\mathbf{A}_2$. Let y be a point of U different from x and $\{v_n\}$ be an increasing sequence of $\mathfrak{U}$ such that $u_n \leqslant v_n$ and

$$\lim_{n\to\infty} v_n(y) = \sup_{u\in\mathfrak{U}} u(y).$$

We denote

$$v_0 = \lim_{n\to\infty} v_n.$$

Since v_0 is finite at x it is harmonic. Obviously $u_0 \leqslant v_0$. Since u_0 and v_0 are equal at x they coincide everywhere. It follows that

$$u_0(y) = \sup_{u\in\mathfrak{U}} u(y).$$

y being arbitrary, u_0 is the least upper bound of $\mathfrak{U}$.
The theorem shows that axiom $\mathbf{A}_2$ is equivalent to axiom 3 [2].

2. A lower semi-continuous numerical function [2] s on an open set V, which does not take the value $-\infty$, is called hyperharmonic if for any regular domain U, $\overline{\mathrm{U}} \subset \mathrm{V}$, and $x \in \mathrm{U}$

$$s(x) \geqslant \int s \, d\omega_x^{\mathrm{U}}.$$

[2] « Numerical function » will mean a function whose values are real numbers or $\pm\infty$.

A hyperharmonic function on the open set V is called superharmonic if it is not identically infinite on any component of V. A function s is called hypoharmonic (resp. subharmonic) if $-s$ is hyperharmonic (resp. superharmonic). A non-negative hyperharmonic function is called a potential if its greatest harmonic minorant is zero. A set A is called polar if for any $x \in X$ there exists a positive superharmonic function on a neighbourhood U of x infinite on $U \cap A$.

THEOREM 2. — *Let* $U \subset X$ *be an open non-compact* (3) *set on which there exists a positive superharmonic function. Then any superharmonic function on* U *non-negative outside a compact subset of* U *is non-negative.*

Let s_0 be a positive superharmonic function on U and s be a superharmonic function on U non-negative outside a compact set. For any positive real number α we denote

$$K_\alpha = \{x \in U \mid s(x) + \alpha s_0(x) \leqslant 0\}.$$

K_α is a compact set and we have $K_\alpha \subset K_\beta$ for $\alpha \geqslant \beta$ and

$$K_\alpha = \bigcap_{\alpha > \beta} K_\beta.$$

Suppose s negative at a point. Then since $\inf s > -\infty$ there exists a real number $\alpha > 0$ such that $K_\alpha \neq \emptyset$ and $K_\beta = \emptyset$ for $\beta > \alpha$. The function $s + \alpha s_0$ is superharmonic and non-negative. Let x be a point of K_α and V the component of U which contains x. Since $s + \alpha s_0$ vanishes at x it vanishes on the whole V ([2] Theorem 3 (i)). It follows $V \subset K_\alpha$ which is a contradiction since V is non-compact. Hence s is non-negative.

COROLLARY 1. — *Let* $U \subset X$ *be an open non-compact set and* s_0 *a superharmonic function on* U *such that*

$$\inf s_0 > 0.$$

Then any superharmonic function s *on* U *for which*

$$\liminf_{x \to a_U} s(x) \geqslant 0,$$

where a_U *is the Alexandroff point of* U, *is non-negative.*

(3) This means either X non-compact or $U \neq X$.

Indeed for any $\varepsilon > 0$ the function $s + \varepsilon s_0$ is non-negative outside a compact set and therefore non-negative on U.

THEOREM 3. — *Let f be a lower semi-continuous numerical function on X, which does not take the value $-\infty$. The greatest lower bound of the set of hyperharmonic functions which dominate f is hyperharmonic, continuous at any point x where f is continuous; if, moreover, it is superharmonic and different at x from $f(x)$ then it is harmonic on a neighbourhood of x*(4).

Let $\mathcal{S}$ denote the set of hyperharmonic functions which dominate f, s_0 be its greatest lower bound, U a regular domain and $y \in \mathrm{U}$. We have

$$s_0(y) = \inf_{s \in \mathcal{S}} s(y) \geqslant \inf_{s \in \mathcal{S}} \int s \, d\omega_y^{\mathrm{U}} \geqslant \overline{\int} s_0 \, d\omega_y^{\mathrm{U}}.$$

From this relation it follows that the regularised function $\hat{s}_0$ is hyperharmonic ([2] Theorem 7). Since f is lower semi-continuous $\hat{s}_0 \in \mathcal{S}$, $\hat{s}_0 = s_0$.

Let x be a point at which f is continuous and $s \in \mathcal{S}$ such that

$$s(x) \neq f(x).$$

There exists a harmonic functions u, defined on a neighbourhood of x, for which

$$f(x) < u(x) < s(x).$$

Let U be a regular neighbourhood of x, where these inequalities still hold. For any $y \in \mathrm{U}$ we have

$$\int s \, d\omega_y^{\mathrm{U}} \geqslant \int u \, d\omega_y^{\mathrm{U}} = u(y) > f(y)$$

and therefore the balayaged function of s relative to

$$\mathrm{X} - \mathrm{U}, \qquad \hat{\mathrm{R}}_s^{\mathrm{X}-\mathrm{U}},$$

belongs to $\mathcal{S}$. Herefrom it follows that if s_0 is superharmonic and $s_0(x) \neq f(x)$, then s_0 is harmonic on a neighbourhood of x. Further we get

$$\limsup_{y \to x} \; s_0(y) \leqslant \limsup_{y \to x} \; \hat{\mathrm{R}}_s^{\mathrm{X}-\mathrm{U}}(y) = \hat{\mathrm{R}}_s^{\mathrm{X}-\mathrm{U}}(y) \leqslant s(x).$$

(4) This theorem was proved in the classical case by M. BRELOT, *Journ. de Math. Pures et Appl.*, 24, 1945, 1-32.

Let U′ be a domain, K a compact set in U′ and f' the function defined on U′ equal to f on K and equal to s_0 on U′ — K. We denote by $\mathcal{S}'$ the set of hyperharmonic functions on U′ which dominate f' and by s'_0 its greatest lower bound. Obviously $s'_0 \leqslant s_0$ on U′ and $s'_0 = s_0$ on U′ — K. The function on X equal to s_0 on X — K and equal to s'_0 on U′ is hyperharmonic ([2] Theorem 4) and dominates f. Hence $s_0 = s'_0$ on U′.

We take U′ as being a regular neighbourhood of x and K a compact neighbourhood of x. For any $\varepsilon > 0$ we have

$$s'_0 + \varepsilon H_1^{U'} \in \mathcal{S}'$$

and

$$s'_0(x) + \varepsilon H_1^{U'}(x) \neq f'(x).$$

From the preceding considerations we have, since f' is continuous at x,

$$\limsup_{y \to x} s_0(y) = \limsup_{y \to x} s'_0(y) \leqslant s'_0(x) + \varepsilon H_1^{U'}(x) = s_0(x) + \varepsilon H_1^{U'}(x).$$

ε being arbitrary s is continuous at x.

Corollary 2. — *A superharmonic function which dominates a continuous function is equal to the least upper bound of the set of its continuous finite superharmonic minorants.*

Corollary 3([2] Proposition 12). — *Let* F *be a closed set with a non-empty interior. If there exists a potential on* X *then there exists a continuous positive potential on* X *harmonic on* X — F.

It is sufficient to take f as being a continuous non-negative function, $f \not\equiv 0$, whose carrier lies in F.

3. *We shall denote by* $\mathfrak{P}$ (*resp.* $\mathfrak{H}$) *the class of spaces* (X, $\mathcal{H}$) *for which there exists at least a positive potential* (*resp. a positive harmonic function*) *on* X. The type $\mathfrak{P}$ (resp. $\mathfrak{H}$) of X is not altered by the multiplication of all the functions of $\mathcal{H}$ by a positive continuous function. An open set $U \subset X$ is said to be of *type* $\mathfrak{P}$ (*resp.* $\mathfrak{H}$) *if any component of* U *belongs to* $\mathfrak{P}$ (*resp.* $\mathfrak{H}$). The spaces of type $\mathfrak{P} \cup \mathfrak{H}$ (resp. $\mathfrak{P}$) are exactly those on which there exists a positive superharmonic (resp. positive superharmonic non-harmonic) function. On a space of type $\mathfrak{H} - \mathfrak{P}$ any two positive superharmonic functions are

proportional. If $X \in \mathfrak{P}$ (resp. $\mathfrak{H}$) and U is a domain in X, then $U \in \mathfrak{P}$ (resp. $\mathfrak{H}$). This is trivial for $\mathfrak{H}$ and for $\mathfrak{P}$ it results from the fact that there exits a positive superharmonic function on X which is not harmonic on U. If $U \subset X$, $U \in \mathfrak{P}$ and $X - U$ is polar, then X also belongs to $\mathfrak{P}$ since any potential can be extended to a superharmonic function on X, ([2] page 125) which is obviously a potential. This result does not hold if we take $\mathfrak{H}$ instead of $\mathfrak{P}$.

Let X_1 denote the real axis and X_2 the unit circumference in the complex plane and let $\mathcal{H}_1$ (resp. $\mathcal{H}_2$) be the sheaf of solutions of the equation $u'' + \alpha u = 0$ on X_1 (resp. X_2), where α is a real number. If α is positive, no positive superharmonic function exists on the spaces $(X_1, \mathcal{H}_1)$, $(X_2, \mathcal{H}_2)$. If α is positive and irrational, the only harmonic function on X_2 is identically zero. For $\alpha = 0$ we obtain examples of spaces of the type $\mathfrak{H} - \mathfrak{P}$. For $\alpha < 0$, X_1 belongs to $\mathfrak{P} \cap \mathfrak{H}$ and X_2 belongs to $\mathfrak{P} - \mathfrak{H}$. We do not know if there exists non-compact spaces of the type $\mathfrak{P} - \mathfrak{H}$.

LEMMA 1 [5]. — *If $\mathcal{H}$ satisfies axiom A_1 the axiom A_2 is equivalent to the following assertion. Let U be a domain, $V \subset U$ an open set, $K \subset V$ a compact set and $x \in U$. There exists a positive number $\alpha = \alpha\ (U, V, K, x)$ such that for any non-negative superharmonic function s on U harmonic on V*

$$\sup_{y \in K} s(y) \leqslant \alpha s(x).$$

Suppose A_2 fulfilled. If α does not exist there exists for any natural number n a non-negative superharmonic function s_n on U, harmonic on V and such that

$$\sup_{y \in K} s_n(y) > n, \qquad s_n(x) < \frac{1}{n^2}.$$

This leads to a contradiction since the function

$$\sum_{n=1}^{\infty} s_n$$

is superharmonic on U and infinite on a component of V.

[5] This lemma was inspired by a similar result of R.-M. HERVÉ ([3] n° 2; propriété 7).

Suppose now the existence of an α asserted in the lemma, and let $\{u_n\}$ be an increasing sequence of harmonic functions on U. If there exists a point $x \in U$ for which $\{u_n(x)\}$ is convergent, then $\{u_n(x) - u_{n-1}(x)\}$ converges to zero. Hence

$$\{u_n - u_{n-1}\}$$

converges to zero uniformly on any compact set of U. It follows immediately that $\lim_{n \to \infty} u_n$ is harmonic.

The axiom $\mathbf{A}_2$ can be strengthened by requiring

$$\lim_{K \to x} \alpha(U,\ U,\ K,\ x) = 1.$$

This assertion was called axiom 3′ ([2], page 147). From this axiom it follows that the positive harmonic functions on U equal to 1 at a point of U form an equicontinuous set of functions.

LEMMA 2. — *Let $x \in X$, I be an increasingly directed ordered set and for any $\iota \in I$ let U_ι be a domain on X containing x, V_ι be an open subset of U_ι and s_ι a non-negative superharmonic function on U_ι harmonic on V_ι equal to 1 at x. We suppose $U_\iota \subset U_\varkappa$, $V_\iota \subset V_\varkappa$ for any $\iota \leqslant \varkappa$. Let $\mathfrak{U}$ be an ultrafilter on I finer than the filter of sections of I* (6). *If $\mathcal{H}$ satisfies the axiom 3′ then s_ι converges uniformely along $\mathfrak{U}$ on any compact subset of $\bigcup_{\iota \in I} V_\iota$ to a harmonic function.*

Let K be a compact set in $\bigcup_{\iota \in I} V_\iota$ and $\varkappa \in I$ such that $K \subset V_\varkappa$. Since $\{s_\iota | \iota \geqslant \varkappa\}$ is an equicontinuous family of functions on K, s_ι converges uniformly along $\mathfrak{U}$ on K. Its limit is therefore harmonic on $\bigcup_{\iota \in I} V_\iota$.

THEOREM 4. — *Let X be non-compact. If $\mathcal{H}$ satisfies the axiom 3′ and any relatively compact domain of X belongs to $\mathfrak{H}$ then X belongs to $\mathfrak{H}$.*

Let $x \in X$, I be the set of relatively compact domains containing x ordered by the inclusion relation and for any $\iota \in I$ denote $U_\iota = V_\iota = \iota$ and let s_ι be a positive harmonic

(6) This is filter generated by the family of sets $\{\iota \in I | \iota \geqslant \varkappa\}_{\varkappa \in I}$.

function on U_ι equal to 1 at x. By means of lemma 2 one can construct a positive harmonic function on X.

Corollary 4 [3] (7). — *If $\mathcal{H}$ satisfies the axiom 3′ and X belongs to $\mathfrak{P}$ and is non-compact, then X belongs to $\mathfrak{H}$.*

Theorem 5 [3] (7). — *If $\mathcal{H}$ satisfies the axiom 3′ and X belongs to $\mathfrak{P}$ then there exists for any point $x \in X$ a positive potential on X harmonic on $X - \{x\}$.*

Let p be a positive potential on X and I be the set of compact neighbourhoods of x ordered by the inverse inclusion relation. For any $\iota \in I$ we denote

$$U_\iota = X, \qquad V_\iota = X - \iota,$$
$$s_\iota = \frac{\hat{R}_p^\iota}{\hat{R}_p^\iota(x_0)},$$

where x_0 is a fixed point different from x. Let $\mathfrak{U}$ be an ultrafilter on I finer than the filter of sections of I and for any $y \in X$

$$s(y) = \lim_{\iota, \mathfrak{U}} s_\iota(y).$$

By lemma 2 s is harmonic on $X - \{x\}$.

Let U be a regular neighbourhood of x and $y \in U$. Since by lemma 2 s_ι converges uniformly along $\mathfrak{U}$ to u on ∂U we have

$$s(y) = \lim_{\iota, \mathfrak{U}} s_\iota(y) \geqslant \lim_{\iota, \mathfrak{U}} \int s_\iota \, d\omega_y^U = \int s \, d\omega_y^U.$$

The regularised function $\hat{s}$ of s is therefore superharmonic. From the above uniform convergence we deduce the existence of a positive number α and a $\varkappa \in I$ such that

$$s_\iota \leqslant \alpha p$$

on ∂U for any $\iota \geqslant \varkappa$. It follows ([3] Lemma 3.1)

$$s_\iota \leqslant \alpha p, \qquad s \leqslant \alpha p$$

on $X - U$. Hence $\hat{s}$ is a potential. It cannot be harmonic on a neighbourhood of x since then it would be harmonic on X and therefore zero.

(7) Théorème 16-1.

THEOREM 6. — *On a compact space of the type $\mathfrak{P}$ any superharmonic function is a potential. Particularly any superharmonic function is non-negative.*

If s is a superharmonic function then $-\min(s, o)$ is a subharmonic function. Since the space is compact and of type $\mathfrak{P}$ it is dominated by a potential. Hence it vanishes and s is non-negative. The greatest harmonic minorant of s vanishes being dominated by a potential. s is therefore a potential.

REMARK. — *A space of the type $\mathfrak{P} \cap \mathfrak{H}$ is non-compact.*

THEOREM 7. — *Let $X \in \mathfrak{P} \cup \mathfrak{H}$ and U be a domain on X. If $X - U$ is non-polar then $U \in \mathfrak{P} \cap \mathfrak{H}$.*

Let s be a positive superharmonic function on X. Suppose its restriction on U is a potential. There exists then a positive superharmonic function s' on U such that

$$\lim_{U \ni x \to \partial U} s'(x) = \infty$$

([1] Lemma 1). If we extend s' to a function on X equal to $+\infty$ on $X - U$ we obtain a superharmonic function. This is a contradiction since $X - U$ is non -polar. Hence the restriction of s on U is not a potential and $U \in \mathfrak{H}$.

∂U is non-polar. This is obvious if $\partial U = X - U$. If

$$\partial U \neq X - U,$$

∂U is non-polar since $X - \partial U$ is non-connected. There exists therefore a point $x \in \partial U$ such that the intersection of any neighbourhood V of x with ∂U is non-polar in V. Let V be a regular domain which contains x and K be a compact non-polar set, $K \subset V \cap \partial U$. The reduced function $(R_s^K)_V$ of s relative to K, where the operation is made on V, does not vanish, it converges to zero at the boundary of V and is harmonic on $V - K$. The function s' on U equal to s on $U - V$ and equal to $s - (R_s^K)_V$ on $V \cap U$ is superharmonic and non-proportional to s. Hence $U \in \mathfrak{P}$.

COROLLARY 5. — *If $X \in \mathfrak{P} \cup \mathfrak{H}$ and U is an open non-connected set, then U is of the type $\mathfrak{P} \cap \mathfrak{H}$.*

Let V be a component of U. Since $X - V$ has interior points it is non-polar. V is therefore of the type $\mathfrak{P} \cap \mathfrak{H}$.

4. LEMMA 3. — *The sum of a sequence of potentials convergent at a point is a potential.*

Let $\{p_n\}$ be a sequence of potentials such that

$$s = \sum_{n=1}^{\infty} p_n$$

be finite at a point. Let u be a harmonic minorant of s. We shall prove inductively that

$$u \leqslant \sum_{n=m}^{\infty} p_n.$$

Suppose

$$u \leqslant \sum_{n=m}^{\infty} p_n.$$

Then $u - \sum_{n=m+1}^{\infty} p_n$ is a subharmonic minorant of p_n and therefore non-positive.

A sequence $\{U_n\}$ of relatively compact domains on X is called a pseuso-exhaustion of X if

$$\overline{U}_n \subset U_{n+1}$$

for any n and

$$X - \bigcup_{n=1}^{\infty} U_n$$

is polar.

THEOREM 8. — *Any space of the type $\mathfrak{P} \cup \mathfrak{H}$ possesses a pseudo-exhaustion.*

Let $K \subset X$ be a compact non-polar set such that $X - K$ contains only a finite number of components, let p be a positive potential on $X - K$ and $\mathcal{G}$ be the set of functions $(\hat{R}_p^{X-U})_{X-K}$, where U is a relatively compact domain which contains K. The greatest lower bound of $\mathcal{G}$, being a non-negative harmonic minorant of p, is equal to zero. There exists therefore a sequence $\{U_n\}$ of relatively compact domains containing K, such that for any n $\overline{U}_n \subset U_{n+1}$ and

$$\sum_{n=1}^{\infty} (\hat{R}_p^{X-U_n})_{X-K}$$

is a superharmonic function on $X - K$, infinite on $X - \bigcup_{n=1}^{\infty} U_n$.

THEOREM 9. — *If* $X \in \mathfrak{P}$ *and* $\{U_n\}$ *is a pseudo-exaustion of* X *there exists a continuous potential on* X, *which is infinite exactly on* $X - \bigcup_{n=1}^{\infty} U_n$.

Let p be a continuous finite potential on X and, for any n, let f_n denote a continuous non-negative function on X equal to 0 on U_n, equal at most to p on $U_{n+1} - U_n$ and equal to p on $X - U_{n+1}$. The function

$$p_n = R_{f_n}^X$$

is a continuous finite potential, harmonic on U_n (Theorem 3) and $p_n \geqslant p_{n+1}$. Let u denote the limit of the sequence $\{p_n\}$. The function u is locally bounded and harmonic on $\bigcup_{n=1}^{\infty} U_n$. Since $X - \bigcup_{n=1}^{\infty} U_n$ is polar there exists a harmonic function on X equal to u on $\bigcup_{n=1}^{\infty} U_n$. Being a harmonic minorant of p it vanishes. Hence u is equal to zero on $\bigcup_{n=1}^{\infty} U_n$. We may therefore assume that the function

$$p_0 = \sum_{n=1}^{\infty} p_n$$

is finite at a certain point. Since p_n is harmonic on U_n, p_0 is continuous and finite on $\bigcup_{n=1}^{\infty} U_n$. According to lemma 3 p is a potential and it is equal to infinite on $X - \bigcup_{n=1}^{\infty} U_n$.

COROLLARY 6. — *Let* f *be a finite non-negative upper semi-continuous function on* $X \in \mathfrak{P}$. R_f^X *is the greatest lower bound of the set of continuous hyperharmonic majorants of* f. *If* $\hat{R}_f^X$ *is a potential,* R_f^X *is the greatest lower bound of the set of continuous potentials which dominate* f.

Let $x \in X$ and s be a superharmonic majorant of f. Let further $\{U_n\}$ be a pseudo-exhaustion of X, $U_1 \ni x$, $U = \bigcup_{n=1}^{\infty} U_n$ and let p be a continuous potential on X finite at x and equal to ∞ on $X - U$. Since U is a normal space there exists a continuous finite function g on U, $f \leqslant g \leqslant s$. Let g_0 be the lower semi-continuous function on X equal to g on U and equal to 0 on X — U. The function $R_{g_0}^X$ is superharmonic and continuous on U according to theorem 3. Hence the function $s_0 = R_{g_0}^X + \varepsilon p$ is a continuous superharmonic majorant of f for any $\varepsilon > 0$ and we have

$$s_0(x) \leqslant s(x) + \varepsilon p(x).$$

In order to prove the last assertion it is sufficient to show that there exists a potential which dominates f. The function

$$u = \lim_{n \to \infty} \hat{R}_f^{X-U_n}$$

is a harmonic function on U. Since $\hat{R}_f^X$ is locally bounded u is locally bounded. There exists therefore a harmonic function on X equal to u on U. This function is a minorant of $\hat{R}_f^X$. Hence u vanishes on U. We may therefore assume that

$$\sum_{n=1}^{\infty} R_f^{X-U_n}(x)$$

is convergent. Let us denote

$$U_0 = \emptyset, \qquad G_n = U_{n+1} - \overline{U}_{n-1}.$$

We have

$$\sum_{n=1}^{\infty} \hat{R}_f^{G_n}(x) < \infty.$$

The function

$$p + \sum_{n=1}^{\infty} \hat{R}_f^{G_n}$$

is a potential which dominates f.

5. LEMMA 4. — *Let* X *be a locally compact locally connected space,* F *a closed nowhere disconnecting set in* X, *and* $\aleph$ *a cardinal number. If* X — F *possesses a basis whose cardinal is at most equal to* $\aleph$ *and if there exists a set of continuous functions on* X *whose cardinal is at most equal to* $\aleph$ *and which separates the points of* F, *then* X *possesses a basis whose cardinal is at most equal to* $\aleph$.

Let U be an open set on X and $\{U_\iota\}_{\iota \in I}$ the family of components of U. Since F is nowhere disconnecting, $\{U_\iota - F\}_{\iota \in I}$

are exactly the family of components of $U - F$. Since $X - F$ possesses a basis whose cardinal is most equal to $\aleph$, the cardinal of I is at most equal to $\aleph$.

There exists a set $\mathcal{F}$ whose cardinal is at most equal to $\aleph$ of continuous functions on X which separates the points of X. For any $f \in \mathcal{F}$ and any two rational numbers α, β we denote

$$U(f;\ \alpha,\ \beta) = \{x \in X | \alpha < f(x) < \beta\}.$$

Let $\mathfrak{B}'$ denote the family

$$\mathfrak{B}' = \{U(f;\ \alpha,\ \beta) | f \in \mathcal{F},\ \alpha,\ \beta \text{ rational numbers}\}.$$

The cardinal number of $\mathfrak{B}'$ is at most $\aleph$. Let us denote by $\mathfrak{B}$ the system of components of the sets of the form $\bigcap_{i=1}^{n} U_i$, $U_i \in \mathfrak{B}'$. According to the above remark the cardinal number of $\mathfrak{B}$ is at most equal to $\aleph$.

We want to prove that $\mathfrak{B}$ is a basis of X. Let x be a point of X and U be a relatively compact neighbourhood of x. For any $y \in \partial U$ let f_y be a function of $\mathcal{F}$ such that

$$f_y(x) \neq f_y(y).$$

There exist two rational numbers α_y, β_y such that

$$x \in U(f_y;\ \alpha_y,\ \beta_y),\ y \notin \overline{U(f_y;\ \alpha_y,\ \beta_y)}.$$

Since ∂U is compact we may find a finite number of points $\{y_i | i = 1, \ldots n\}$ on ∂U such that

$$x \in \bigcap_{i=1}^{n} U(f_{y_i};\ \alpha_{y_i},\ \beta_{y_i}), \qquad \left(\bigcap_{i=1}^{n} U(f_{y_i};\ \alpha_{y_i},\ \beta_{y_i})\right) \cap \partial U = \emptyset.$$

Let V denote the component of $\bigcap_{i=1}^{n} U(f_{y_i}; \alpha_{y_i}, \beta_{y_i})$ which contains x. Since $V \cap \partial U = \emptyset$ we have $V \subset U$. $\mathfrak{B}$ is hence a basis, since $V \in \mathfrak{B}$.

Theorem 10. — *Let $X \in \mathfrak{P} \cup \mathfrak{H}$, F be a closed polar set on X and let $\aleph$ be a cardinal number. If for any point of $X - F$ there exists a neighbourhood which possesses a basis whose cardinal is at most equal to $\aleph$, then X possesses a basis whose cardinal is at most equal to $\aleph$.*

Since any space of the type $\mathfrak{P} \cup \mathfrak{H}$ can be covered with a finite system of domains of the type $\mathfrak{P}$, it is sufficient to prove the theorem for the case $X \in \mathfrak{P}$.

Let $\{U_n\}$ be a pseudo-exhaustion on $X - F$. Since F is polar $\{U_n\}$ is a pseudo-exhaustion of X. We denote $U = \bigcup_{n=1}^{\infty} U_n$ and assume $F = X - \bigcup_{n=1}^{\infty} U$. Let p be a continuous potential on X such that p is infinite exactly on F. Since any U_n possesses a basis whose cardinal is at most equal to $\aleph$, U possesses a basis $\mathfrak{B}$ whose cardinal is at most equal to $\aleph$. For any two relatively compact sets $V, W \in \mathfrak{B}$, $\overline{V} \subset W$, let $f_{V,W}$ denote a continuous function on X, $0 \leqslant f_{V,W} \leqslant 1$, equal to 1 on V and equal to 0 on $X - W$. We denote by $\mathcal{F}$ the set of functions of the form

$$\max_{1 \leqslant i \leqslant n} f_{V_i, W_i}.$$

The cardinal number of $\mathcal{F}$ is at most equal to $\aleph$. We denote further for any $f \in \mathcal{F}$

$$s_f = R^X_{fp}$$

and

$$\mathcal{S} = \{s_f | f \in \mathcal{F}\}.$$

The cardinal number of $\mathcal{S}$ is at most equal to $\aleph$. Hence, according to the preceding lemma, it remains only to prove that $\mathcal{S}$ separates the points of F.

Let $x, y \in F$, $x \neq y$ and V be a neighbourhood of x, $y \notin \overline{V}$. We denote by $\mathcal{S}_V$ the family of functions $s_f \in \mathcal{S}$ for which the carrier of f is contained in V. $\mathcal{S}_V$ is an upper directed family of superharmonic functions. Its least upper bound s is therefore superharmonic. We have

$$s \leqslant R^V_p, \qquad s(y) \leqslant R^V_p(y) < \infty$$

and $s = p$ on $V - F$. Since F is polar we have $s = p$ on V and therefore $s(x) = \infty$. There exists therefore an $s_f \in \mathcal{S}_V$ such that

$$s_f(x) > s_f(y).$$

Corollary 7. — *If* $X \in \mathfrak{P} \cup \mathfrak{H}$ *and any point of* X *possesses a neighbourhood with a countable basis,* X *possesses a countable basis. Particularly if* X *is a manifold, and* $X \in \mathfrak{P} \cup \mathfrak{H}$, X *possesses a countable basis.*

There exist for any cardinal number $\aleph$ examples of spaces on which the constants are harmonic and which possess points for which the cardinal number of any fundamental system of neighbourhoods is at least equal to $\aleph$. Let M be a set whose cardinal is $\aleph$ and Γ the set of points of the complex plane $\{e^{i\theta} | \theta \text{ real number}\}$. For any finite set $I \subset M$ we denote by X_I the topological space obtained from the topological space $\Gamma \times I$, where I is considered with the discrete topology, identifying the points $(1, \iota)$ with $\iota \in I$. We denote by a_I this point of X_I. The harmonic functions on $X_I - \{a_I\}$ will be the functions which are linear in θ_ι. A continuous function u defined on a neighbourhood of a_I is harmonic if it is harmonic outside $\{a_I\}$ and for sufficiently small $\varepsilon > 0$

$$u(a_I) = \frac{1}{2n} \sum_{\iota \in I} [u(e^{i\varepsilon}, \iota) + u(e^{-i\varepsilon}, \iota)],$$

where n is the cardinal number of I. It is easy to verify that the harmonic functions satisfy the axions $\mathbf{A}_1$, $\mathbf{A}_2$.

For any $I \subset J$ we denote by φ_{IJ} the map $X_J \to X_I$ defined by

$$\varphi_{I,J}(z, \iota) = \begin{cases} (z, \iota) & \text{if } \iota \in I \\ a_I & \text{if } \iota \notin I \end{cases}; \qquad \varphi_{I,J}(a_J) = a_I.$$

The system $\{X_I, \varphi_{I,J}\}$ is a projective system of topological spaces. Let $\{X, \varphi_I\}$, be its projective limit and a the point of X corresponding to the points a_I. X is compact and the cardinal number of any fundamental system of neighbourhoods of a is at least equal to $\aleph$. The harmonic functions on X will be the functions of the form $u \circ \varphi_I$, where u is a harmonic function on X_I. It can be verified that the sheaf of harmonic functions on X satisfies the required axioms (and even the axiom 3′).

THEOREM 11. — *The set of non-relatively compact components of an open set on* $X \in \mathfrak{P} \cup \mathfrak{H}$ *is at most countable.*

Let $\{G_\iota\}_{\iota \in I}$ be a family of pairwise disjoint domains on X and U be a relatively compact domain on X. We denote by I_U the set of $\iota \in I$ for which

$$G_\iota \cap U \neq \emptyset, \qquad G_\iota - \overline{U} \neq \emptyset.$$

For any $\iota \in I_U$ we denote by f_ι the function on ∂U equal to 1 on $G_\iota \cap \partial U$ and equal to 0 on $\partial U - G_\iota$. This function is resolutive with respect to U [1] and let $H^U_{f_\iota}$ denote its solution. This function doesn't vanish since in the contrary case there would exist a non-negative superharmonic function s on U converging to infinite at any point of $G_\iota \cap \partial U$. The function on $U \cup G_\iota$ equal to s on U and equal to infinite on $G_\iota - U$ would be a superharmonic function infinite on an open set. This is a contradiction. From

$$\sum_{\iota \in I_U} H^U_{f_\iota} \leqslant H^U_1$$

is follows that I_U is at most countable.

Let G be an open set $\{G_\iota\}_{\iota \in I}$ be the family of its non-relatively compact components and $\{U_n\}$ be a pseudo-exhaustion of X. From the above proof its follows that $I = \bigcup_{n=1}^{\infty} I_{U_n}$ is at most countable.

BIBLIOGRAPHY

[1] N. Boboc, C. Constantinescu and A. Cornea. On the Dirichlet problem in the axiomatic theory of harmonic functions, *Nagoya Math. Journ.*, (1964), to appear.

[2] M. Brelot. *Lectures on potential theory*, Part. IV. Tata Institute of Fundamental Research, Bombay, (1960).

[3] R.-M. Hervé. Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel, *Ann. Inst. Fourier*, 12 (1962), 415-571.