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Uniform rectifiability and $\varepsilon$-approximability of harmonic functions in $L^p$

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UNIFORM RECTIFIABILITY AND
\(\varepsilon\)-APPROXIMABILITY OF HARMONIC FUNCTIONS
IN \(L^p\)

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Abstract. — Suppose that \(E \subset \mathbb{R}^{n+1}\) is a uniformly rectifiable set of codimension 1. We show that every harmonic function is \(\varepsilon\)-approximable in \(L^p(\Omega)\) for every \(p \in (1, \infty)\), where \(\Omega := \mathbb{R}^{n+1} \setminus E\). Together with results of many authors this shows that pointwise, \(L^\infty\) and \(L^p\) type \(\varepsilon\)-approximability properties of harmonic functions are all equivalent and they characterize uniform rectifiability for codimension 1 Ahlfors–David regular sets. Our results and techniques are generalizations of recent works of T. Hytönen and A. Rosén and the first author, J. M. Martell and S. Mayboroda.

Résumé. — Soit \(E\) un ensemble uniformément rectifiable de codimension 1 dans un espace euclidien de dimension \(n + 1\) et soit \(\Omega\) son complémentaire. Nous montrons que toute fonction harmonique est \(\varepsilon\)-approchable dans \(L^p(\Omega)\) pour tout \(p\) fini strictement plus grand que 1. Cela montre, compte tenu de résultats précédents par différents auteurs, que ponctuellement, les propriétés d’\(\varepsilon\)-approximation de type \(L^\infty\) et \(L^p\) de fonctions harmoniques sont équivalentes et elles caractérisent la rectifiabilité uniforme des ensembles réguliers au sens d’Ahlfors–David de codimension 1. Nos résultats et techniques sont des généralisations de travaux récents de T. Hytönen, A. Rosén et du premier auteur, J. M. Martell et S. Mayboroda.

1. Introduction

In many branches of analysis, Carleson measure estimates are powerful tools that are deeply connected to e.g. elliptic partial differential equations and geometric measure theory. These estimates are particularly useful for

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measures of the type $|\nabla u(Y)|dY$ (see e.g. [11, 12]) but the problem is that even strong analytic properties of the function $u$ are not enough to guarantee that the distributional gradient defines a measure of this type. The idea behind $\varepsilon$-approximability is that although a function may fail this Carleson measure property, it can sometimes be approximated arbitrarily well in the $L^\infty$ sense (typically, if it is the solution to an elliptic partial differential equation) by a function $\varphi$ such that $|\nabla \varphi(Y)|dY$ is a Carleson measure. Starting from the work of N. Th. Varopoulos [25] and J. Garnett [12], this approximation technique has had an important role in the development of the theory of elliptic partial differential equations. It has been used to e.g. explore the absolute continuity properties of elliptic measures [15, 22] and, very recently, give a new characterization of uniform rectifiability [13, 18].

In this article, we extend the recent results of the first author, J. M. Martell and S. Mayboroda [18] and show that if $E \subset \mathbb{R}^{n+1}$ is a uniformly rectifiable (UR) set of codimension $1$, then every harmonic function is $\varepsilon$-approximable in $L^p(\Omega)$ for every $\varepsilon \in (0,1)$ and every $p \in (1,\infty)$, where $\Omega := \mathbb{R}^{n+1} \setminus E$. The $L^p$ version of $\varepsilon$-approximability was recently introduced by T. Hytönen and A. Rosén [20] who showed that any weak solution to certain elliptic partial differential equations in $\mathbb{R}^{n+1}_+$ is $\varepsilon$-approximable in $L^p$ for every $\varepsilon \in (0,1)$ and every $p \in (1,\infty)$.

Let us be more precise and recall the definition of $\varepsilon$-approximability:

**Definition 1.1.** Suppose that $E \subset \mathbb{R}^{n+1}$ is an $n$-dimensional ADR set (see Definition 1.7) and let $\Omega := \mathbb{R}^{n+1} \setminus E$ and $\varepsilon \in (0,1)$. We say that a function $u$ such that $\|u\|_{L^\infty(\Omega)} \leq 1$ is $\varepsilon$-approximable if there exists a constant $C_\varepsilon$ and a function $\varphi = \varphi^\varepsilon \in BV_{\text{loc}}(\Omega)$ satisfying

$$\|u - \varphi\|_{L^\infty(\Omega)} < \varepsilon \quad \text{and} \quad \sup_{x \in E, r > 0} \frac{1}{r^n} \int_{B(x,r) \cap \Omega} |\nabla \varphi(Y)|dY \leq C_\varepsilon.$$

Here $\int_{B(x,r) \cap \Omega} |\nabla \varphi|dY$ stands for the total variation of $\varphi$ over $B(x,r) \cap \Omega$ (see Section 1.5).

Sometimes $W^{1,1}$ [15] or $C^\infty$ [12, 22] is used in the definition instead of $BV_{\text{loc}}$. The first results about $\varepsilon$-approximability showed that every bounded harmonic function $u$, normalized so that $\|u\|_{L^\infty} \leq 1$, enjoys this approximation property for every $\varepsilon \in (0,1)$ in the upper half-space $\mathbb{R}^{n+1}_+$ [12, 25] and in Lipschitz domains [6]. This is a highly non-trivial property since there exist bounded harmonic functions $u$ such that $|\nabla u(Y)|dY$ is not a Carleson measure [12]. The $L^p$ version of the property was defined only recently in [20]:

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Definition 1.2. — Suppose that \( E \subset \mathbb{R}^{n+1} \) is an \( n \)-dimensional ADR set and let \( \Omega := \mathbb{R}^{n+1} \setminus E \), \( \varepsilon \in (0, 1) \) and \( p \in (1, \infty) \). We say that a function \( u \) is \( \varepsilon \)-approximable in \( L^p \) if there exists a function \( \varphi = \varphi^\varepsilon \in BV_{\text{loc}}(\Omega) \) and constants \( C_p \) and \( D_{p, \varepsilon} \) such that

\[
\begin{cases}
\| N^*(u - \varphi) \|_{L^p(E)} \lesssim \varepsilon C_p \| N^* u \|_{L^p(E)} \\
\| C(\nabla \varphi) \|_{L^p(E)} \lesssim D_{p, \varepsilon} \| N^* u \|_{L^p(E)},
\end{cases}
\]

where \( N^* \) is the non-tangential maximal operator (see Definition 1.24) and

\[
C(\nabla \varphi)(x) := \sup_{r > 0} \frac{1}{r^n} \int_{B(x,r) \cap \Omega} |\nabla \varphi| \, dY.
\]

Here, as above, we have written \( \int_{B(x,r) \cap \Omega} |\nabla \varphi| \, dY \) to denote the total variation of \( \varphi \) over \( B(x,r) \cap \Omega \); we ask the reader to forgive this abuse of notation. See Section 1.5 for details.

In [20], the authors showed that if \( \Omega = \mathbb{R}^{n+1} \) and \( A \in L^\infty(\mathbb{R}^n; L(\mathbb{R}^{n+1})) \) satisfies \( \langle A(x)v, v \rangle \geq \lambda_A |v|^2 \) for almost every \( x \in \mathbb{R}^n \) and all \( v \in \mathbb{R}^{n+1} \setminus \{0\} \), then any weak solution \( u \) to the \( t \)-independent real scalar (but possibly non-symmetric) divergence form elliptic equation \( \text{div}_{x,t} A(x) \nabla_{x,t} u(x,t) = 0 \) is \( \varepsilon \)-approximable in \( L^p \) for any \( \varepsilon \in (0, 1) \) and any \( p \in (1, \infty) \).

If we move from \( \mathbb{R}^{n+1} \) to the UR context (see Definition 1.8) with no assumptions on connectivity, things will not only get more complicated but we also lose many powerful tools. For example, constructing objects like Whitney regions and Carleson boxes becomes considerably more difficult and the harmonic measure no longer necessarily belongs to the class weak-\( A_\infty \) with respect to the surface measure [3]. Despite these difficulties, there exists a rich theory of harmonic analysis and many results on elliptic partial differential equations on sets with UR boundaries. Uniform rectifiability can be characterized in numerous different ways and many of these characterizations are valid in all codimensions (see the seminal work of G. David and S. Semmes [7, 8]). For example, UR sets are precisely those ADR sets for which certain types of singular integral operators are bounded from \( L^2 \) to \( L^2 \). Recently, the first author, Martell and Mayboroda showed that if \( E \) is a UR set of codimension 1, then every bounded harmonic function in \( \mathbb{R}^{n+1} \setminus E \) is \( \varepsilon \)-approximable for every \( \varepsilon \in (0, 1) \) [18]. After this, it was shown by Garnett, Mourgoglou and Tolsa that \( \varepsilon \)-approximability of bounded harmonic functions implies uniform rectifiability for \( n \)-ADR sets [13]. This characterization result was then generalized for a class of elliptic operators by Azzam, Garnett, Mourgoglou and Tolsa [1].

Our main result is the following generalization of the Hytönen–Rosén approximation theorem [20, Theorem 1.3]:

\[
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\]
**Theorem 1.3.** — Let $E \subset \mathbb{R}^{n+1}$ be a UR set of codimension 1 and denote $\Omega := \mathbb{R}^{n+1} \setminus E$. Then every harmonic function in $\Omega$ is $\varepsilon$-approximable in $L^p$ for every $\varepsilon \in (0, 1)$ and every $p \in (1, \infty)$ with $C_p = \|M\|_{L^p \to L^p}$ and $D_p = C_p \|M\|_{L^p \to L^p}/\varepsilon^2$, where $M$ is the Hardy–Littlewood maximal operator and $M_D$ is its dyadic version (see Section 1.1).

In fact, the key ideas of Hytönen and Rosén allow us to construct $p$-independent approximating functions. To be more precise, let us consider the following pointwise approximating property:

**Definition 1.4.** — Suppose that $E \subset \mathbb{R}^{n+1}$ is an $n$-dimensional ADR set and let $\Omega := \mathbb{R}^{n+1} \setminus E$ and $\varepsilon \in (0, 1)$. We say that a function $u$ is pointwise $\varepsilon$-approximable if there exists a function $\varphi = \varphi^\varepsilon \in BV_{\text{loc}}(\Omega)$ and a constant $D_\varepsilon$ such that

$$\begin{align*}
N_{\star}(u - \varphi)(x) &\lesssim \varepsilon M_D(N_{\star}u)(x) \\
C_D(\nabla \varphi)(x) &\lesssim D_\varepsilon M(M_D(N_{\star}u))(x)
\end{align*}$$

for almost any $x \in E$, where $C_D$ is a dyadic version of $D$ (see Section 1.6).

Since $C(\nabla \varphi)$ and $C_D(\nabla \varphi)$ are $L^p$-equivalent by Lemma 1.23, Theorem 1.3 is an immediate corollary of the following result and the $L^p$-boundedness of the Hardy–Littlewood maximal operator and its dyadic versions:

**Theorem 1.5.** — Suppose that $E \subset \mathbb{R}^{n+1}$ is an $n$-dimensional UR set and let $\Omega := \mathbb{R}^{n+1} \setminus E$ and $\varepsilon \in (0, 1)$. Then every harmonic function in $\Omega$ is pointwise $\varepsilon$-approximable.

Although the $L^p$ version of $\varepsilon$-approximability seems like the weakest one of all the properties, it is equivalent with the other properties in the codimension 1 ADR context provided that $p$ is large enough. This follows from the recent results of S. Bortz and the second author [4]. Hence, combining our results with the results in [4, 13, 18] gives us the following characterization theorem:

**Theorem 1.6.** — Suppose that $E \subset \mathbb{R}^{n+1}$ is an $n$-dimensional ADR set and let $\Omega := \mathbb{R}^{n+1} \setminus E$. The following conditions are equivalent:

1. $E$ is UR.
2. Bounded harmonic functions in $\Omega$ are $\varepsilon$-approximable for every $\varepsilon \in (0, 1)$.
3. Harmonic functions in $\Omega$ are pointwise $\varepsilon$-approximable for every $\varepsilon \in (0, 1)$.
4. Harmonic functions in $\Omega$ are $\varepsilon$-approximable in $L^p$ for some $p > n/(n - 1)$ and every $\varepsilon \in (0, 1)$.
(5) Harmonic functions in $\Omega$ are $\varepsilon$-approximable in $L^p$ for all $p \in (1, \infty)$ and every $\varepsilon \in (0, 1)$.

To prove the implication $(1) \Rightarrow (3)$, we combine some techniques of the proof of the Hytönen–Rosén theorem with the tools and techniques from [18]. Some of the techniques can be used in a straightforward way but with the rest of them we have take care of many technicalities and be careful with the details.

We start by recalling the basic definitions and some results needed in our statements and proofs. For the most part, our notation and terminology agrees with [18].

1.1. Notation

We use the following notation.

1. The set $E \subset \mathbb{R}^{n+1}$ will always be a closed set of Hausdorff dimension $n$. We denote $\Omega := \mathbb{R}^{n+1} \setminus E$.

2. The letters $c$ and $C$ denote constants that depend only on the dimension, the ADR constant (see Definition 1.7), the UR constants (see Definition 1.8) and other similar parameters. We call them structural constants. The values of $c$ and $C$ may change from one occurrence to another. We do not track how our bounds depend on these constants and usually just write $\lambda_1 \lesssim \lambda_2$ if $\lambda_1 \leq c \lambda_2$ for a structural constant $c$ and $\lambda_1 \approx \lambda_2$ if $\lambda_1 \approx c \lambda_2 \approx \lambda_1$.

3. We use capital letters $X, Y, Z$, and so on to denote points in $\Omega$ and lowercase letters $x, y, z$, and so on to denote points in $E$.

4. The $(n + 1)$-dimensional Euclidean open ball of radius $r$ will be denoted $B(x, r)$ or $B(X, r)$ depending on whether the center point lies on $E$ or $\Omega$. We denote the surface ball of radius $r$ centered at $x$ by $\Delta(x, r) := B(x, r) \cap E$.

5. Given a Euclidean ball $B := B(X, r)$ or a surface ball $\Delta := \Delta(x, r)$ and constant $\kappa > 0$, we denote $\kappa B := B(X, \kappa r)$ and $\kappa \Delta := \Delta(x, \kappa r)$.

6. For every $X \in \Omega$ we set $\delta(X) := \text{dist}(X, E)$.

7. We let $\mathcal{H}^n$ be the $n$-dimensional Hausdorff measure and denote $\sigma := \mathcal{H}^n|_E$. The $(n+1)$-dimensional Lebesgue measure of a measurable set $A \subset \Omega$ will be denoted by $|A|$.

8. For a set $A \subset \mathbb{R}^{n+1}$, we let $1_A$ be the indicator function of $A$: $1_A(x) = 0$ if $x \notin A$ and $1_A(x) = 1$ if $x \in A$.

9. The interior of a set $A$ will be denoted by $\text{int}(A)$. The closure of a set $A$ will be denoted by $\bar{A}$. 

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For $\mu$-measurable sets $A$ with positive and finite measure we set
\[ \int_A f \, d\mu := \frac{1}{\mu(A)} \int_A f \, d\mu. \]

The Hardy–Littlewood maximal operator and its dyadic version (see Section 1.3) in $E$ will be denoted $M$ and $M_D$, respectively:
\[ Mf(x) := \sup_{\Delta(y,r) \ni x} \int_{\Delta(y,r)} |f(z)| \, d\sigma(z), \]
\[ M_Df(x) := \sup_{Q \in D, Q \ni x} \int_Q |f(z)| \, d\sigma(z). \]

### 1.2. ADR, UR and NTA sets

**Definition 1.7.** — We say that a closed set $E \subset \mathbb{R}^{n+1}$ is an $n$-ADR (Ahlfors–David regular) set if there exists a uniform constant $C$ such that
\[ \frac{1}{C} r^n \leq \sigma(\Delta(x,r)) \leq C r^n \]
for every $x \in E$ and every $r \in (0, \text{diam}(E))$, where $\text{diam}(E)$ may be infinite.

**Definition 1.8.** — Following [7, 8], we say that an $n$-ADR set $E \subset \mathbb{R}^{n+1}$ is UR (uniformly rectifiable) if it contains “big pieces of Lipschitz images” (BPLI) of $\mathbb{R}^n$: there exist constants $\theta, \Lambda > 0$ such that for every $x \in E$ and $r \in (0, \text{diam}(E))$ there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz norm no larger than $\Lambda$, such that
\[ \mathcal{H}^n(E \cap B(x,r) \cap \rho(\{y \in \mathbb{R}^n : |y| < r\})) \geq \theta r^n. \]

**Definition 1.9.** — Following [21], we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ is NTA (nontangentially accessible) if
\begin{enumerate}
  \item $\Omega$ satisfies the Harnack chain condition: there exists a uniform constant $C$ such that for every $\rho > 0$, $\Lambda \geq 1$ and $X, X' \in \Omega$ with $\delta(X, \delta(X')) \geq \rho$ and $|X - X'| < \Lambda \rho$ there exists a chain of open balls $B_1, \ldots, B_N \subset \Omega$, $N \leq C(\Lambda)$, with $X \in B_1, X' \in B_N, B_k \cap B_{k+1} \neq \emptyset$ and $C^{-1} \text{diam}(B_k) \leq \text{dist}(B_k, \partial \Omega) \leq C \text{diam}(B_k)$,
  \item $\Omega$ satisfies the corkscrew condition: there exists a uniform constant $c$ such that for every surface ball $\Delta := \Delta(x,r)$ with $x \in \partial \Omega$ and $0 < r < \text{diam}(\partial \Omega)$ there exists a point $X_\Delta \in \Omega$ such that $B(X_\Delta, cr) \subset B(x,r) \cap \Omega$,
  \item $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfies the corkscrew condition.
\end{enumerate}
1.3. Dyadic cubes; Carleson and sparse collections

**Theorem 1.10** (E.g. [5, 19, 24]). — Suppose that $E$ is an ADR set. Then there exists a countable collection $\mathcal{D}$,

$$
\mathcal{D} := \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k, \quad \mathcal{D}_k := \{Q^k_\alpha : \alpha \in \mathcal{A}_k\}
$$

of Borel sets (that we call dyadic cubes) such that

1. the collection $\mathcal{D}$ is nested: if $Q, P \in \mathcal{D}$, then $Q \cap P \in \{\emptyset, Q, P\}$,
2. $E = \bigcup_{Q \in \mathcal{D}_k} Q$ for every $k \in \mathbb{Z}$ and the union is disjoint,
3. there exist constants $c_1 > 0$ and $C_1 \geq 1$ with the following property: for any cube $Q^k_\alpha$, there exists a point $z^k_\alpha \in Q^k_\alpha$ (that we call the center point of $Q^k_\alpha$) such that

$$
\Delta(z^k_\alpha, c_1 2^{-k}) \subseteq Q^k_\alpha \subseteq \Delta(z^k_\alpha, C_1 2^{-k}) =: \Delta_Q^k,
$$

4. if $Q, P \in \mathcal{D}$ and $Q \subseteq P$, then $\Delta_Q \subseteq \Delta_P$,

5. for every cube $Q^k_\alpha$ there exists a uniformly bounded number of disjoint cubes $Q^{k+1}_{\beta_i}$ such that $Q^k_\alpha = \bigcup_{i} Q^{k+1}_{\beta_i}$, where the uniform bound depends only on the ADR constant of $E$,
6. the cubes form a connected tree under inclusion: if $Q, P \in \mathcal{D}$, then there exists a cube $R \in \mathcal{D}$ such that $Q \cup P \subseteq R$.

**Remark 1.11.** — The last property in the previous theorem does not appear in the constructions in [5, 19, 24], but it is easy to modify the construction to get this property. The basic idea in the construction in [19] is to choose first the center points $z^k_\alpha$, then define a partial order among those points and finally build the cubes by using density arguments. Thus, if we simply choose the center points $z^k_\alpha$ in such a way that there exists a point $z_0 \in \bigcap_{k \in \mathbb{Z}} \{z^k_\alpha\}_\alpha$, then by (1.1) for any $r > 0$ there exists a cube $Q_r$ that contains the ball $B(z_0, r)$. This implies the last property in the previous theorem.

**Notation 1.12.**

1. Since the set $E$ may be bounded or disconnected, we may encounter a situation where $Q^k_\alpha = Q^l_\beta$, although $k \neq l$. In particular, in the second to last property of Theorem 1.10 there might exist only one cube $Q^{k+1}_{\beta_i}$ which equals $Q^k_\alpha$ as a set. Thus, we use the notation $\mathcal{D}(E)$ for the collection of all relevant cubes $Q \in \mathcal{D}$, i.e. if $Q^k_\alpha \in \mathcal{D}(E)$, then $C_1 2^{-k} \lesssim \text{diam}(E)$ and the number $k$ is maximal in the sense
that there does not exist a cube $Q^l_β \in D$ such that $Q^l_β = Q^k_α$ for some $l > k$. Notice that the number $k$ is bounded for each cube since the ADR condition excludes the presence of isolated points in $E$. This way in $D(E)$ it is natural to talk about the children of a cube $Q$ (i.e. the largest cubes $P \subset Q$) and the parent of a cube $Q$ (i.e. the smallest cube $R \supseteq Q$).

(2) For every cube $Q^k_α := Q \in D$, we denote $\ell(Q) := 2^{-k}$ and $z_Q := z^k_α$. We call $\ell(Q)$ the side length of $Q$.

(3) For every $Q \in D$, we denote the collection of dyadic subcubes of $Q$ by $D_Q$.

**Definition 1.13.** — Suppose that $\Lambda \geq 1$. We say that a collection $A \subset D$ is $\Lambda$-Carleson (or that it satisfies a Carleson packing condition) if

$$\sum_{Q \in A, Q \subset Q_0} \sigma(Q) \leq \Lambda \sigma(Q_0)$$

for every cube $Q_0 \in D$.

**Definition 1.14.** — Suppose that $\lambda \in (0, 1)$. We say that a collection $A \subset D$ is $\lambda$-sparse if for every cube $Q \in A$ there exists a subset $E_Q \subset Q$ satisfying

1. $E_Q \cap E_{Q'} = \emptyset$ if $Q \neq Q'$ and
2. $\sigma(E_Q) \geq \lambda \sigma(Q)$.

The following result will be useful for us with some technical estimates.

**Theorem 1.15.** — A collection $A \subset D$ is $\Lambda$-Carleson if and only if it is $\frac{1}{\Lambda}$-sparse.

Although it is very easy to show that sparseness implies the Carleson property, the other implication is not obvious. For dyadic cubes in $\mathbb{R}^n$, it was first proven by I. Verbitsky [26, Corollary 2] and the result was later rediscovered by A. Lerner and F. Nazarov with a different proof [23, Lemma 6.3]. For general Borel sets, the result was proven by T. Hänninen [14, Theorem 1.3]. Since the dyadic cubes in Theorem 1.10 are Borel sets, the result of Hänninen is suitable for us.

In addition to sparseness arguments, we use a discrete Carleson embedding theorem (Theorem A.1) to prove that local bounds imply global bounds. In fact, we could use the embedding theorem instead of sparseness arguments throughout the paper but this would give us slightly weaker estimates.
Definition 1.16. — Let $A \subset D$ be any collection of dyadic cubes. We say that a cube $P \in A$ is an $A$-maximal subcube of $Q_0$ if there do not exist any cubes $P' \in A$ such that $P \subset P' \subset Q_0$.

1.4. Corona decomposition, Whitney regions and Carleson boxes

Definition 1.17. — We say that a subcollection $S \subset D(E)$ is coherent if the following three conditions hold.

1. There exists a maximal element $Q(S) \in S$ such that $Q \subset S$ for every $Q \in S$.
2. If $Q \in S$ and $P \in D(E)$ is a cube such that $Q \subset P \subset Q(S)$, then also $P \in S$.
3. If $Q \in S$, then either all children of $Q$ belong to $S$ or none of them do.

If $S$ satisfies only conditions (1) and (2), then we say that $S$ is semicoherent.

In this article, we do not work directly with Definition 1.8 but use the bilateral corona decomposition instead:

Lemma 1.18 ([18, Lemma 2.2]). — Suppose that $E \subset \mathbb{R}^{n+1}$ is a uniformly rectifiable set of codimension 1. Then for any pair of positive constants $\eta \ll 1$ and $K \gg 1$ there exists a disjoint decomposition $D(E) = \mathcal{G} \cup \mathcal{B}$ satisfying the following properties:

1. The “good” collection $\mathcal{G}$ is a disjoint union of coherent stopping time regimes $S$.
2. The “bad” collection $\mathcal{B}$ and the maximal cubes $Q(S)$ satisfy a Carleson packing condition: for every $Q \in D(E)$ we have
   $$\sum_{Q' \subset Q, Q' \in \mathcal{B}} \sigma(Q') + \sum_{S : Q(S) \subset Q} \sigma(Q(S)) \leq C_{\eta,K} \sigma(Q).$$
3. For every $S$, there exists a Lipschitz graph $\Gamma_S$, with Lipschitz constant at most $\eta$, such that for every $Q \in S$ we have
   $$\sup_{x \in \Delta_Q^*} \text{dist}(x, \Gamma_S) + \sup_{y \in B_Q^* \cap \Gamma_S} \text{dist}(y, E) < \eta \ell(Q),$$
   where $B_Q^* := B(z_Q, K\ell(Q))$ and $\Delta_Q^* := B_Q^* \cap E$.

The proof of this decomposition is based on the use of both the unilateral corona decomposition [7] and the bilateral weak geometric lemma [8] of David and Semmes. The decomposition plays a key role in this paper.
In [18, Section 3], the bilateral corona decomposition is used to construct Whitney regions $U_Q$ and Carleson boxes $T_Q$ with respect to the dyadic cubes $Q \in \mathbb{D}(E)$ using a dyadic Whitney decomposition of $\mathbb{R}^{n+1} \setminus E$. The Whitney regions are a substitute for the dyadic Whitney tiles $Q \times (\ell(Q)/2, \ell(Q))$ and the Carleson boxes are a substitute for the dyadic boxes $Q \times (0, \ell(Q))$ in $\mathbb{R}^{n+1}$. We list some of their important properties in the next lemma which we use constantly without specifically referring to it each time.

**Lemma 1.19.** — The Whitney regions $U_Q$, $Q \in \mathbb{D}(E)$, satisfy the following properties.

1. The region $U_Q$ is a union of a bounded number of slightly fattened Whitney cubes $I^* := (1+\tau)I$ such that $\ell(Q) \approx \ell(I)$ and $\text{dist}(Q, I) \approx \ell(Q)$. We denote the collection of these Whitney cubes by $\mathcal{W}_Q$.
2. The regions $U_Q$ have a bounded overlap property. In particular, we have $\sum_i |U_{Q_i}| \lesssim |\bigcup_i U_{Q_i}|$ for cubes $Q_i$ such that $Q_i \neq Q_j$ if $i \neq j$.
3. If $U_Q \cap U_P \neq \emptyset$, then $\ell(Q) \approx \ell(P)$ and $\text{dist}(Q, P) \lesssim \ell(Q)$.
4. For every $Y \in U_Q$, we have $\delta(Y) \approx \ell(Q)$.
5. For every $Q \in \mathbb{D}(E)$, we have $|U_Q| \approx \ell(Q)^{n+1} \approx \ell(Q) \cdot \sigma(Q)$.
6. If $Q \in \mathcal{G}$, then $U_Q$ breaks into exactly two connected components $U_+^Q$ and $U_-^Q$ such that $|U_+^Q| \approx |U_-^Q|$.
7. If $Q \in \mathcal{B}$, then $U_Q$ breaks into a bounded number of connected components $U_i^Q$ such that $|U_i^Q| \approx |U_j^Q|$ for all $i$ and $j$.
8. If $\text{diam}(E) = \infty$, then $\bigcup_{Q \in \mathbb{D}(E)} U_Q = \Omega$.
9. If $\text{diam}(E) < \infty$, then there exists a point $z_0 \in E$ and a constant $C \geq 1$ such that $B(z_0, C \cdot \text{diam}(E)) \setminus E \subset \bigcup_{Q \in \mathbb{D}(E)} U_Q$. The constant $C$ can be made large but this makes the implicit constant in the bounded overlap property large as well.

For every $Q \in \mathcal{G}$, the components $U_+^Q$ and $U_-^Q$ have “center points” that we denote by $X_+^Q$ and $X_-^Q$, respectively. We also set $Y_\pm^Q := X_\pm^\tilde{Q}$, where $\tilde{Q}$ is the dyadic parent of $Q$ unless $Q = Q(\mathcal{S})$, in which case we set $\tilde{Q} = Q$. We use these points in the construction in Section 5.1. For any cube $Q \in \mathcal{G}$, the collection $\mathcal{W}_Q$ breaks naturally into two disjoint subcollection $\mathcal{W}_Q^+$ and $\mathcal{W}_Q^-$. For every $Q \in \mathbb{D}(E)$, we define the Carleson box as the set

$$T_Q := \text{int} \left( \bigcup_{Q' \in \mathbb{D}Q} U_Q \right).$$
For each $A \subset D(E)$, we set
$$
\Omega_A := \text{int} \left( \bigcup_{Q' \in A} U_{Q'} \right).
$$

1.5. Local $BV$

**Definition 1.20.** — We say that a function $f \in L^1_{\text{loc}}(\Omega)$ has locally bounded variation (denote $f \in BV_{\text{loc}}(\Omega)$) if for any bounded open set $U \subset \Omega$ such that $\overline{U} \subset \Omega$ we have
$$
\sup_{\Psi \in C^1_0(U), \|\Psi\|_{L^\infty} \leq 1} \iint_U f(Y) \text{div} \Psi(Y) dY < \infty.
$$

The latter expression can be shown to define a measure, by the Riesz representation theorem. We have the following:

**Theorem 1.21** ([10, Section 5.1]). — Suppose that $f \in BV_{\text{loc}}(\Omega)$. Then there exists a Radon measure $\mu$ on $\Omega$ such that
$$
\mu(U) = \sup_{\Psi \in C^1_0(U), \|\Psi\|_{L^\infty} \leq 1} \iint_U f(Y) \text{div} \Psi(Y) dY,
$$
for any open set $U \subset \Omega$; we call $\mu(U)$ the total variation of $f$ on $U$.

Abusing notation, for an open set $U \subset \Omega$, we shall write
$$
\mu(U) := \iint_U |\nabla f(Y)| dY,
$$
which should not be mistaken for a usual Lebesgue integral. Indeed, we may have situations where $A \subset B$ and $|A| = |B|$ but $\iint_A |\nabla f(Y)| dY \ll \iint_B |\nabla f(Y)| dY$.

In particular, if $f \in BV_{\text{loc}}(\Omega)$, the sets $U, U_1, \ldots, U_k \subset \Omega$ are open and $U \subset \bigcup_i U_i$, then
$$
\iint_U |\nabla f(Y)| dY \leq \sum_i \iint_{U_i} |\nabla f(Y)| dY. \quad (1.3)
$$

**Remark 1.22.** — We emphasize that we write $|\nabla f| dY$ to indicate the variation measure of $f$, which is denoted by $\|Df\|$ in [10]; thus, for $f \in BV_{\text{loc}}(\Omega)$, and for any open set $U \subset \Omega$, we let $\iint_U |\nabla f| dY$ denote the total variation of $f$ over $U$. We shall continue to use this (mildly abusive) notational convention in the sequel, when working with elements of $BV_{\text{loc}}(\Omega)$. 

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**TOME 0 (0), FASCICULE 0**
1.6. $C$ and $C_D$

For every $k \in \mathbb{N}$, we let $F_k$ be the ordered pair $(E, k)$. In this section, we let $Q_0 = E$ be the maximal dyadic cube if $E$ is a bounded set. We define the operators $C$ and $C_D$ by setting

$$C(f)(x) := \sup_{r > 0} \frac{1}{r^n} \int_{B(x, r) \setminus E} |f(Y)|dY,$$

$$C_D(f)(x) := \sup_{Q \in D^*, x \in Q} \frac{1}{\ell(Q)^n} \int_{T_Q} |f(Y)|dY,$$

where

$$D^* := \begin{cases} D(E), & \text{if } \text{diam}(E) = \infty \\ D(E) \cup \{F_k : k = \Lambda_0, \Lambda_0 + 1, \ldots\}, & \text{if } \text{diam}(E) < \infty \end{cases}$$

and

$$T_{F_k} := B(z_0, 2^k \text{diam}(E)), \quad \ell(F_k) := 2^k \text{diam}(E)$$

for some fixed point $z_0 \in E$ and a number $\Lambda_0$ such that $T_{Q_0} \subset T_{F_{\Lambda_0}}$. We will call also the pairs $F_k$ cubes although their actual structure is irrelevant and we will interpret $x \in F_k$ simply as $x \in E$.

Usually these functions are not pointwise equivalent but we only have the estimate $C_D(f)(x) \lesssim C(f)(x)$ for every $x \in E$ (this follows from the ADR property of $E$ and the fact that $T_Q \subset B(z_Q, C\ell(Q))$ for a uniform constant $C$). However, in $L^p$ sense, these functions are always comparable. This can be seen easily from the level set comparison formula that we prove next. This comparability is convenient for us since we construct the approximating function $\varphi$ in Theorem 1.3 with the help of the dyadic Whitney regions. Thus, it is more natural for us to prove the desired $L^p$ bound for $C_D(\nabla \varphi)$ instead of $C(\nabla \varphi)$. We prove the comparison formula by using well-known techniques from the proof of the corresponding formula for the Hardy–Littlewood maximal function and its dyadic version [9, Lemma 2.12].

Lemma 1.23. — Suppose that $f \in BV_{loc}(\Omega)$. Then there exist uniform constants $A_1$ and $A_2$ (depending on the dimension and the ADR constant) such that for every $\lambda > 0$ we have

$$\sigma(\{x \in E : C(\nabla f)(x) > A_1 \lambda\}) \leq A_2 \cdot \sigma(\{x \in E : C_D(\nabla f)(x) > \lambda\}).$$

In particular, $\|C(f)\|_{L^p(E)} \leq A_1 A_2^{1/p} \|C_D(f)\|_{L^p(E)}$ for every $p \in (1, \infty)$.

Proof. — We first note that if $r \gg \text{diam}(E)$, then by the definition of $C_D$ we have the bound $\frac{1}{r^n} \int_{B(x, r) \setminus E} |\nabla f(Y)|dY \lesssim C_D(\nabla f)(x)$. Thus, we may
assume that the balls in this proof have uniformly bounded radii \( \lesssim \text{diam}(E) \) and the cubes belong to \( D(E) \). Naturally, we may also assume that the right hand side of the inequality is finite.

We notice that if \( C_D(f)(x) > \lambda \), then there exists a cube \( Q \in D(E) \) such that \( x \in Q \) and \( \frac{1}{\sigma(Q)} \int_{T_Q} |\nabla f(Y)|dY > \lambda \). By the definition of \( C_D(f) \), we also have \( C_D(f)(y) > \lambda \) for every \( y \in Q \). In particular, we have

\[
\{ x \in E : C_D(\nabla f)(x) > \lambda \} = \bigcup_i Q_i
\]

for disjoint dyadic cubes \( Q_i \). We now claim that if \( A_1 \) is large enough, then

\[
(1.4) \quad \{ x \in E : C(\nabla f)(x) > A_1 \lambda \} \subseteq \bigcup_i 2\Delta_i
\]

where \( \Delta_i \) is the surface ball (1.1). Suppose that \( y \notin \bigcup_i 2\Delta_i \), and let \( r > 0 \). Let us choose \( k \in \mathbb{Z} \) so that \( 2^{k-1} < r < 2^k \). Now there exist at most \( K \) dyadic cubes \( R_1, R_2, \ldots, R_m \) such that \( \ell(R_j) = 2^k \) and \( R_j \cap \Delta(y, r) \neq \emptyset \) for every \( j = 1, 2, \ldots, m \). We notice that none of the cubes \( R_j \) can be contained in any of the cubes \( Q_i \) since otherwise we would have \( y \in 2\Delta_{R_j} \subset 2\Delta_i \) by (1.2). Thus, we have \( \frac{1}{\ell(R_j)^n} \int_{T_{R_j}} |\nabla f(Y)|dY < \lambda \) for every \( j \). We can use a straightforward geometric argument to show that \( B(y, r) \subseteq \bigcup_j T_{R_j} \) (see [18, p. 2353–2354]). Hence, since \( r \approx \ell(R_j) \) for every \( j \), we have

\[
\frac{1}{r^n} \int_{B(y, r)} |\nabla f(Y)|dY \lesssim \sum_{j=1}^m \frac{1}{\ell(R_j)^n} \int_{T_{R_j}} |\nabla f(Y)|dY \lesssim \lambda
\]

and \( y \notin \{ x \in E : C(\nabla f)(x) > A_1 \lambda \} \) for a large enough \( A_1 \). In particular, (1.4) holds and we have

\[
\sigma(\{ x \in E : C(\nabla f)(x) > A_1 \lambda \}) \leq \sum_i \sigma(2\Delta_i) 
\lesssim \sum_i \sigma(Q_i) 
= \sigma \left( \bigcup_i Q_i \right) = \sigma(\{ x \in E : C_D(\nabla f)(x) > \lambda \}).
\]

The \( L^p \) comparability \( C(\nabla f) \) and \( C_D(\nabla f) \) follows immediately:

\[
\| C(\nabla f) \|_{L^p(E)}^p = p \int_0^\infty \lambda^{p-1} \sigma(\{ x \in E : C(\nabla f)(x) > \lambda \})d\lambda 
\leq A_2 p \int_0^\infty \lambda^{p-1} \sigma(\{ x \in E : A_1 C_D(\nabla f)(x) > \lambda \})d\lambda 
= A_1^p A_2 \| C_D(\nabla f) \|_{L^p(E)}^p.
\]

\( \square \)
1.7. Cones, non-tangential maximal functions and square functions

We recall from [18, Section 3] that the Whitney regions $U_Q$ and the fattened Whitney regions $\hat{U}_Q$, $Q \in \mathbb{D}$, are defined using fattened Whitney boxes $I^* := (1 + \tau)I$ and $I^{**} := (1 + 2\tau)I$ respectively, where $\tau$ is a suitable positive parameter. Let us define the regions $\hat{U}_Q$ using even fatter Whitney boxes $I^{***} := (1 + 3\tau)W$.

**Definition 1.24.** — For any $x \in E$, we define the cone at $x$ by setting

$$\Gamma(x) := \bigcup_{Q \in \mathbb{D}(E), Q \ni x} \hat{U}_Q.$$  

We define the non-tangential maximal function $N_* u$ and, for $u \in W^{1,2}_{\text{loc}}(\Omega)$, the square function $S u$ as follows:

$$N_* u(x) := \sup_{Y \in \Gamma(x)} |u(Y)|, \quad x \in E,$$

$$S u(x) := \left( \int_{\Gamma(x)} |\nabla u(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{1/2}, \quad x \in E.$$  

The Hytönen–Rosén techniques in [20, Section 6] rely on the use of local $S \lesssim N$ and $N \lesssim S$ estimates from [15]. Although a local $S \lesssim N$ estimate holds also in our context [17], a local $N \lesssim S$ estimate does not hold without suitable assumptions on connectivity. Thus, we cannot apply the Hytönen–Rosén techniques directly but we have to combine them with the techniques created in [18].

In Section 5 we consider the following modified versions of $\Gamma(x)$ and $N_* u$ to bypass some additional technicalities:

**Definition 1.25.** — For every $x \in E$ and $\alpha > 0$ we define the cone of $\alpha$-aperture at $x$ $\Gamma_\alpha(x)$ by setting

$$\Gamma_\alpha(x) := \bigcup_{Q \in \mathbb{D}(E), Q \ni x} \bigcup_{P \in \mathbb{D}(E), \ell(P) = \ell(Q), \alpha \Delta_Q \cap P \neq \emptyset} \hat{U}_P.$$  

Using the cones $\Gamma_\alpha(x)$, we define the non-tangential maximal function of $\alpha$-aperture $N_*^\alpha u$ by setting $N_*^\alpha u(x) := \sup_{Y \in \Gamma_\alpha(x)} |u(Y)|$.

**Remark 1.26.** — If the set $E$ is bounded, then the cones (1.5) and (1.6) are also bounded since we only constructed Whitney regions $U$ such that
diam(\(U\)) \lesssim \text{diam}(E). \) Thus, if \(E\) is bounded, we use the cones
\[
\tilde{\Gamma}(x) := \Gamma(x) \cup B(z_0, C \cdot \text{diam}(E)) \]
and
\[
\tilde{\Gamma}_\alpha(x) := \Gamma_\alpha(x) \cup B(z_0, C_\alpha \cdot \text{diam}(E))
\]
for a suitable point \(z_0 \in E\) and suitable constants \(C\) and \(C_\alpha\) instead.

The usefulness of these modified cones and non-tangential maximal functions lies in the fact that for a suitable choice of \(\alpha\) the cone \(\tilde{\Gamma}_\alpha(x)\) contains some crucial points that may not be contained in \(\Gamma(x)\) and in the \(L^p\) sense the function \(N_\alpha^* u\) is not too much larger than \(N_\ast u\). We prove the latter claim in the next lemma but postpone the proof of the first claim to Section 5.

**Lemma 1.27.** — Suppose that \(u\) is a continuous function and let \(\alpha \geq 1\). Then \(\|N_\ast u\|_{L^p(E)} \approx_\alpha \|N_\alpha^* u\|_{L^p(E)}\) for every \(p \in (0, \infty)\).

**Proof.** — We only prove the claim for the case \(\text{diam}(E) = \infty\) as the proof for the case \(\text{diam}(E) < \infty\) is almost the same.

Since the set \(E\) is ADR, measures of balls with comparable radii are comparable. Using this property makes it simple and straightforward to generalize the classical proof of C. Fefferman and E. Stein [11, Lemma 1] from \(\mathbb{R}^{n+1}_+\) to \(\Omega\) to show that
\[
\|N_\gamma u\|_{L^p(E)} \approx \|N_\gamma^* u\|_{L^p(E)}
\]
where
\[
N_\gamma u(x) := \sup_{Y \in \tilde{\Gamma}_\gamma(x)} |u(Y)|, \quad \tilde{\Gamma}_\gamma(x) := \{Y \in \Omega: \text{dist}(x, Y) < \gamma \cdot \delta(Y)\}.
\]

By the definition of the cones \(\Gamma(x)\), there exists \(\gamma_0 > 0\) such that \(\tilde{\Gamma}_{\gamma_0}(x) \subset \Gamma(x)\) for every \(x \in E\). Thus, we only need to show that \(\Gamma_\alpha(x) \subset \tilde{\Gamma}_{\gamma}(x)\) for some uniform \(\gamma = \gamma(\alpha)\) for all \(x \in E\) since this gives us the estimate \((*)\) in the chain
\[
\|N_\ast u\|_{L^p(E)} \leq \|N_\alpha^* u\|_{L^p(E)} \overset{(*)}{\leq} \|N_\gamma u\|_{L^p(E)} \approx_{\gamma, \gamma_0} \|N_\gamma^* u\|_{L^p(E)} \leq \|N_\ast u\|_{L^p(E)}.
\]

Suppose that \(Q, P \in \mathbb{D}(E)\), \(x \in Q\), \(\ell(Q) = \ell(P)\) and \(\alpha \Delta_Q \cap P \neq \emptyset\). By the construction of the Whitney regions, for every \(Y \in \hat{U}_P\) we have
\[
\delta(Y) \approx \ell(P) \approx \text{dist}(Y, P).
\]

On the other hand, since \(\alpha \Delta_Q \cap P \neq \emptyset\) and \(\ell(P) = \ell(Q)\), we know that for any \(y \in P\) we have
\[
\text{dist}(x, y) \lesssim \alpha \ell(Q) = \alpha \ell(P).
\]
Let us take any \( z \in \mathcal{P} \). Now for every \( Y \in \hat{U}_P \) we have
\[
\operatorname{dist}(x, Y) \leq \operatorname{dist}(x, z) + \operatorname{dist}(z, Y) \lesssim \alpha \ell(P) + \ell(P) \lessapprox \alpha \ell(P) \approx \alpha \cdot \delta(Y).
\]
In particular, there exists a uniform constant \( \gamma = \gamma(\alpha) \) such that \( \Gamma_{\alpha}(x) \subset \hat{\Gamma}_\gamma(x) \). \( \square \)

2. Principal cubes

As in [20], we define the numbers \( M_D(N_*u)(Q) \) by setting
\[
M_D(N_*u)(Q) := \sup_{Q \subseteq R \in D} \int_R N_*u(y) d\sigma(y)
\]
for every \( Q \in D(E) =: D \). We shall use a collection \( \mathcal{I} \subset D(E) = D \) such that
\[
(2.1) \quad \mathcal{I} := \left\{ Q_i : i \in \tilde{N} \right\}, \quad Q_i \subset Q_{i+1} \quad \forall \ i, \quad \bigcup_i Q_i = E,
\]
where \( \tilde{N} = \{1, 2, \ldots, n_0\} \) for some \( n_0 \in \mathbb{N} \) if \( E \) is bounded, and \( \tilde{N} = \mathbb{N} \) otherwise. This type of a collection exists by the last property in Theorem 1.10 and by the properties of dyadic cubes, the collection is Carleson. Let us construct a collection \( \mathcal{P} \subset D \) of "stopping cubes" using the construction described in [20, Section 6.1]. We set \( \mathcal{P}_0 := \mathcal{I} \) and consider all the cubes \( Q' \in D(E) \setminus \mathcal{P}_0 \) such that
(1) for some \( Q \in \mathcal{P}_0 \) we have \( Q' \subset Q \) and
(2.2) \( M_D(N_*u)(Q') = \sup_{Q' \subseteq R \in D} \int_R N_*u(y) d\sigma(y) > 2M_D(N_*u)(Q), \)

(2) \( Q' \) is not contained in any such \( Q'' \subset Q \) such that either \( Q'' \in \mathcal{P}_0 \) or (2.2) holds for the pair \( (Q'', Q) \).

We denote by \( \mathcal{P}_1 \) the collection we get by adding all the cubes \( Q' \) satisfying both (1) and (2) to \( \mathcal{P}_0 \). We then continue this process for \( \mathcal{P}_1 \) in place of \( \mathcal{P}_0 \) and so on. We set \( \mathcal{P} := \bigcup_{k=0}^\infty \mathcal{P}_k \). We also set
\[
\pi_\mathcal{P} Q = \text{the smallest cube } Q_0 \in \mathcal{P} \text{ such that } Q \subseteq Q_0.
\]
Here we mean smallest with respect to the side length. Naturally, we have \( \pi_\mathcal{P} Q = Q \) for every \( Q \in \mathcal{P} \), and since \( \mathcal{I} \subset \mathcal{P} \), for every cube \( Q \in D \) there exists some cube \( P_Q \in \mathcal{P} \) such that \( Q \subset P_Q \).
Remark 2.1. — The collection $\mathcal{P}$ is an auxiliary collection that helps us to simplify the proofs of several claims. We use it in the following way. Suppose that we have a subcollection $\mathcal{W} \subset \mathcal{D}$ and we want to show that $\mathcal{W}$ satisfies a Carleson packing condition. Let $Q_0 \in \mathcal{D}$. Now for every $Q \in \mathcal{W}$ such that $Q \subset Q_0$, we have either $\pi_{\mathcal{P}}Q = \pi_{\mathcal{P}}Q_0$ or $\pi_{\mathcal{P}}Q = P = \pi_{\mathcal{P}}P$ for some $P \in \mathcal{P}$ such that $P \subseteq \pi_{\mathcal{P}}Q_0$. In particular, we have

$$\sum_{Q \in \mathcal{W}, Q \subseteq Q_0} \sigma(Q) = \sum_{Q \in \mathcal{W}, \pi_{\mathcal{P}}Q = \pi_{\mathcal{P}}Q_0} \sigma(Q) + \sum_{P \in \mathcal{P}, P \subseteq \pi_{\mathcal{P}}Q_0} \sum_{Q \in \mathcal{W}, \pi_{\mathcal{P}}Q = P} \sigma(Q)$$

$$=: I_{Q_0} + \sum_{P \in \mathcal{P}, P \subseteq \pi_{\mathcal{P}}Q_0} I_P.$$ 

We prove in Lemma 2.2 below that the collection $\mathcal{P}$ satisfies a Carleson packing condition. Thus, if we can show that $I_{Q_0} \lesssim \sigma(Q_0)$ for an arbitrary cube $Q_0 \in \mathcal{P}$, we get

$$\sum_{P \in \mathcal{P}, P \subseteq \pi_{\mathcal{P}}Q_0} I_P \lesssim \sum_{P \in \mathcal{P}, P \subseteq \pi_{\mathcal{P}}Q_0} \sigma(P) \lesssim \sigma(Q_0).$$

Thus, to show that the collection $\mathcal{W}$ satisfies a Carleson packing condition, it is enough to show that $I_{Q_0} \lesssim \sigma(Q_0)$ for every cube $Q_0 \in \mathcal{D}$. The usefulness of this simplification is that if $Q \in \mathcal{D} \setminus \mathcal{P}$ and $\pi_{\mathcal{P}}Q = P$, then by the construction of the collection $\mathcal{P}$ we have

$$M_\mathcal{D}(N_*u)(Q) \leq 2M_\mathcal{D}(N_*u)(P).$$

We use this property several times in the proofs.

For any cube $Q_0 \in \mathcal{D}$, we say that $R \in \mathcal{P}$ is a $\mathcal{P}$-proper subcube of $Q_0$ if we have $M_\mathcal{D}(N_*u)(R) > 2M_\mathcal{D}(N_*u)(Q_0)$ and $M_\mathcal{D}(N_*u)(R') \leq 2M_\mathcal{D}(N_*u)(Q_0)$ for every intermediate cube $R \subsetneq R' \subsetneq Q_0$.

**Lemma 2.2.** — For every $Q_0 \in \mathcal{D}(E)$ we have

$$\sum_{P \in \mathcal{P}, P \subseteq Q_0} \sigma(P) \lesssim \sigma(Q_0).$$

**Proof.** — Let us start by noting that we may assume that $Q_0 \in \mathcal{P}$ since otherwise we can simply consider the $\mathcal{P}$-maximal subcubes of $Q_0$. To be more precise, the $\mathcal{P}$-maximal subcubes of $Q_0$ are disjoint by definition and thus, if we sum their measures together, it is at most $\sigma(Q_0)$. Now, if $Q \in \mathcal{P}$ and $Q \subset Q_0$, we know that $Q$ is one of the $\mathcal{P}$-maximal subcubes of $Q_0$ or it is contained properly in one of them. Hence, if we prove the estimate (2.3) for the case $Q_0 \in \mathcal{P}$, it implies the same estimate even with the same implicit constant for the case $Q_0 \notin \mathcal{P}$. 
Suppose first that we have a collection of disjoint cubes $Q' \subset Q$ that satisfy $M_D(N_u)(Q') > 2M_D(N_u)(Q)$. Then, for every such cube $Q'$ we have $M_D(N_u)(Q') > f_Q N_u d\sigma$ and thus, for every point $x \in Q'$ we get

$$M_D(1_Q N_u)(x) = \sup_{R \in D, x \in R \subseteq Q} f_R N_u d\sigma$$

$$\geq \sup_{R \in D, Q' \subseteq R \subseteq Q} f_R N_u d\sigma$$

$$= M_D(N_u)(Q') > 2M_D(N_u)(Q).$$

In particular, by the $L^1 \to L^{1,\infty}$ boundedness of $M_D$ we have

$$\sum_{Q'} \sigma(Q') \leq \sigma(\{x \in E : M_D(1_Q N_u)(x) > 2M_D(N_u)(Q)\})$$

$$\leq \frac{1}{2M_D(N_u)(Q)} \|1_Q N_u \|_{L^1(\sigma)} = \frac{\int_{Q'} N_u d\sigma}{M(N_u)(Q)} \leq \frac{\sigma(Q)}{2}.$$  \hspace{1cm} (2.4)

We notice that if $R \in \mathcal{P} \setminus \mathcal{I}$, then $R$ is a $\mathcal{P}$-proper subcube of some cube $Q \in \mathcal{P}$. To be more precise, if $R \in \mathcal{P} \setminus \mathcal{I}$, then there exists a chain of cubes $R = R_1 \subset R_2 \subset \ldots \subset R_k$, $R_i \in \mathcal{P}$, such that for every $i = 1, 2, \ldots, k - 1$ $R_i$ is a $\mathcal{P}$-proper subcube of $R_{i+1}$ and $R_k \in \mathcal{I}$. If such a chain of length $k$ from $R$ to $Q$ exists, we denote $R \in \mathcal{P}_Q^k$. By using the property (2.4) $k$ times, we see that for each $Q \in \mathcal{P}$ we have

$$\sum_{R \in \mathcal{P}_Q^k} \sigma(R) \leq \sum_{R \in \mathcal{P}_Q^{k-1}} \sum_{S \in \mathcal{P}_Q, S \subseteq R} \sigma(S) \leq \frac{1}{2} \sum_{R \in \mathcal{P}_Q^{k-1}} \sigma(R)$$

$$\leq \cdots \leq \frac{1}{2^{k-1}} \sum_{R \in \mathcal{P}_Q} \sigma(R) \leq \frac{\sigma(Q)}{2^k}.$$  \hspace{1cm} (2.5)

Now it is straightforward to prove the packing condition. We have

$$\sum_{P \in \mathcal{P}, P \subseteq Q_0} \sigma(P) = \sum_{P \in \mathcal{I}, P \subseteq Q_0} \sigma(P) + \sum_{P \in \mathcal{P} \setminus \mathcal{I} \subseteq Q_0} \sigma(P)$$

$$\leq C_{\mathcal{I}} \sigma(Q_0) + \sum_{Q \in \mathcal{I}, Q \subseteq Q_0} \sum_{k=1}^{\infty} \sum_{P \in \mathcal{P}_Q^k} \sigma(P)$$

$$\leq \sum_{Q \in \mathcal{I}, Q \subseteq Q_0} \sum_{k=1}^{\infty} \frac{\sigma(Q)}{2^k}$$

$$= C_{\mathcal{I}} \sigma(Q_0) + \sum_{Q \in \mathcal{I}, Q \subseteq Q_0} \sigma(Q) \leq C_{\mathcal{I}} \sigma(Q_0) + C_{\mathcal{I}} \sigma(Q_0)$$

which proves the claim. \hspace{1cm} \Box
3. “Large Oscillation” cubes

Before constructing the approximating function, we consider two collections of cubes that will act as the basis of our construction. In this section, we show that the union of the collection of “large oscillation” cubes

$$\mathcal{R} := \left\{ Q \in \mathbb{D} : \text{osc}_{U^i_Q} u > \varepsilon M_{\mathbb{D}}(N_*u)(Q) \text{ for some } i \right\}.$$  

and the collection of “bad” cubes from the corona decomposition satisfies a Carleson packing condition. We apply this property in the technical estimates in Section 5.

**Lemma 3.1.** — For every $Q_0 \in \mathbb{D}(E)$ we have

$$\sum_{R \in \mathcal{R}, R \subseteq Q_0} \sigma(R) \lesssim \frac{1}{\varepsilon^2} \sigma(Q_0).$$  

**Proof.** — We break the proof into three parts.

**Part 1: Simplification.** — First, by Remark 2.1, it is enough to show that

$$\sum_{R \in \mathcal{R}, R \subseteq Q_0, \pi R = \pi Q_0} \sigma(R) \lesssim \frac{1}{\varepsilon^2} \sigma(Q_0).$$  

Also, since the “bad” collection in the bilateral corona decomposition is Carleson, it suffices to consider the “good” cubes in $\mathcal{R}$, i.e. the collection $\mathcal{R} \cap \mathcal{G}$. Thus, we may assume that $Q_0 \in \mathcal{R} \cap \mathcal{G}$ since otherwise we may simply consider the $(\mathcal{R} \cap \mathcal{G})$-maximal subcubes of $Q_0$ similarly as with the collection $\mathcal{P}$ in the proof of Lemma 2.2. Furthermore, since the Whitney regions $U_R$ of the “good” cubes $R$ break into two components $U^+_R$ and $U^-_R$, it is enough to bound the sum

$$\sum_{R \in \mathcal{R}^+, R \subseteq Q_0, \pi R = \pi Q_0} \sigma(R) \lesssim \sigma(Q_0),$$  

where $\mathcal{R}^+ := \{ Q \in \mathcal{R} \cap \mathcal{G} : \text{osc}_{U^+_Q} u > \varepsilon M_{\mathbb{D}}(N_*u)(Q) \}$, as the arguments for the corresponding collection $\mathcal{R}^-$ are the same.

Since $Q_0 \in \mathcal{G}$, there exists a stopping time regime $\mathcal{S}_0 = \mathcal{S}_0(Q_0)$ such that $Q_0 \in \mathcal{S}_0$. We note that if we have $Q \subset Q_0$ for a cube $Q \in \mathcal{R}^+$, then either $Q \in \mathcal{S}_0$ or, by the coherency and disjointness of the stopping time regimes, $Q_0 \in \mathcal{S}$ for such a $\mathcal{S}$ that $Q(\mathcal{S}) \subsetneq Q_0$. Let $\mathcal{S} = \mathcal{S}(Q_0)$ be the collection of
the stopping time regimes $S$ such that $Q(S) \subsetneq Q_0$. Then we have

$$
\sum_{R \in \mathcal{R}^+, R \subseteq Q_0} \sigma(R) = \sum_{R \in \mathcal{R}^+, R \subseteq Q_0} \sigma(R) + \sum_{S \in \mathcal{S}} \sum_{R \in \mathcal{R}^+ \cap S, R \subseteq Q_0} \sigma(R) =: I_{Q_0} + II_{Q_0}.
$$

Let us show that if $I_{Q_0} \lesssim \sigma(Q_0)$ for every $Q_0 \in \mathbb{D}$, then $II_{Q_0} \lesssim \sigma(Q_0)$ for every $Q_0 \in \mathbb{D}$. Suppose that $Q \in \mathcal{S} \in \mathcal{G}$. Since $Q(S) \subsetneq Q_0$, we have $\pi_p Q = \pi_p Q_0$ only if $\pi_p Q = \pi_p Q(S) = \pi_p Q_0$. Thus, it holds that

$$
II_{Q_0} = \sum_{S \in \mathcal{S}} \sum_{R \in \mathcal{R}^+ \cap S, R \subseteq Q_0} \sigma(R) \leq \sum_{S \in \mathcal{S}} \sum_{R \in \mathcal{R}^+ \cap S, R \subseteq Q_0} \sigma(R) =: \sum_{S \in \mathcal{S}} \sigma(Q(S)) \lesssim \sigma(Q_0)
$$

by the Carleson packing property of the collection $\{Q(S)\}_S$. Hence, to prove (3.1), it suffices to show $I_{Q_0} \lesssim \sigma(Q_0)$.

**Part 2:** $\delta(Y) \lesssim D_A(Y)$ in $\hat{U}_P^+$. — Let $A \subset \mathcal{G}$ be a collection of cubes and set

$$
\Omega_A^* := \text{int} \left( \bigcup_{Q \in A} \hat{U}_Q^+ \right) = \text{int} \left( \bigcup_{Q \in A} \bigcup_{I \in \mathcal{W}_Q^+} I^{***} \right)
$$

and $D_A(Y) := \text{dist}(Y, \partial \Omega_A^*)$. Recall the definitions of $I^{**}$ and $I^{***}$ from Section 1.7. Let us fix a cube $P \in A$ and a point $Y \in \hat{U}_P^+ = \bigcup_{I \in \mathcal{W}_P^+} I^{**}$. We now claim that $\delta(Y) \lesssim D_A(Y)$. We notice first that although the regions $\hat{U}_Q^+$ may overlap, we have $\ell(Q) \approx \ell(Q') \approx \ell(P)$ for all overlapping regions $\hat{U}_Q^+$ and $\hat{U}_{Q'}^+$, such that $Y \in \hat{U}_Q^+ \cap \hat{U}_{Q'}^+$ (see (3.2), (3.8) and related estimates in [18]). Also, the fattened Whitney boxes $I^{***}$ may overlap, but we have $\ell(I^{***}) \approx \ell(I) \approx \ell(J) \approx \ell(J^{***}) \approx \ell(P)$ if $Y \in I^{***} \cap J^{***}$. By a simple geometrical consideration we know that

$$
\text{dist}(Y, \partial I^{***}) \approx_{\tau} \ell(I).
$$
It now holds that $D_A(Y) = \text{dist}(Y, \partial I^{**})$ for some $I^{**} \ni Y$ or $D_A(Y) \geq \text{dist}(Y, \partial I^{**})$ for every such $I^{**}$. In particular, we have

$$D_A(Y) \geq \inf_{Q \in A, Y \in \hat{U}_Q^+} \inf_{I \in W_Q^+} \text{dist}(Y, \partial I^{**}) \approx \inf_{Q \in A, Y \in \hat{U}_Q^+} \ell(I) \approx \inf_{Q \in A, Y \in \hat{U}_Q^+} \ell(Q) \approx \ell(P).$$

Now we can take any $I \in W_P^+$ such that $Y \in \partial I^{**}$ and notice that $\ell(P) \approx \ell(I^{**}) \approx \text{dist}(I^{**}, \partial \Omega) \approx \text{dist}(Y, \partial \Omega)$. Hence $D_A(Y) \geq \delta(Y)$ for every $Y \in \hat{U}_P^+$.

**Part 3: The sum $I_{Q_0}$**. — To simplify the notation, let us write $R_0^+ := \{R \in R^+ : R \subset Q_0, \pi_P R = \pi_P Q_0\}$.

We consider the region $\Omega^{**}$,

$$\Omega^{**} := \text{int} \left( \bigcup_{R \in R_0^+} \hat{U}_R^+ \right)$$

and set $D(Y) := \text{dist}(Y, \partial \Omega^{**})$ for every $Y \in \Omega$. Suppose that $R \in R_0^+$. By Part 2, we know that

$$\delta(Y) \lesssim D(Y) \quad \text{for every } Y \in \hat{U}_R^+. \quad (3.2)$$

We also notice that

$$\Omega^{**} = \text{int} \left( \bigcup_{R \in R_0^+} \hat{U}_R^+ \right) \subset \text{int} \left( \bigcup_{R \in R_0^+} \bigcup_{x \in R} \Gamma(x) \right),$$

so we have

$$\sup_{X \in \Omega^{**}} |u(X)| = \sup_{R \in R_0^+} \sup_{X \in \hat{U}_R^+} |u(X)| \leq \sup_{R \in R_0^+} \inf_{x \in R} N_* u(x) \leq \sup_{R \in R_0^+} M_D(N_* u)(R) \leq M_D(N_* u)(\pi_P Q_0). \quad (3.3)$$

In the last inequality we used the definition of $R_0^+$ (see Remark 2.1).

By [18, (5.8)] (or [16, Section 4]), we have

$$\left( \text{osc}_u \frac{u}{U_R^+} \right)^2 \lesssim \ell(R)^{-n} \int_{\hat{U}_R^+} |\nabla u(Y)|^2 \delta(Y) dY \quad (3.4)$$

for every $R \in R^+$. Notice also that if $R \in R_0^+$, then by the definition of the numbers $M_D(N_* u)(Q)$ we have $M_D(N_* u)(\pi_P Q_0) \leq M_D(N_* u)(R)$.
simply because $R \subset \pi P Q_0$. Thus, using (A) the definition of the numbers $M_D(N_s u)(Q)$, (B) the ADR property of $E$, (C) the definition of the collection $\mathcal{R}^+$ and (D) the bounded overlap of the regions $\mathcal{U}_R^+$ we get

\begin{equation}
M_D(N_s u)(\pi P Q_0)^2 I_{Q_0} \overset{(A)}{\leq} \sum_{R \in \mathcal{R}_0^+} M_D(N_s u)(R)^2 \sigma(R)
\end{equation}

\begin{equation}
\overset{(B)}{\leq} \sum_{R \in \mathcal{R}_0^+} M_D(N_s u)(R)^2 \ell(R)^n
\end{equation}

\begin{equation}
\overset{(C), (3.4)}{\leq} \frac{1}{\varepsilon^2} \sum_{R \in \mathcal{R}_0^+} \iint_{\mathcal{U}_R^+} |\nabla u(Y)|^2 \delta(Y) dY
\end{equation}

\begin{equation}
\overset{(3.2)}{\leq} \frac{1}{\varepsilon^2} \sum_{R \in \mathcal{R}_0^+} \iint_{\mathcal{U}_R^+} |\nabla u(Y)|^2 D(Y) dY
\end{equation}

\begin{equation}
\overset{(D)}{\leq} \frac{1}{\varepsilon^2} \iint_{\Omega^{**}} |\nabla u(Y)|^2 D(Y) dY
\end{equation}

Since $Q_0 \in \mathcal{R}$, we notice that the collection $\mathcal{R}_0^+$ forms a semi-coherent subregime of $S_0$. Thus, by [18, Lemma 3.24], the set $\Omega^{**}$ is a chord-arc domain (i.e. NTA domain with ADR boundary). Furthermore, by [2, Theorem 1.2], $\partial \Omega^{**}$ is UR. Since $\Omega^{**} \subset B(x_{Q_0}, C\ell(Q_0))$ for a suitable structural constant $C$ (see [18, (3.14)]), the ADR property of $\partial \Omega$ and [18, Theorem 1.1] give us

\begin{equation}
\frac{1}{\varepsilon^2} \iint_{\Omega^{**}} |\nabla u(Y)|^2 D(Y) dY \lesssim \frac{1}{\varepsilon^2} \|u\|_{L^\infty(\Omega^{**})}^2 \cdot \sigma(Q_0)
\end{equation}

\begin{equation}
\overset{(3.3)}{\lesssim} \frac{1}{\varepsilon^2} M_D(N_s u)(\pi P Q_0)^2 \cdot \sigma(Q_0).
\end{equation}

Since the numbers $M_D(N_s u)(\pi P Q_0)^2$ cancel from (3.5) and (3.6), this concludes the proof of the lemma. \qed

Since the bad collection $\mathcal{B}$ in the bilateral corona decomposition satisfies a Carleson packing condition, we immediately get the following corollary:

**Corollary 3.2.** For every $Q_0 \in \mathcal{D}(E)$ we have

\begin{equation}
\sum_{R \in (\mathcal{R} \cup \mathcal{B}), R \subseteq Q_0} \sigma(R) \lesssim \frac{1}{\varepsilon^2} \sigma(Q_0).
\end{equation}
4. Generation cubes

For every stopping time regime $S$, we construct a collection of generation cubes $G(S)$ as in [18, Section 5] but with modified stopping conditions. For clarity, let us repeat the key details and definitions from [18, Section 5] here. We set $Q^0 := Q(S)$ and $G_0 := \{Q^0\}$, start subdividing $Q^0$ dyadically and stop when we reach a cube $Q \in \mathbb{D}_{Q^0}$ for which at least one of the following conditions holds:

1. $Q$ is not in $S$,
2. $|u(Y_Q^+) - u(Y_{Q^0}^+)| > \varepsilon M_D(N_u)(Q)$,
3. $|u(Y_Q^-) - u(Y_{Q^0}^-)| > \varepsilon M_D(N_u)(Q)$.

The points $Y_Q^\pm$ were defined in Section 1.4. We denote the collection of maximal subcubes of $Q^0$ extracted by these stopping time conditions by $F_1 = F_1(Q^0)$ and we let $G_1 = G_1(Q^0) := F_1 \cap S$ be the collection of first generation cubes. We notice that the collection of subcubes of $Q^0$ that are not contained in any stopping cube $Q \in F_1$ form a semicoherent subregime of $S$. We denote this subregime by $S' = S'(Q^0)$.

If $G_1$ is non-empty, we repeat the construction above for the cubes $Q^1 \in G_1$ but replace $Y_{Q^0}^\pm$ by $Y_{Q^1}^\pm$ in conditions (2) and (3). Continuing like this gives us collections $G_k$ for $k \geq 0$ (notice that starting from some $k$ the collections might be empty), where

$$G_{k+1}(Q^0) := \bigcup_{Q^k \in G_k(Q^0)} G_1(Q^k).$$

To emphasize the dependency on $S$, we denote

$$G_k(S) := G_k(Q(S)),$$

and we set the collection of all generation cubes to be

$$G^* := \bigcup_{S} G(S).$$

By this construction, we have

$$S = \bigcup_{Q \in G(S)} S'(Q)$$

for each stopping time regime $S$, where $S'(Q)$ is a semicoherent subregime of $S$ with maximal element $Q$ and the subregimes $S'(Q)$ are disjoint.

Our next goal is to prove that the collection $G^*$ satisfies a Carleson packing condition:
**Lemma 4.1.** — For every $Q_0 \in \mathbb{D}$ we have

\begin{equation}
(4.2) \quad \sum_{S \in G^* \setminus S \subseteq Q_0} \sigma(S) \lesssim \frac{1}{\varepsilon^2} \sigma(Q_0).
\end{equation}

Before the proof, let us make two observations that help us to simplify the proof.

(1) By arguing as in the proof of Lemma 3.1, we may assume that $Q_0 \in G^*$ and it suffices to show that

\begin{equation}
\sum_{S \in G^* \cap S_0, S \subset Q_0} \sigma(S) \lesssim \frac{1}{\varepsilon^2} \sigma(Q_0),
\end{equation}

where $S_0$ is the unique stopping time regime such that $Q_0 \in S_0$.

(2) For every $k \geq 0$ and $S \in G_k(S_0)$, let $G_1(S) \subset G(S_0)$ be the $G^*$-children of $S$, i.e. the cubes $P \in G_{k+1}(S_0)$ such that $P \subset S$. For each such $S$ we have

\begin{equation}
(4.3) \quad M_D(N_*u)(S)^2 \sum_{Q \in G_1(S)} \sigma(Q) \lesssim \frac{1}{\varepsilon^2} \iint_{\Omega_{\mathcal{S}}(S)} |\nabla u(Y)|^2 \delta(Y) dY,
\end{equation}

where $\mathcal{S} := S'(S) \cap \{Q \in \mathbb{D} : \pi_P Q = \pi_P Q_0\}$ is a semi-coherent subregime of $S_0$ and $\Omega_{\mathcal{S}}(S)$ is the associated sawtooth region (see (1.4)). The estimate (4.3) is a counterpart of [18, Lemma 5.11] and it follows easily from the original proof. To be a little more precise, instead of having $\varepsilon^2 \leq 100 |u(Y_Q^+) - u(Y_S^+)|^2$ for every $Q \in G_1(S)$ as in [18, (5.13)], we have

\begin{align*}
\varepsilon^2 M_D(N_*u)(S)^2 & \leq \varepsilon^2 M_D(N_*u)(Q)^2 \leq |u(Y_Q^+) - u(Y_S^+)|^2
\end{align*}

for every $Q \in G_1(S)$. The rest of the proof works as it is.

**Proof of Lemma 4.1.** — Let us follow the arguments in the proof of [18, Lemma 5.16] and write

\begin{equation}
\sum_{S \in G^* \cap S_0, S \subset Q_0} \sigma(S) = \sum_{k \geq 0} \sum_{S \in G_k(Q_0)} \sigma(S) + \sum_{k \geq 1} \sum_{S' \in G_{k-1}(Q_0)} \sum_{S \in G_1(S')} \sigma(S) =: \sigma(Q_0) + I.
\end{equation}
Using (4.3) and the definition of the sawtooth regions gives us

\[
M_D(N_* u)(\pi_p Q_0)^2 I \lesssim \frac{1}{\varepsilon^2} \sum_{k \geq 1} \sum_{S' \in G_{k-1}(Q_0)} \sum_{S \in S'(S')} \int_{\Omega_{\mathcal{I}}(S')} |\nabla u(Y)|^2 \delta(Y) dY
\]

\[(4.4)\]

Let us denote \(\Omega_0 := \bigcup_{S \in G_{Q_0}^*} U_S\) where \(G_{Q_0}^* := \{S \in \mathbb{D} : \pi_p S = \pi_p Q_0\} \cap \bigcup_{k \geq 1} \bigcup_{S' \in G_{k-1}(Q_0)} S'(S').\) By the construction, \(\bigcup_{k \geq 1} \bigcup_{S' \in G_{k-1}(Q_0)} S'(S')\) is a coherent subregime of \(S_0\) with maximal element \(Q_0\) and thus, \(G_{Q_0}^*\) is a semicoherent subregime of \(S_0.\) In particular, the sawtooth region \(\Omega_0\) splits into two chord-arc domains \(\Omega_{0,0}^\pm\) by [18, Lemma 3.24]. Furthermore, by [2, Theorem 1.2], both \(\partial \Omega_{0,0}^-\) and \(\partial \Omega_{0,0}^+\) are UR. We also note that \(\Omega_0 \subset B(x_{Q_0}, C\ell(Q_0))\) (see [18, (3.14)]). Thus, since the triple sum in (4.4) runs over a collection of disjoint cubes, we can use the bounded overlap of the Whitney regions, [18, Theorem 1.1] and the ADR property of \(E\) to show that

\[
\sum_{k \geq 1} \sum_{S' \in G_{k-1}(Q_0)} \sum_{S \in S'(S')} \int_{U_S} |\nabla u(Y)|^2 \delta(Y) dY \lesssim \frac{1}{\varepsilon^2} \int_{\Omega_0} |\nabla u(Y)|^2 \delta(Y) dY \lesssim \frac{1}{\varepsilon^2} \|u\|^2_{L^\infty(\Omega_0)} \sigma(Q_0).
\]

Since \(\pi_p S = \pi_p Q_0\) for every \(S \in G_{Q_0}^*\), by (2.2) we have \(M_D(N_* u)(S) \leq 2M_D(N_* u)(\pi_p Q_0)\) for every \(S \in G_{Q_0}^*.\) In particular:

\[
\|u\|^2_{L^\infty(\Omega_0)} \leq \sup_{S \in G_{Q_0}^*} \sup_{Y \in U_S} |u(Y)|^2 \leq \sup_{S \in G_{Q_0}^*} \inf_{x \in S} N_* u(x)^2 \leq \sup_{S \in G_{Q_0}^*} M_D(N_* u)(S)^2 \lesssim M_D(N_* u)(\pi_p Q_0)^2.
\]

Since the numbers \(M_D(N_* u)(\pi_p Q_0)^2\) cancel out, we have proven the Carleson packing condition of \(G^*\).

\(\square\)
5. Construction of the approximating function

Before we construct the function, we prove the following technical lemma related to the modified cones $\Gamma_\alpha(x)$ that we defined in Section 1.7. Recall that

$$\Gamma_\alpha(x) = \bigcup_{Q \in \mathcal{D}(E), Q \ni x} \bigcup_{P \in \mathcal{D}(E), \ell(P) = \ell(Q), \alpha \Delta_Q \cap P \neq \emptyset} \tilde{U}_P.$$  

Lemma 5.1. — There exists a uniform constant $\alpha_0 > 0$ such that the following holds: if $Q \in \mathcal{D}(E)$ is any cube and $P \in G^*$ is a generation cube such that $\ell(Q) \leq \ell(P)$ and $\Omega_{S'(P)} \cap T_Q \neq \emptyset$, then $X_P^\pm, Y_P^\pm \in \Gamma_{\alpha_0}(x)$ for every $x \in Q$.

Proof. — We start by noticing that there exists $\alpha > 0$ (depending only on the structural constants) such that

$$\text{(5.2)} \quad \text{if } P \text{ appears in the union (5.1), then also } \tilde{P} \text{ appears in the same union},$$

where $\tilde{P}$ is the dyadic parent of $P$. Indeed, if we have $Q, P \in \mathcal{D}(E), x \in Q$, $\ell(Q) = \ell(P)$ and $\alpha \Delta_Q \cap P \neq \emptyset$, then also $x \in \tilde{Q}, \ell(\tilde{Q}) = \ell(\tilde{P})$ and $\alpha \Delta_{\tilde{Q}} \cap \tilde{P} \neq \emptyset$. The last claim follows from the fact that $\emptyset \neq \alpha \Delta_Q \cap P \subset \alpha \Delta_{\tilde{Q}} \cap \tilde{P}$.

Let us then prove the claim of the lemma by following the argument in the proof of [18, Lemma 5.20]. Since $\Omega_{S'(P)} \cap T_Q \neq \emptyset$, there exist cubes $P' \in S'(P)$ and $Q' \subset Q$ such that $U_{P'} \cap U_{Q'} \neq \emptyset$. By the properties of the Whitney regions, we have $\operatorname{dist}(Q', P') \leq \ell(Q') \approx \ell(P')$. Let us consider two cases:

1) Suppose that $\ell(P') \geq \ell(Q)$. Then there exists a cube $Q''$ such that $Q \subset Q''$ and $\ell(Q'') = \ell(P')$. Since $Q' \subset Q''$, we have $\operatorname{dist}(Q'', P') \leq \operatorname{dist}(Q', P') \leq \ell(Q') \leq \ell(Q'')$. Thus, for a large enough $\alpha_0$, we have $\tilde{U}_{P'} \subset \Gamma_{\alpha_0}(x)$ for every $x \in Q$ and the claim follows from (5.2).

2) Suppose that $\ell(P') < \ell(Q)$. Then by the semicoherency of $S'(P)$, there exists a cube $P'' \in S'(P)$ such that $P' \subset P'' \subset P$ and $\ell(P'') = \ell(Q)$. Since $P'' \subset P''$ and $Q' \subset Q$, we know that $\operatorname{dist}(P'', Q) \leq \operatorname{dist}(P', Q') \leq \ell(Q') \leq \ell(Q)$. Thus, for a large enough $\alpha_0$, we have $\tilde{U}_{P''} \subset \Gamma_{\alpha_0}(x)$ for every $x \in Q$. Again, the claim follows now from (5.2).
5.1. Constructing the function in $T_{Q_0}$

In this section we adopt the terminology from other papers (including [18]) and say that a component $U^i_Q$ is blue if $\text{osc}_{U^i_Q} u \leq \varepsilon M_B(N_u)(Q)$ and red if $\text{osc}_{U^i_Q} u > \varepsilon M_B(N_u)(Q)$.

We recall the construction of the local functions $\varphi_0$, $\varphi_1$ and $\varphi$ from [18, Section 5]. We start by defining an ordered family of good cubes $\{Q_k\}_{k \geq 1}$ relative to a fixed cube $Q_0 \in \mathcal{D}$. If $Q_0 \in \mathcal{G}$, then $Q_0 \in \mathcal{S}$ for some stopping time regime $\mathcal{S}$ and thus, $Q_0 \in S'_1$ for some subregime in (4.1). In this case, we set $Q_1 = Q(S'_1)$. If $Q_0 \notin \mathcal{G}$, then we let $Q_1$ be any good subcube of $Q_0$ such that $Q_1$ is maximal with respect to the side length; such a cube much exist since $\mathcal{B}$ is Carleson. Since $Q_1 \in \mathcal{G}$, we have $Q_1 \in \mathcal{S}$ for some stopping time regime $\mathcal{S}$, and by the coherency of $\mathcal{S}$, we have $Q_1 = Q(S'_1)$ for some subregime in (4.1). Once the cube $Q_1$ has been chosen in these two cases, we let $Q_2$ be a subcube of maximum side length in $(\mathbb{D}_{Q_0} \cap \mathcal{G}) \setminus S'_1$ and so on. This gives us a sequence of cubes $Q_k \in \mathcal{G}$ such that $\ell(Q_1) \geq \ell(Q_2) \geq \ell(Q_3) \geq \cdots$, $Q_k = Q(S'_k)$ and $\mathcal{G} \cap \mathbb{D}_{Q_0} \subset \bigcup_{k \geq 1} S'_k$.

We define recursively

$$A_1 := \Omega_{S'_1}, \quad A_k := \Omega_{S'_k} \setminus \left( \bigcup_{j=1}^{k-1} A_j \right), \quad k \geq 2.$$ 

and

$$A_1^\pm := \Omega_{S'_1}^\pm, \quad A_k^\pm := \Omega_{S'_k}^\pm \setminus \left( \bigcup_{j=1}^{k-1} A_j \right), \quad k \geq 2,$$

where

$$\Omega_{S'_k}^\pm := \text{int} \left( \bigcup_{Q \in S'_k} U^\pm_Q \right).$$

We also set

$$\Omega_0 := \bigcup_k \Omega_{S'_k} = \bigcup_k A_k \quad \text{and} \quad \Omega_0^\pm := \bigcup_k A_k^\pm.$$ 

We now define $\varphi_0$ on $\Omega_0$ by setting

$$\varphi_0 := \sum_k \left( u(Y_{Q_k}^+) 1_{A_k^+} + u(Y_{Q_k}^-) 1_{A_k^-} \right).$$

As for the rest of the subcubes of $\mathbb{D}_{Q_0}$, we let $\{Q(k)\}_k$ be some fixed enumeration of the cubes $(\mathcal{R} \cup \mathcal{B}) \cap \mathbb{D}_{Q_0}$ and define recursively

$$V_1 := U_{Q(1)}, \quad V_k := U_{Q(k)} \setminus \left( \bigcup_{j=1}^{k-1} V_j \right), \quad k \geq 2.$$
Each Whitney region $U_{Q(k)}$ splits into a uniformly bounded number of connected components $U^i_{Q(k)}$. Thus, we may further split

$$V^i_1 := U^i_{Q(1)}, \quad V^i_k := U^i_{Q(k)} \setminus \left( \bigcup_{j=1}^{k-1} V^j \right), \quad k \geq 2$$

and then define

$$\varphi_1(Y) := \begin{cases} u(Y), & \text{if } U^i_{Q(k)} \text{ is red} \\ u(X_I), & \text{if } U^i_{Q(k)} \text{ is blue}, \end{cases} \quad Y \in V^i_k,$$

on each $V^i_k$, where $X_I$ is the center of a fixed Whitney cube $I \subset U^i_{Q(k)}$. We then denote $\Omega_1 := \text{int} \left( \bigcup_{Q \in (B \cup R) \cap D_{Q_0}} U_Q \right) = \text{int} \left( \bigcup_{k} V_k \right)$, set the values of $\varphi_0$ and $\varphi_1$ to be 0 outside their original domains of definition and define the function $\varphi$ on the Carleson box $T_{Q_0}$ as

$$\varphi(Y) := \begin{cases} \varphi_0(Y), & Y \in T_{Q_0} \setminus \overline{\Omega_1} \\ \varphi_1(Y), & Y \in \Omega_1, \end{cases}$$

From the point of view of $C_D$, the values of $\varphi$ on the boundary of $\Omega_1$ are not important since the $(n + 1)$-dimensional measure of $\partial \Omega_1$ is 0. Thus, we may simply set $\varphi|_{\partial \Omega_1} = u$ since this is convenient from the point of view of $N^*(u - \varphi)$.

### 5.2. Verifying the estimates on $Q_0$

Let us fix a cube $Q_0 \in D(E)$. We start by verifying the following three estimates on $Q_0$.

**Lemma 5.2.** — Suppose that $x \in Q_0$, $Q' \in D_{Q_0}$ and $\overrightarrow{\Psi} \in C^1_{0}(W_{Q'})$ with $\|\overrightarrow{\Psi}\|_{L^\infty} \leq 1$, where $W_{Q'} \subset \Omega$ is any bounded and open set satisfying $T_{Q'} \subset W_{Q'}$. Then the following estimates hold:

1. $N^*(1_{T_{Q_0}}(u - \varphi))(x) \leq \varepsilon M_D(N^* u)(x)$,
2. $\int_{T_{Q'}} \varphi_0 \text{ div } \overrightarrow{\Psi} \lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta_{Q'}} N^*_u \text{ d} \sigma$,
3. $\int_{T_{Q'}} \varphi_1 \text{ div } \overrightarrow{\Psi} \lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta_{Q'}} N^*_u \text{ d} \sigma$,

where $\beta > 0$ is a uniform constant and $\alpha_0 > 0$ is the constant in Lemma 5.1.

**Proof.** —

1. Let us estimate the quantity $|u(Y) - \varphi(Y)|$ for different $Y \in T_{Q_0}$.
(1) Suppose that \( Y \in V^i_k \) such that \( U^i_{Q(k)} \) is a red component. Then we have \( \varphi(Y) = u(Y) \) and \( |u(Y) - \varphi(Y)| = 0 \).

(2) Suppose that \( Y \in V^i_k \) such that \( U^i_{Q(k)} \) is a blue component. Then \( \varphi(Y) = u(X_I) \) for a Whitney cube \( I \subseteq U^i_{Q(k)} \) and \( |u(Y) - \varphi(Y)| \leq \text{osc}_{U^i_{Q(k)}} u \leq \varepsilon M_D(N_*u(Q(k))). \)

(3) Suppose that \( Y \in T^0 \setminus \overline{\Omega}_1 \). Then \( Y \in A^{\pm}_k \) for some \( k \) such that \( Q_k \notin \mathcal{R} \). Without loss of generality, we may assume that \( Y \in A^+_k \). Now \( \varphi(Y) = u(Y^+_Q) \) and, since \( Q_k \notin \mathcal{R} \), we have \( |u(Y) - \varphi(Y)| \leq \text{osc}_{U^+_Q} u \leq \varepsilon M_D(N_*u(Q_k)). \)

Combining the previous estimates gives us

\[
N_*(1_{T^0_Q(u - \varphi)})(x) = \sup_{Y \in \Gamma(x) \cap T^0_Q} |u(Y) - \varphi(Y)|
= \sup_{Q \in D_{T^0_Q} \ Y \in U_Q} \sup_{Q \ni x} |u(Y) - \varphi(Y)|
\leq \sup_{Q \in D_{T^0_Q}} \varepsilon M_D(N_*u(Q))
\leq \varepsilon M_D(N_*u(x)).
\]

(2) — We first notice that since \( \Psi \) is compactly supported in \( \Omega \), we have \( \text{dist}(\text{supp } \Psi, E) > 0 \). Thus, for each \( A_k \), the set \((T^i_Q \setminus A_k \cap \text{supp } \Psi) \setminus \overline{\Omega}_1\) consists of a union of boundedly overlapping sets that are “nice” enough for integration by parts. The divergence theorem gives us

\[
\iint_{T^i_Q \setminus \overline{\Omega}_1} \varphi_0 \text{div } \overrightarrow{\Psi} \leq \sum_k \iint_{(T^i_Q \setminus A_k) \setminus \overline{\Omega}_1} \varphi_0 \text{div } \overrightarrow{\Psi}
= \sum_k \iint_{(T^i_Q \setminus A_k) \setminus \overline{\Omega}_1} \text{div}(\varphi_0 \overrightarrow{\Psi})
\leq \sum_k \left( \iint_{\partial((T^i_Q \setminus A^+_k) \setminus \overline{\Omega}_1)} \varphi_0 \overrightarrow{\Psi} \cdot \overrightarrow{N}
+ \iint_{\partial((T^i_Q \setminus A^-_k) \setminus \overline{\Omega}_1)} \varphi_0 \overrightarrow{\Psi} \cdot \overrightarrow{N} \right)
\leq \sum_k |u(Y^+_Q)| : \mathcal{H}^n(T^i_Q \setminus \partial(A^+_k \setminus \overline{\Omega}_1))
+ \sum_k |u(Y^-_Q)| : \mathcal{H}^n(T^i_Q \setminus \partial(A^-_k \setminus \overline{\Omega}_1)) =: I^+ + I^-.
\]
We only consider the sum $I^+$ since the sum $I^-$ can be handled the same way as $I^+$. We get
\[
\mathcal{H}^n(T_{Q'} \cap \partial(A^+_k \setminus \overline{Q_k})) \leq \mathcal{H}^n(T_{Q'} \cap \partial A^+_k) + \mathcal{H}^n(T_{Q'} \cap A^+_k \cap \partial \Omega_1)
\]
and thus, we have
\[
I^+ \leq \sum_k |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q'} \cap \partial A^+_k)
\]
\[
+ \sum_k |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q'} \cap A^+_k \cap \partial \Omega_1) =: J_1^+ + J_2^+.
\]
Let us consider the sum $I_1^+$ first. We split
\[
I_1^+ = \sum_{k : Q_k \subset Q'} |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q'} \cap \partial A^+_k)
\]
\[
+ \sum_{k : Q_k \not\subset Q'} |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q'} \cap A^+_k \cap \partial \Omega_1) =: J_1^+ + J_2^+.
\]
By [18, Proposition A.2, (5.21)] we know that the boundary $\partial A^+_k$ satisfies an upper ADR bound. Thus, since $\partial(T_{Q'} \cap A^+_k) \subset \overline{Q_k}$ and $\text{diam}(\Omega_{S_k'}) \lesssim \ell(Q_k)$, we get
\[
J_1^+ \lesssim \sum_{k : Q_k \subset Q'} |u(Y_{Q_k}^+)| \cdot \ell(Q_k)^n \approx \sum_{k : Q_k \subset Q'} |u(Y_{Q_k}^+)| \cdot \sigma(Q_k)
\]
\[
\leq \sum_{k : Q_k \subset Q'} \inf_{Q_k} N_* u \cdot \sigma(Q_k).
\]
Since the collection of generation cubes is $C\varepsilon^{-2}$-Carleson by Lemma 4.1, it is $C\varepsilon^2$-sparse by Theorem 1.15. Thus, we get
\[
\sum_{k : Q_k \subset Q'} \inf_{Q_k} N_* u \cdot \sigma(Q_k) \lesssim \frac{1}{\varepsilon^2} \sum_{k : Q_k \subset Q'} \inf_{Q_k} N_* u \cdot \sigma(E_{Q_k})
\]
\[
\leq \frac{1}{\varepsilon^2} \sum_{k : Q_k \subset Q'} \int_{E_{Q_k}} N_* u d\sigma
\]
\[
\leq \frac{1}{\varepsilon^2} \int_{Q'} N_* u d\sigma.
\]
Let us then consider the sum $J_2^+$. By the same argument as in [18, p. 2370], we know that the number of the cubes $Q_k$ such that $T_{Q'} \cap \partial A^+_k \neq \emptyset$ and $\ell(Q_k) \geq \ell(Q')$ is uniformly bounded. Thus, by Lemma 5.1 and the fact
that $\partial A_k^+$ satisfies an upper ADR bound (as we noted above), we get
\[
\sum_{k: Q_k \in Q', \ell(Q') \leq \ell(Q_k)} |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q'} \cap \partial A_k^+) \leq \sum_{k: Q_k \in Q'} \inf_{Q_k} N_*^{\alpha_0} u \cdot \mathcal{H}^n(T_{Q'} \cap \partial A_k^+)
\]
\[
\leq \inf_{Q'} N_*^{\alpha_0} u \cdot (\text{diam}(T_{Q'}))^n \approx \inf_{Q'} N_*^{\alpha_0} u \cdot \sigma(Q')
\]
\[
\leq \int_{Q'} N_*^{\alpha_0} u d\sigma.
\]
For the cubes $Q_k$ in $J_1^+$ such that $\ell(Q_k) \leq \ell(Q')$ we may use the same argument as in [18, p. 2370] to see that every such cube is contained in some nearby cube $Q''$ of the same side length as $Q'$ with $\text{dist}(Q', Q'') \lesssim \ell(Q')$. The number of such $Q''$ is uniformly bounded. By using the same techniques as with the sum $J_1^+$, we get
\[
\sum_{k: Q_k \in Q', \ell(Q') \geq \ell(Q_k)} |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q'} \cap \partial A_k^+) \lesssim \int_{Q''} N_*^{\alpha_0} u d\sigma
\]
for some uniform constant $\beta_0$. Thus, we get
\[
J_2^+ \lesssim \frac{1}{\varepsilon^2} \int_{\beta_0 \Delta_{Q'}} N_*^{\alpha_0} u d\sigma.
\]
Let us then consider the sum $I_2^+$. We first notice that
\[
\mathcal{H}^n(T_{Q'} \cap A_k^+ \cap \partial \Omega_1) \leq \sum_m \mathcal{H}^n(T_{Q'} \cap A_k^+ \cap \partial V_m).
\]
Thus, we get
\[
I_2^+ \leq \sum_{k} \sum_m |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q'} \cap A_k^+ \cap \partial V_m)
\]
\[
= \sum_{k: Q_k \in Q'} \sum_m |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q'} \cap A_k^+ \cap \partial V_m)
\]
\[
+ \sum_{k: Q_k \in Q' \supset Q_k} \sum_m |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q'} \cap A_k^+ \cap \partial V_m)
\]
\[
= J_3^+ + J_4^+.
\]
Suppose that $A_k^+ \cap \partial V_m \neq \emptyset$. Then, by the construction, we have $\ell(Q(m)) \lesssim \ell(Q_k)$ and $\text{dist}(Q(m), Q_k) \lesssim \ell(Q_k)$. Thus, there exists a uniform constant $\beta_1 > 0$ such that $Q(m) \subset \beta_1 \Delta Q_k$ and the set $\beta_1 \Delta Q_k$ can be covered by a uniformly bounded number of disjoint cubes with approximately the same side length as $Q_k$. In particular, since $T_{Q'} \cap A_k^+ \cap \partial V_m$ satisfies an upper ADR bound for every $m$ by the construction and \[18, (5.25), \text{Proposition A.2}], we get

$$J_3^+ = \sum_{k:Q_k \subset Q'} \sum_m |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q'} \cap A_k^+ \cap \partial V_m) \lesssim \sum_{k:Q_k \subset Q'} |u(Y_{Q_k}^+)| \sum_{m:Q(m) \subset \beta_1 \Delta Q_k} \ell(Q(m))^n \lesssim \sum_{k:Q_k \subset Q'} |u(Y_{Q_k}^+)| \sum_{m:Q(m) \subset \beta_1 \Delta Q_k} \sigma(Q(m)) \leq \frac{1}{\varepsilon^2} \sum_{k:Q_k \subset Q'} |u(Y_{Q_k}^+)| \cdot \sigma(Q_k).$$

Now we can use exactly the same arguments as with the sum $J_1^+$ to see that

$$J_3^+ \lesssim \frac{1}{\varepsilon^2} \int_{Q'} N_* u \, d\sigma.$$

Finally, let us handle the sum $J_4^+$. Just as above with the sum $J_3^+$, for some uniform constant $\beta_2 > 0$ we get

$$\sum_{k:Q_k \not\subset Q'} \sum_{m} |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q'} \cap A_k^+ \cap \partial V_m) \lesssim \sum_{k:Q_k \not\subset Q'} \sum_{m:V_m \subset \beta_2 \Delta Q'} \sigma(Q(m)) \lesssim \frac{1}{\varepsilon^2} \sum_{k:Q_k \not\subset Q'} |u(Y_{Q_k}^+)| \cdot \sigma(Q') \leq \frac{1}{\varepsilon^2} \int_{Q'} \inf_{Q'} N_*^{\alpha_0} u \cdot \sigma(Q').$$

Lem. 5.1
where we used the fact that there exists only a uniformly bounded number of cubes $Q_k$ that satisfy the condition of the sum by [18, Lemma 5.20]. By using the same argument as with the latter half of the sum $J_2^+$, we get the bound

$$
\sum_{k : Q_k \notin Q'} \sum_m |u(Y_{Q_k}^+)| \cdot \mathcal{H}^n(T_{Q_k}^+ \cap A_k^+ \cap \partial V_m) \lesssim \frac{1}{\varepsilon^2} \int_{\beta_3 \Delta Q'} N_s u d\sigma
$$

for some uniform constant $\beta_3 > 0$. Thus, we have

$$
J_4^+ \lesssim \frac{1}{\varepsilon^2} \int_{\beta_3 \Delta Q'} N_s u d\sigma.
$$

Combining the estimates for $J_1^+, J_2^+, J_3^+$ and $J_4^+$ gives us the claim.

(3). — By [18, (5.25)], we have

$$
(5.3) \quad \mathcal{H}^n(\partial V_i^k) \leq \mathcal{H}^n(\partial V_k) \lesssim \ell(Q(k))^n \approx \sigma(Q(k))
$$

for every $Q(k)$ and $i$. We also note that $\partial T_{Q'}$ satisfies an upper ADR bound [18, Proposition A.2]. Recall that the function $\varphi_1$ is supported on $\Omega_1$. Thus, since the sets $V_i$ are disjoint, we get

$$
\iint_{T_{Q'}} \varphi_1 \, \text{div} \varphi_1 \, \text{div} \Psi = \sum_{l} \iint_{T_{Q'} \cap V_i} \varphi_1 \, \text{div} \Psi
$$

$$
= \sum_{l} \sum_{i} \iint_{T_{Q'} \cap V_i} \varphi_1 \, \text{div} \Psi
$$

$$
= \sum_{l} \sum_{i} \left( \iint_{T_{Q'} \cap V_i} \text{div}(\varphi_1 \Psi) - \iint_{T_{Q'} \cap V_i} \nabla \varphi_1 \cdot \Psi \right)
$$

$$
\lesssim \sum_{l} \sum_{i} \left( \iint_{T_{Q'} \cap V_i} |\text{div}(\varphi_1 \Psi)| + \iint_{T_{Q'} \cap V_i} |\nabla \varphi_1| \right).
$$

Let us first assume that $U^i_{Q(l)}$ is a blue component. Recall that since the collection $\mathcal{R} \cup \mathcal{B}$ is $C\varepsilon^{-2}$-Carleson by Corollary 3.2, it is $C\varepsilon^2$-sparse by Theorem 1.15. Thus, by the definition of $\varphi_1$ and the divergence theorem,
we have

$$
\left| \int_{T_{Q'} \cap V_i^i} \text{div}(\varphi_1 \vec{\Psi}) \right| + \int_{T_{Q'} \cap V_i^i} |\nabla \varphi_1| = \left| \int_{T_{Q'} \cap V_i^i} \text{div}(\varphi_1 \vec{\Psi}) \right|
$$

\[ \leq \int_{T_{Q'} \cap \partial V_i^i} |u(X_{I(l,i)})| \]

\[ \overset{(5.3)}{\leq} \inf_{Q(l)} N \ast u \cdot \sigma(Q(l)) \]

\[ \lesssim \frac{1}{\varepsilon^2} \inf_{Q(l)} N \ast u \cdot \sigma(E_{Q(l)}). \]

Suppose then that $U_{Q(l)}^i$ is a red component. Since $\partial V_i^i \subset \Gamma(y)$ for every $y \in Q(l)$, we get

$$
\left| \int_{T_{Q'} \cap V_i^i} \text{div}(u \vec{\Psi}) \right| \leq \frac{1}{\varepsilon^2} \inf_{Q(l)} N \ast u \cdot \sigma(E_{Q(l)})
$$

by the same argument as above. Also, by the definition of the function $\varphi_1$, Caccioppoli’s inequality and the sparseness arguments, we have

$$
\left| \int_{T_{Q'} \cap V_i^i} |\nabla \varphi_1| \right| \overset{\text{by (5.3)}}{\lesssim} \left( \int_{V_i^i} |\nabla u|^2 \right)^{1/2} \ell(Q(l))^{(n+1)/2}
$$

$$
\lesssim \frac{1}{\ell(Q(l))} \left( \int_{\tilde{U}_{Q(l)}} |u|^2 \right)^{1/2} \ell(Q(l))^{(n+1)/2}
$$

$$
\lesssim \frac{1}{\ell(Q(l))} \left( \int_{\tilde{U}_{Q(l)}} \inf_{Q(l)} (N \ast u)^2 \right)^{1/2} \ell(Q(l))^{(n+1)/2}
$$

$$
\lesssim \frac{1}{\ell(Q(l))} \inf_{Q(l)} (N \ast u) \cdot \ell(Q(l))^{n+1}
$$

$$
\approx \sigma(Q(l)) \cdot \inf_{Q(l)} (N \ast u)
$$

$$
\lesssim \frac{1}{\varepsilon^2} \sigma(E_{Q(l)}) \cdot \inf_{Q(l)} N \ast u.
$$

Thus, since every Whitney region $U_Q$ has only a uniformly bounded number of components $U_Q^i$, we get

$$
\left| \int_{T_{Q'}} |\nabla \varphi_1| \right| \lesssim \sum_l \frac{1}{\varepsilon^2} \sigma(E_{Q(l)}) \cdot \inf_{Q(l)} N \ast u.
$$

Since $V_i$ meets $T_{Q'}$, we know that $\text{dist}(Q(l), Q') \lesssim \ell(Q')$. In particular, all the relevant cubes $Q(l)$ are contained in some nearby cubes $Q''$ such
that $\ell(Q'') \approx \ell(Q')$ and $\text{dist}(Q'', Q') \lesssim \ell(Q')$. The number of such $Q''$ is uniformly bounded. Thus, since the sets $E_{Q(l)}$ are disjoint, we get

$$\sum_l \frac{1}{\varepsilon^2} \sigma(E_{Q(l)}) \cdot \inf_{Q(l)} N_* u \lesssim \frac{1}{\varepsilon^2} \sum Q'' \int_{Q''} N_* u \lesssim \frac{1}{\varepsilon^2} \int_{\beta_0 \Delta Q'} N_* u$$

for some uniform constant $\beta_0$.

Let us then consider the dyadic total variation of the whole approximating function $\varphi$:

**Proposition 5.3.** Suppose that $Q' \in D_{Q_0}$ and $\vec{\Psi} \in C^1_0(W_{Q'})$ with $\|\vec{\Psi}\|_{L^\infty} \leq 1$, where $W_{Q'} \subset \Omega$ is any bounded and open set satisfying $T_{Q'} \subset W_{Q'}$. Then

$$\iint_{T_{Q'}} \varphi \text{div} \vec{\Psi} \lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta Q'} N_*^\alpha u d\sigma,$$

where $\beta > 0$ is a uniformly bounded constant and $\alpha_0 > 0$ is the constant in Lemma 5.1.

**Proof.** We start by splitting the integral with respect to $\varphi_0$ and $\varphi_1$.

$$\iint_{T_{Q'}} \varphi \text{div} \vec{\Psi} = \iint_{T_{Q'} \setminus \Omega_1} \varphi \text{div} \vec{\Psi} + \iint_{T_{Q'} \cap \Omega_1} \varphi_1 \text{div} \vec{\Psi}.$$

For the first integral, we can simply use the part (2) of Lemma 5.2. For the second integral we get

$$\iint_{T_{Q'} \cap \Omega_1} \varphi_1 \text{div} \vec{\Psi} = \sum_k \iint_{V_k \cap T_{Q'}} \varphi_1 \text{div} \vec{\Psi}$$

$$= \sum_k \left( \iint_{V_k \cap T_{Q'}} \text{div}(\varphi_1 \vec{\Psi}) - \iint_{V_k \cap T_{Q'}} \nabla \varphi_1 \cdot \vec{\Psi} \right)$$

$$\leq \sum_k \left| \iint_{V_k \cap T_{Q'}} \text{div}(\varphi_1 \vec{\Psi}) \right| + \sum_k \iint_{V_k \cap T_{Q'}} |\nabla \varphi_1|.$$
theorem and Theorem 1.15 and get
\[
\sum_k \left| \int_{V_k \cap T_{Q'}} \text{div}(\varphi_1 \overline{\Psi}) \right| \leq \sum_k \int_{\partial (V_k \cap T_{Q'})} \left| \varphi_1 \overline{\Psi} \cdot \vec{N} \right|
\leq \sum_k \sup_{U_Q(k)} |u| \cdot H^n(V_k \cap \partial T_{Q'})
\leq \sum_{k: \text{dist}(Q(k), Q'') \leq \ell(Q(k))} \inf_{Q(k)} N_u \cdot \sigma(Q(k))
\leq \frac{1}{\varepsilon^2} \sum_{k: \text{dist}(Q(k), Q'') \leq \ell(Q(k))} \inf_{Q(k)} N_u \cdot \sigma(E_Q(k))
\]

By the structure of the Whitney regions, we know $V_k \cap T_{Q'} = \emptyset$ if $\ell(Q(k)) \gg \ell(Q'')$ or $\text{dist}(Q(k), Q'') \gg \ell(Q'')$. Thus, there exists a uniform constant $\beta_1 > 0$ such that $Q(k) \subset \beta_1 \Delta_{Q''}$ for every $k$ in the sum above. We may cover $\beta_1 \Delta_{Q''}$ by a uniformly bounded number of disjoint cubes $P_j$ such that
\[
\sum_{k: \text{dist}(Q(k), Q'') \leq \ell(Q(k))} \text{dist}(Q(k), Q'') \lesssim \ell(Q'') \inf_{Q(k)} N_u \cdot \sigma(E_Q(k))
\]
for some uniform constant $\beta_2 \geq \beta_1$. Combining the previous bounds finishes the proof.

\[\square\]

\textbf{Remark 5.4.} — We notice that the previous proposition holds also in the following form: If we have cubes $Q', Q_1, Q_2 \in \mathbb{D}_{Q_0}$ and $\overline{\Psi} \in C^1_0(W_{Q'})$ with $\|\overline{\Psi}\|_{L^\infty} \leq 1$ for an open and bounded set $W_{Q'}$ containing $T_{Q'}$, then
\[
\int_{(T_{Q'} \cap T_{Q_1}) \setminus T_{Q_2}} \varphi \text{ div } \overline{\Psi} \lesssim \frac{1}{\varepsilon^2} \min \left\{ \int_{\beta_2 \Delta_{Q'}} N_u u \text{ d}\sigma, \int_{\beta_2 \Delta_{Q_2}} N_u u \text{ d}\sigma \right\}
\]
for some uniform constant $\beta_2$. Indeed, in the previous two proofs, we needed only the upper ADR estimates for the boundaries of $A_m$ and $V_k$ and these estimates remain valid if we remove a finite number of pieces whose boundaries satisfy an upper ADR estimate. By [18, Proposition A.2], $\partial T_Q$ is ADR for every $Q \in \mathbb{D}(E)$. Also, by the structure of the regions, these modified sets are “nice” enough to justify integration by parts that we used in the proofs.
5.3. From local to global

Let us now construct the global approximating function. Although our construction is a little different than the construction in [18, p. 2373], the basic ideas are the same.

5.3.1. $E$ is a bounded set

Let us first assume that $\text{diam}(E) < \infty$. In this case, we have a cube $Q_0 \in \mathbb{D}(E)$ such that $E = Q_0$ and $\ell(Q_0) \approx \text{diam}(E)$. We now set

$$\varphi(X) := \begin{cases} \varphi_{Q_0}(X), & \text{if } X \in T_{Q_0} \\ u(X), & \text{if } X \in \Omega \setminus T_{Q_0}, \end{cases}$$

where $\varphi_{Q_0}$ is the function constructed in Section 5.1. By part (1) of Lemma 5.2, we have

$$N^* (u - \varphi) (x) \leq \varepsilon M_D (N_* u)(x)$$

on $E$. As for the $C_D$ bound, we first notice that for any $Q \in \mathbb{D}_{Q_0}$ Proposition 5.3 gives us

\begin{equation}
\frac{1}{\sigma(Q)} \iint_{T_Q} |\nabla \varphi| \lesssim \frac{1}{\varepsilon^2} M(N_*^\alpha u)(x)
\end{equation}

for every $x \in Q$ since $\sigma(Q) \approx \sigma(\beta \Delta_Q)$. Let us now fix a cube $F_k \in \mathbb{D}^*$ (recall the definition of $\mathbb{D}^*$ in Section 1.6), take any $\overrightarrow{\Psi} \in C_0^1(T_{F_k})$ with $\|\overrightarrow{\Psi}\|_{L^\infty} \leq 1$ and modify the argument in [18, p. 2353]. We denote $R := 2^k \text{diam}(E)$ and thus have $T_{F_k} = B(z_0, R)$. By a suitable choice of parameters in the construction of the Whitney regions in [18], the Carleson box $T_{Q_0}$ is so large that we may fix a ball $B(z_0, r) \subset T_{Q_0}$ such that $r \geq \frac{1}{2} \text{diam}(E)$. Because of this, we may fix a uniform constant $\alpha_1$ such that a small enlargement of $B(z_0, R) \setminus B(z_0, r)$ is contained in $\widehat{\Gamma}_{\alpha_1} (x)$ (recall the definition of $\widehat{\Gamma}_{\alpha_1} (x)$ in Section 1.7) for every $x \in E$. We split

$$\frac{1}{\ell(F_k)^n} \iint_{T_{F_k}} \varphi \text{ div } \overrightarrow{\Psi} = \frac{1}{\ell(F_k)^n} \iint_{T_{Q_0}} \varphi \text{ div } \overrightarrow{\Psi} + \frac{1}{\ell(F_k)^n} \iint_{T_{F_k} \setminus T_{Q_0}} \varphi \text{ div } \overrightarrow{\Psi}.$$ 

By Proposition 5.3, we can bound the first integral by $M(N_*^\alpha u)(x)$ for any $x \in Q_0$. As for the second integral, we use the smoothness of $u$, Hölder’s
inequality and Caccioppoli’s inequality to get
\[
\iint_{T_{F_k} \setminus T_{Q_0}} \varphi \, \text{div} \, \overrightarrow{\Psi} - \int_{T_{F_k} \setminus T_{Q_0}} u \, \text{div} \, \overrightarrow{\Psi} \\
\leq \int_{B(z_0, R) \setminus T_{Q_0}} |\nabla u| \\
\leq \int_{B(z_0, R) \setminus B(z_0, r)} |\nabla u| \\
\leq \left( \int_{B(z_0, R) \setminus B(z_0, r)} |\nabla u|^2 \right)^{1/2} R^{n+1} \\
\leq \left( \sum_{0 \leq j \leq \log_2 (R/r)} \int_{2^j r \leq |z_0 - X| < 2^{j+1} r} |\nabla u(X)|^2 \right)^{1/2} R^{n+1} \\
\leq \inf_{E} N^\alpha_{\ast} u \cdot \left( \sum_{0 \leq j \leq \log_2 (R/r)} (2^j r)^{n-1} \right)^{1/2} R^{n+1} \\
\leq \inf_{E} N^\alpha_{\ast} u \cdot R^{n-1} R^{n+1} \\
\leq R^n M(N^\alpha_{\ast} u)(x)
\]
for every $x \in Q_0$. Combining the calculations and the cases gives us the desired $C_D$ bound.

5.3.2. $E$ is an unbounded set

Suppose then that $\text{diam}(E) = \infty$. We fix a sequence of cubes $Q_i \in \mathbb{D}(E)$, $i \in \mathbb{N}$, such that $\bigcup_i Q_i = E$ and $Q_i \subseteq Q_{i+1}$ and $\ell(Q_i) < \gamma_0 \ell(Q_{i+1})$ for every $i$, where we fix the value of the constant $\gamma_0$ later. We set
\[
W_1 := T_{Q_1}, \quad W_k := T_{Q_k} \setminus T_{Q_{k-1}}
\]
and
\[
\varphi_k := 1_{W_k} \varphi_{Q_k}, \quad \varphi := \sum_k \varphi_k.
\]
Here $\varphi_{Q_k}$ is the function constructed in Section 5.1 for the cube $Q_k$. The sets $W_k$ cover the whole space $\Omega$ and since $T_{Q_i} \subseteq T_{Q_{i+1}}$ for every $i$, they are also pairwise disjoint. Let us consider the pointwise bound for $N_{\ast}(u - \varphi)$. Fix $x \in E$ and let $Q_m$ be the smallest of the previously chosen cubes such that $x \in Q_m$. Now, if $\Gamma(x) \cap T_{Q_j} = \emptyset$ for every $j = 1, 2, \ldots, m-1$, then the pointwise bound follows directly from part (2) of Lemma 5.2. Suppose then
that there exists a point \( Y \in \Gamma(x) \cap T_{Q_j} \) for some \( j < m \). We may assume that \( Y \notin T_{Q_i} \) for all \( i < j \). By the structure of the sets, there exist now cubes \( P_1 \subset Q_m \) and \( P_2 \subset Q_j \) such that \( \ell(P_1) \approx \ell(P_2) \), \( \text{dist}(P_1, P_2) \lesssim \ell(P_1) \), \( Y \in U_{P_1} \cap U_{P_2} \) and \( \varphi(Y) = \varphi|_{U_{P_2}}(Y) \). By the considerations in the proof of part (1) of Lemma 5.2, we know that \( |u(Y) - \varphi(Y)| \lesssim \varepsilon M_D(N_*u)(P_2) \). By the properties of \( P_1 \) and \( P_2 \), there exists a uniform constant \( \beta_0 \) such that \( P_1 \subset \beta_0 Q \) for any \( Q \in D(E) \) such that \( Q \supset P_2 \). In particular,

\[
\varepsilon M_D(N_*u)(P_2) = \varepsilon \sup_{Q \in D(E), P_2 \subseteq Q} \int_Q N_* u d\sigma \\
\lesssim \varepsilon \sup_{Q \in D(E), P_2 \subseteq Q} \int_{\beta_0 Q} N_* u d\sigma \\
\lesssim \varepsilon M(N_*u)(x).
\]

Thus,

\[
N_*(u - \varphi)(x) = \sup_{Y \in \Gamma(x)} |u(Y) - \varphi(Y)| \\
= \sup_{k \in \mathbb{N}} \sup_{Y \in \Gamma(x) \cap W_k} |u(Y) - \varphi(Y)| \\
\lesssim \varepsilon M_D(N_*u)(x).
\]

Let us then prove the \( C_D \) estimate. We fix a point \( x \in E \) and a cube \( Q \in D(E) \) such that \( x \in Q \) and split the proof to three different cases. Below, \( \beta \) and \( \alpha \) are uniform constants and \( m \) is the smallest such number that \( T_Q \subset T_{Q_m} \).

(1) \( T_Q \subset T_{Q_m} \) such that \( T_Q \cap T_{Q_k} = \emptyset \) for every \( k < m \). Now we simply have

\[
\iint_{T_Q} |\nabla \varphi| = \iint_{T_Q} |\nabla \varphi_m| \lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta Q} N_*^\alpha u d\sigma
\]

by Proposition 5.3.

(2) \( T_Q \subset T_{Q_m} \) and \( Q_k \subset Q \) for every \( k < m \). Take any \( \nabla \Psi \in C_0^1(T_Q) \) with \( \|\nabla \Psi\|_{L^\infty} \lesssim 1 \). We get

\[
\iint_{T_Q} \varphi \text{ div } \nabla \Psi = \iint_{T_Q \setminus T_{Q_{m-1}}} \varphi_m \text{ div } \nabla \Psi + \sum_{i=1}^{m-2} \iint_{T_{Q_{m-i}} \setminus T_{Q_{m-(i+1)}}} \varphi_{m-i} \text{ div } \nabla \Psi \\
+ \iint_{T_{Q_1}} \varphi_1 \text{ div } \nabla \Psi \\
\lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta Q} N_*^\alpha u d\sigma + \sum_{i=1}^{k-1} \frac{1}{\varepsilon^2} \int_{\beta \Delta Q_i} N_*^\alpha u d\sigma
\]
by Remark 5.4. We note that the balls $\beta \Delta Q_i$ form an increasing sequence with respect to inclusion. If we choose the constant $\gamma_0$ to be large enough, the balls $\beta \Delta Q_i$ satisfy a Carleson packing condition independent of $m$. Thus, for a large enough $\gamma_0$, we get

$$
\frac{1}{\varepsilon^2} \int_{\beta \Delta Q} N_\alpha^u d\sigma + \sum_{i=1}^{k-1} \frac{1}{\varepsilon^2} \int_{\beta \Delta Q_i} N_\alpha^u d\sigma \lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta Q} M_D(N_\alpha^u) d\sigma.
$$

by a simple dyadic covering argument and the discrete Carleson embedding theorem (Theorem A.1).

(3) $T_Q \subset T_{Q_m}$, $Q_k \not\subset Q$ for every $k < m$ and $T_Q \cap T_{Q_{m-1}} \neq \emptyset$. Without loss of generality, we may assume that $\ell(Q) \approx \ell(Q_{m-1})$. Take any $\vec{\Psi} \in C^1_0(T_Q)$ with $\|\vec{\Psi}\|_{L^\infty} \leq 1$. We get

\[
\begin{aligned}
\int_{T_Q} \varphi \nabla \cdot \vec{\Psi} &= \int_{T_Q \setminus T_{Q_{m-1}}} \varphi_m \nabla \cdot \vec{\Psi} + \sum_{i=1}^{m-2} \int_{(T_Q \cap T_{Q_{m-1}}) \setminus T_{Q_{m-(i+1)}}} \varphi_{m-i} \nabla \cdot \vec{\Psi} \\
&\quad + \int_{T_Q \cap T_{Q_1}} \varphi_1 \nabla \cdot \vec{\Psi}
\end{aligned}
\]

\[
\lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta Q} N_\alpha^u d\sigma + \sum_{i=1}^{k-1} \frac{1}{\varepsilon^2} \int_{\beta \Delta Q_i} N_\alpha^u d\sigma
\]

by Remark 5.4. Again, if we choose the constant $\gamma_0$ to be large enough, we get

\[
\frac{1}{\varepsilon^2} \int_{\beta \Delta Q} N_\alpha^u d\sigma + \sum_{i=1}^{k-1} \frac{1}{\varepsilon^2} \int_{\beta \Delta Q_i} N_\alpha^u d\sigma \lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta Q} M_D(N_\alpha^u) d\sigma
\]

by a simple dyadic covering argument and the discrete Carleson embedding theorem (Theorem A.1).

Since $\sigma(Q) \approx \sigma(\beta \Delta Q)$, combining the three cases gives us

\[
\begin{aligned}
\frac{1}{\sigma(Q)} \int_{T_Q} |\nabla \varphi| &\lesssim \frac{1}{\varepsilon^2} \frac{1}{\sigma(Q)} \int_{\beta \Delta Q} M_D(N_\alpha^u) d\sigma \\
&\lesssim \frac{1}{\varepsilon^2} M(M_D(N_\alpha^u))(x)
\end{aligned}
\]

for almost every $x \in Q$. This completes the proof of Theorem 1.5.
Appendix A. Discrete Carleson embedding theorem

For the convenience of the reader, we prove here the version of the Carleson embedding theorem that we used in Section 5.3.2.

**Theorem A.1.** — Suppose that $\mu$ is a locally finite doubling Borel measure in a (quasi)metric space $X$ satisfying $\mu(B(x, r)) > 0$ for any $r > 0$ and $\mathbb{D}$ is a dyadic system in $X$. Let $f \geq 0$ be a locally integrable function. If $A \subset \mathbb{D}$ is a collection that satisfies a Carleson packing condition with a constant $\Lambda \geq 1$, then

$$\sum_{Q \in A, Q \subset Q_0} \int_Q f \, d\mu \leq \Lambda \int_{Q_0} M_{\mathbb{D}} f \, d\mu$$

for any $Q_0 \in \mathbb{D}$.

**Proof.** — For every $m \in \mathbb{Z}$, we define the averaging operator $T_m$ by setting

$$T_m f(x) = \sum_{Q \in \mathbb{D}, \ell(Q) = 2^{-m}} 1_Q(x) \int_Q f \, d\mu,$$

and we define the measure $\nu$ by setting

$$d\nu(x, m) = \left( \sum_{Q \in A, \ell(Q) = 2^{-m}} 1_Q(x) \right) d\mu(x).$$

Now we have

$$\sum_{Q \in A, Q \subset Q_0} \int_Q f \, d\mu = \sum_{Q \in A, Q \subset Q_0} \mu(Q) \int_Q f \, d\mu$$

$$= \sum_{m: 2^{-m} \leq \ell(Q_0)} \sum_{Q \in A, \ell(Q) = 2^{-m}} \int_{Q_0} 1_Q \left( \int_Q f \right) d\mu$$

$$= \sum_{m: 2^{-m} \leq \ell(Q_0)} \int_{Q_0} T_m f(x) d\nu(x, m)$$

$$= \int_0^{\infty} \nu(E_{\lambda}^*) d\lambda,$$

where $E_{\lambda}^* := \{(x, m) : x \in Q_0, 2^{-m} \leq \ell(Q_0), T_m f(x) > \lambda\}$. Thus, to prove the claim, we only need to show that $\nu(E_{\lambda}^*) \leq \Lambda \mu(E_{\lambda})$, where $E_{\lambda} := \{x \in Q_0 : \sup_m T_m f(x) > \lambda\}$. If $\mu(E_{\lambda}) = \infty$, the claim is trivial. Thus, we may assume that $\mu(E_{\lambda}) < \infty$. 


We notice that if \( x \in E_\lambda \), then there exists a subcube \( Q' \subset Q_0 \) such that \( x \in Q' \) and \( \int_{Q'} f \, d\mu > \lambda \). By the definition of \( T_m \), we also have \( y \in E_\lambda \) for every \( y \in Q' \). In particular, we have maximal disjoint subcubes \( R_j \subset Q_0 \) such that \( E_\lambda = \bigcup_j R_j \). We further observe the following two things:

1. If \( x \in Q_0 \setminus \bigcup_j R_j \), then by the maximality of the cubes \( R_j \) we have \( \sup_m T_m f(x) \leq \lambda \).

2. If \( x \in Q \subset Q_0 \) and \( T_m f(x) > \lambda \) for some \( m \) such that \( 2^{-m} > \ell(Q) \), then there exists a cube \( \tilde{Q} \supseteq Q \) such that \( \int_{\tilde{Q}} f \, d\mu > \lambda \). In particular, \( Q \subset E_\lambda \) but \( Q \) is not a maximal cube.

Based on these observations, we have

\[
E^*_\lambda \subset \bigcup_j R_j \times \{ m : 2^{-m} \leq \ell(R_j) \}.
\]

By the Carleson packing condition, we get

\[
\nu(R_j \times \{ m : 2^{-m} \leq \ell(R_j) \}) = \sum_{m: 2^{-m} \leq \ell(R_j)} \sum_{Q' \subset R_j, Q' \in A} \mu(Q') \leq \Lambda \mu(R_j)
\]

for every \( j \). In particular, since the cubes \( R_j \) are disjoint, we get

\[
\nu(E^*_\lambda) \leq \sum_j \nu(R_j \times \{ m : 2^{-m} \leq \ell(R_j) \}) \leq \sum_j \Lambda \mu(P_j) = \Lambda \mu(E_\lambda),
\]

which completes the proof. \( \square \)

**BIBLIOGRAPHY**


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