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SPECIALNESS AND ISOTRIVIALITY FOR REGULAR ALGEBRAIC FOLIATIONS

by Ekaterina AMERIK & Frédéric CAMPANA (*)

Dedicated to Jean-Pierre Demailly on the occasion of his 60th birthday

ABSTRACT. — We show that an everywhere regular foliation \mathcal{F} on a quasi-projective manifold, such that all of its leaves are compact with semi-ample canonical bundle, has isotrivial family of leaves when the orbifold base of this family is special. The specialness condition means that for any $p > 0$, the p -th exterior power of the logarithmic extension of its conormal bundle does not contain any rank-one subsheaf of maximal possible Kodaira dimension p . This condition is satisfied, for example, in the very particular case when the Kodaira dimension of the determinant of the logarithmic extension of the conormal bundle vanishes. Motivating examples are given by the “algebraically coisotropic” submanifolds of irreducible hyperkähler projective manifolds.

RÉSUMÉ. — Nous montrons l’isotrivialité des feuilles d’un feuilletage \mathcal{F} partout régulier et à feuilles compactes sur une variété quasi-projective lorsque la base orbifold de la famille des feuilles est spéciale. Cette dernière condition signifie que, pour tout $p > 0$, la puissance extérieure p -ième de l’extension logarithmique du fibré conormal de \mathcal{F} ne contient aucun sous-faisceau de rang un de dimension de Kodaira maximale p . Cette condition est satisfaite, par exemple, dans le cas très particulier où la dimension de Kodaira du déterminant de l’extension logarithmique du fibré conormal est nulle. Des exemples de cette situation sont fournis par les sous-variétés « algébriquement coisotropes » des variétés hyperkählériennes irréductibles projectives.

1. Introduction

Smooth algebraic families of canonically polarized manifolds, or, more generally, polarized manifolds with semi-ample canonical bundle over a smooth quasiprojective base have been intensively studied in recent years,

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starting from the work by Viehweg and Zuo [18]. Their main result ([18, Theorem 1.4]) states that if $f : X \rightarrow B$ is such a family and \bar{B} a smooth compactification such that the complement S of B in \bar{B} is a normal crossing divisor, then some symmetric power of the log-cotangent bundle of B has an invertible subsheaf whose Kodaira dimension is at least the number of moduli $\text{Var}(f)$ of the family⁽¹⁾. Viehweg conjectured that the base of a family of canonically polarized manifolds of maximal variation ($\text{Var}(f) = \dim(B)$) must be of log-general type. This conjecture is established in [5] (but see [6] for a simpler argument).

A more general conjecture was stated in [4], asserting that such a family is isotrivial (that is, $\text{Var}(f) = 0$) if B is *special*, which roughly means that B does not admit a map onto a positive-dimensional “orbifold” of general type. We do not recall the precise definition of a special quasi-projective manifold in this introduction, and just mention that (\bar{B}, S) is special if its log-Kodaira dimension is zero. This isotriviality conjecture implies that the moduli map factors through the “core map” (see [4]), and so the variation can be maximal only if the core map is the identity map on B , which is then of log-general type.

The isotriviality conjecture has been proved by Jabbusch and Kebekus in dimensions two and three ([12], [11]). B. Taji ([16]) proved it in general, using [5]. A simplified version of Taji’s proof, based on [6], can be found in [7].

We consider here, more generally, the case when the family $f : X \rightarrow B$ is not smooth but only quasi-smooth, that is, has only multiple fibers with smooth reduction as singularities; B may then acquire quotient singularities.

Such is the case when there is a smooth foliation \mathcal{F} on X such that its leaves are fibers of f . The base B then carries a natural orbifold structure coming from the multiple fibers and one can ask whether the specialness of the orbifold base again implies the isotriviality of the family. The definition of the specialness of the orbifold base in this (mildly) singular context is part of the problem.

In this paper we give two equivalent definitions of the specialness of the orbifold base: first as a property of the relative cotangent of the foliation in Section 4, and then via multiple fibres of fibrations (in the spirit of [4]) in Section 9. Using Viehweg–Zuo sheaves and [6], we prove that if X is a connected quasi-projective complex manifold with an everywhere regular

⁽¹⁾In the semi-ample case [18] has an additional requirement that $\text{Var}(f)$ is maximal, but the general case reduces to this by the argument of [12, Theorem 3.4].

foliation \mathcal{F} with compact leaves which have semi-ample canonical bundle, then the family of its leaves is isotrivial provided that its orbifold base is special. The first step of the proof, in Section 8, is a “tautological” base change which produces a family with non-multiple smooth fibres.

This isotriviality statement should hold for more general fibres, probably when their canonical bundle is pseudo-effective, as the work of Popa and Schnell [15] seems to indicate.

It would also be interesting to extend our main result to the case when X is a complement to a proper subvariety in a compact Kähler manifold. Such manifolds are sometimes called quasi-Kähler by a slight abuse of terminology (indeed, quasi-Kähler manifolds are Kähler). Much of our preliminary discussion is made in the quasi-Kähler case since this seems to be a natural setting.

It is a great pleasure for us to dedicate this paper to Jean-Pierre Demailly. The methods he has developed are important for explicit construction of the Viehweg–Zuo sheaves (see [2]), and his theorem [8] provides a potential source of examples or counterexamples.

2. Regular Algebraic Foliations. Compactification

Let X be a connected complex manifold of complex dimension n , and \mathcal{F} an everywhere regular holomorphic foliation on X . Recall that this means that $\mathcal{F} \subset TX$ is an involutive subbundle, say of rank r , $0 < r < n$. The foliation \mathcal{F} is called *compact*, or *algebraically integrable*, or *algebraic* for short if all of its leaves are compact⁽²⁾. It is well-known that if X is Kähler, the holonomy group of each leaf of a compact foliation \mathcal{F} is finite (see [9], or [14], Corollary, in the case of a non-compact X).

In this paper X is always assumed quasi-Kähler, that is, a complement to a proper subvariety in a compact Kähler manifold \bar{X} , and \mathcal{F} is assumed “algebraic” in the preceding sense. In fact nearly all our results are valid only for a quasiprojective X , but the natural setting seems to be complex analytic. In the C^∞ category, Reeb stability theorem asserts that locally around a compact fiber F with finite holonomy group G and a local transverse T , X is the quotient of $\tilde{F} \times T$, where \tilde{F} is the G -covering of F , by the diagonal action of G , and so the leaves of the foliation are the fibers of a map f which is just the projection to T/G . In the holomorphic situation,

⁽²⁾ the term “algebraic” refers to the Zariski topology as opposed to the metric one. When X is quasi-projective, the associated fibration f defined in this section is projective.

the complex structure on the neighbouring fibers varies, so that X locally around a compact fiber with finite holonomy is a “quotient of a submersion” rather than quotient of a product. There is still a local transverse T with the action of the holonomy group G and the projection f of a saturated neighbourhood of a fiber in X to T/G , with the leaves of \mathcal{F} as fibers (see [10, Theorem 2.4]).

When X is quasi-Kähler, the holonomy group of each leaf of \mathcal{F} is finite, and the existence and compactness of the components of the Chow–Barlet space of the compactification of X implies the existence of a proper and connected holomorphic fibration $f : X \rightarrow B$ onto an irreducible normal complex space B of dimension $n - r$ whose reduced fibres F_b , $b \in B$, are exactly the leaves of f (see for example [1]; the local fibrations around leaves with trivial holonomy glue together to give a map to a component of the Barlet space which by compactness is extended to the whole of X). Conversely, any such fibration $f : X \rightarrow B$ defines an algebraic (everywhere regular) foliation \mathcal{F} which is the saturation of the kernel of df in the tangent bundle T_X .

We define the multiplicity of a fibre F_b in the cycle-theoretic sense, that is, as the number of intersection points of a local transverse T to F_b with a general neighbouring fibre. Clearly this is the same as the order of the holonomy group of F_b .

Our fibration is “orbi-smooth” in the sense that all of its fibres have smooth reduced support, and B has quotient singularities (see [1] for details). Over the complement to a codimension-two subset in B , the map f locally at the point $x \in F_b$ can be written as $(z_1, z_2, \dots, z_{n-r}, \dots, z_n) \mapsto (z_1^m, z_2, \dots, z_{n-r})$ where $m = m_b$ is the multiplicity of F_b , and this multiplicity is also that of the fiber seen as an analytic space (or a scheme in the quasiprojective setting).

We next choose a smooth compactification (\bar{X}, \bar{D}) such that $\bar{D} = \bar{X} - X$ is a simple normal crossing divisor. The compactness of the components of the Chow–Barlet space of analytic cycles on \bar{X} implies that the fibration f extends to a holomorphic fibration $\bar{f} : \bar{X} \rightarrow \bar{B}$ with \bar{B} normal, and such that $\bar{D} = \bar{f}^{-1}(E)$, where $E = \bar{B} - B$ is a divisor on \bar{B} .

Our aim is to give criteria under which an algebraic foliation is isotrivial, that is, all of its generic leaves are isomorphic. Our main result is Theorem 5.1 below. We assume that X is quasiprojective, so that the leaves of \mathcal{F} are naturally polarized, and that the leaves of \mathcal{F} have semi-ample canonical bundle. The criterion we give is expressed in terms of specialness (see

Section 4) of the log-conormal sheaf of \mathcal{F} , which we define in the next section. This property will be shown to be equivalent in Section 9 to another, more geometric property: the specialness of the orbifold base (B, D_B) of the fibration f , defined in Section 9.

3. The log-conormal sheaf of \mathcal{F}

Let \mathcal{F} be an everywhere regular foliation on the connected quasi-Kähler manifold X . Let $\bar{X}, B, \bar{B}, f, \bar{f}$ be as above.

Define the rank r subbundle $\Omega_{X/\mathcal{F}}^1 \subset \Omega_X^1$ as the kernel of the quotient map $\Omega_X^1 \rightarrow \mathcal{F}^* := \Omega_{\mathcal{F}}^1$. Equivalently, $\Omega_{X/\mathcal{F}}^1 = f^*(\Omega_{B^{\text{reg}}}^1)^{\text{sat}}$, the saturation taking place in Ω_X^1 . This bundle is called the conormal bundle of \mathcal{F} . It is also the saturation inside Ω_X^1 of $f^*(\Omega_{B^{\text{reg}}}^1)$ (where B^{reg} denotes the smooth part of B).

On the compactification \bar{X} , we define the extension $\Omega_{\bar{X}/\bar{\mathcal{F}}}^1$ as $(\bar{f}^*(\Omega_{\bar{B}^{\text{reg}}}^1))^{\text{sat}}$. Here the saturation is taken in the logarithmic cotangent bundle $\Omega_{\bar{X}}^1(\text{Log}(\bar{D}))$. In general, we extend sheaves to the compactification by systematically considering their saturations in a suitable larger locally free sheaf. The reason is that a saturated subsheaf of a locally free (or, more generally, reflexive) sheaf is normal (see for example [13, Lemma 1.1.16]), so that Hartogs' theorem applies to prove the birational invariance of appropriate spaces of sections.

So for any $m \geq 0$, we define $(\otimes^m \Omega_{\bar{X}/\bar{\mathcal{F}}}^1)^{\text{sat}}$ as the saturation of $\otimes^m \Omega_{\bar{X}/\bar{\mathcal{F}}}^1$ inside $\otimes^m (\Omega_{\bar{X}}^1(\text{Log}(\bar{D})))$, and similarly for $\text{Sym}^m (\Omega_{\bar{X}/\bar{\mathcal{F}}}^1)^{\text{sat}}$.

To avoid heavy notation, we define $\Omega_{\bar{X}/\bar{\mathcal{F}}}^p$ as being already saturated: $\Omega_{\bar{X}/\bar{\mathcal{F}}}^p := (\wedge^p (\Omega_{\bar{X}/\bar{\mathcal{F}}}^1))^{\text{sat}}, \forall p \geq 0$, where the saturation takes place in the locally free sheaf of logarithmic p -forms. By Hartogs' theorem, the space of sections of $\Omega_{\bar{X}/\bar{\mathcal{F}}}^p$ does not depend on the choice of the compactification.

DEFINITION 3.1. — *For a non-singular algebraic foliation \mathcal{F} on a quasi-Kähler X together with a suitable Kähler compactification $\bar{f} : \bar{X} \rightarrow \bar{B}$, $\Omega_{\bar{X}/\bar{\mathcal{F}}}^1$ is called the Log-conormal sheaf of \mathcal{F} .*

The properties of the conormal sheaf we are interested in will be likewise independent on the chosen compactifications.

Let now $g : B \dashrightarrow Y$ be a dominant rational map, extended to a rational map $\bar{g} : \bar{B} \dashrightarrow \bar{Y}$ on compactifications. We assume throughout the paper that Y and \bar{Y} are smooth (without loss of generality since g is only a

rational map; note that B usually has some singularities). Let $h = g \circ f$, $\bar{h} = \bar{g} \circ \bar{f} : \bar{X} \rightarrow \bar{Y}$. Set $\dim(Y) = p$, $0 \leq p \leq \dim(B) = n - r$.

The map h induces a natural inclusion $h^* \mathcal{O}(K_Y) \subset \Omega_{X/\mathcal{F}}^p := \wedge^p \Omega_{X/\mathcal{F}}^1$, as well as extensions $\bar{h}^* : \otimes^m \Omega_{\bar{Y}}^1 \subset \otimes^m \Omega_{\bar{X}/\bar{\mathcal{F}}}^1, \forall m \geq 0$.

We consider the saturated inverse images by \bar{h}^* of pluridifferentials on \bar{Y} : $(\bar{h}^*(\otimes^m \Omega_{\bar{Y}}^1))^{\text{sat}} \subset (\otimes^m \Omega_{\bar{X}/\bar{\mathcal{F}}}^1)^{\text{sat}}$, and analogously for the sheaf of symmetric differentials and $\Omega_{\bar{Y}}^p$.

The Hartogs' lemma gives the following:

LEMMA 3.2. — *For any $g : B \dashrightarrow Y$, and any $m \geq 0$, $h^0(\bar{X}, (\bar{h}^*(\otimes^m \Omega_{\bar{Y}}^1))^{\text{sat}})$ does not depend on the choices of $\bar{X}, \bar{D}, \bar{B}$. The same property holds for $h^0(\bar{X}, (\bar{h}^*(\text{Sym}^m \Omega_{\bar{Y}}^1))^{\text{sat}})$ and $h^0(\bar{X}, (\bar{h}^* \Omega_{\bar{Y}}^p)^{\text{sat}}), \forall p > 0$.*

DEFINITION 3.3. — *Let \bar{X}, \bar{D} be as above, and $\mathcal{L} \subset \otimes^m \Omega_{\bar{X}}^1(\text{Log}(\bar{D}))$ be a rank-one coherent subsheaf.*

Define: $\kappa^{\text{sat}}(\bar{X}, \mathcal{L}) := \limsup_{k \rightarrow +\infty} \frac{\text{Log}(h^0(\bar{X}, \mathcal{L}^{\otimes k, \text{sat}}))}{\text{Log } k}$, the saturation $\mathcal{L}^{\otimes k, \text{sat}}$ of $\mathcal{L}^{\otimes k}$ being taken in $\otimes^{mk} \Omega_{\bar{X}}^1(\text{Log}(\bar{D}))$.

By the same principle as in 3.2, we see that $\kappa^{\text{sat}}(\bar{X}, \mathcal{L})$ is independent from the birational model (\bar{X}, \bar{D}) chosen; more precisely, $\kappa^{\text{sat}}(\bar{X}, \mathcal{L})$ is equal to the κ^{sat} of the direct or inverse image of \mathcal{L} on a modification of (\bar{X}, \bar{D}) .

It therefore makes sense to consider the restriction of \mathcal{L} to X and talk of $\kappa^{\text{sat}}(X, \mathcal{L})$.

Finally, as usual, κ^{sat} of a divisor means κ^{sat} of the associated line bundle.

We shall also need the following elementary lemma.

LEMMA 3.4. — *Let $\bar{h} : \bar{X} \dashrightarrow \bar{Y}$ be a meromorphic, dominant, and connected fibration with $p = \dim(\bar{Y}) > 0$. Then $\bar{h}^*(\mathcal{O}(K_{\bar{Y}})) \subset \Omega_{\bar{X}/\bar{\mathcal{F}}}^p$ (as subsheaves of $\Omega^p(\text{Log } D)$) if and only if \bar{h} factors through \bar{f} (i.e. if there exists $\bar{g} : \bar{B} \dashrightarrow \bar{Y}$ such that $\bar{h} = \bar{g} \circ \bar{f}$).*

Proof. — In restriction to the open part of X where h is defined and submersive this is classical, and the statement over the compactification follows from the fact that $\Omega_{\bar{X}/\bar{\mathcal{F}}}^p$ is saturated, so that the inclusion over the open part implies the inclusion. \square

4. Specialness

DEFINITION 4.1. — *We say that the orbifold base of f is special if, for every connected dominant rational map $g : B \dashrightarrow Y$ with Y smooth,*

and $\dim(Y) = p > 0$, we have: $\kappa^{\text{sat}}(\overline{X}, \overline{h}^*(K_{\overline{Y}})) < p$ (by Lemma 3.4, the saturation can be taken inside $\Omega_{\overline{X}/\overline{\mathcal{F}}}^p$). This is independent from the choice of $\overline{X}, \overline{D}$, by Lemma 3.2.

The term “orbifold base of f ” will be justified in Section 9, accordingly to the terminology of [4]. Since some additional technicalities are needed, we prefer to introduce some of our results in this and the following section and postpone the proofs until later.

The specialness property will be shown in Theorem 9.18 to be equivalent to other, apparently stronger properties:

THEOREM 4.2. — *The specialness of the orbifold base of f is equivalent to each of the following properties:*

- (1) for any $p > 0$, and any coherent rank-one subsheaf $\mathcal{L} \subset \Omega_{\overline{X}/\overline{\mathcal{F}}}^p$, one has $\kappa^{\text{sat}}(\overline{X}, \mathcal{L}) < p$;
- (2) for any $g : B \dashrightarrow Y$, connected dominant rational map $g : B \dashrightarrow Y$ with Y smooth, and $\dim(Y) = p > 0$, we have: $\kappa^{\text{sat}}(\overline{X}, \mathcal{L}) < p$ for any line bundle $\mathcal{L} \subset \otimes^m \overline{h}^*(\Omega_{\overline{Y}}^1)$, where $\overline{h} : \overline{B} \dashrightarrow \overline{Y}$ is an extension of h to a smooth compactification \overline{Y} of Y .

An important, although very particular example, of specialness holds, is given by the following.

THEOREM 4.3. — *Assume that $\kappa(\overline{X}, \det(\Omega_{\overline{X}/\overline{\mathcal{F}}}^1)) = 0$, the orbifold base of f is then special.*

The proof follows from Theorem 9.19.

5. Isotriviality criterion

We can now formulate our main result.

THEOREM 5.1. — *Let $f : X \rightarrow B$ be the fibration associated to an algebraic and everywhere regular foliation \mathcal{F} on a connected quasi-projective manifold X . Assume that the fibres of f have semi-ample canonical bundle, and that the orbifold base of f is special. Then f is isotrivial.*

This answers positively a question raised in [1] for X quasi-projective (instead of quasi-Kähler there). It is likely that the quasi-Kähler case can be handled by similar arguments, once one constructs Viehweg–Zuo sheaves

in that setting⁽³⁾. The case when X is quasi-projective and f is submersive was established in [16].

For the proof of this theorem, we actually work with another definition of specialness, in terms of the orbifold pairs as in [4]: indeed the base B is equipped with a natural orbifold structure. In this context, Theorem 5.1 becomes Corollary 9.21. The equivalence between various characterisations of specialness is given in Theorem 9.18 and Corollary 9.20, and the connection to the conormal bundle of the foliation is through the Lemma 3.4.

COROLLARY 5.2. — *Let $f : X \rightarrow B$ be the fibration associated to an algebraic and everywhere regular foliation \mathcal{F} on the connected quasi-projective manifold X . Assume that the fibres of f have semi-ample canonical bundle, and that $\kappa(\overline{X}, \det(\Omega_{X/\overline{\mathcal{F}}}^1)) = 0$. Then f is isotrivial.*

6. Viehweg–Zuo sheaves

Let again $f : X \rightarrow B$ be the fibration associated to an everywhere regular and algebraic foliation \mathcal{F} on a connected quasi-projective manifold X . We assume here that its fibres have semi-ample canonical bundle and Hilbert–Samuel polynomial P with respect to the polarization coming from X . Let Mod_P be the quasi-projective scheme constructed in [17], parametrising the manifolds which are polarised with Hilbert–Samuel polynomial P . If $B^* \subset B$ is the (non-empty) Zariski open subset of points over which f is submersive, there is a natural holomorphic map $\mu^* : B^* \rightarrow \text{Mod}_P$ sending b to the isomorphism class of F_b .

Its image M is an algebraic variety of dimension $\text{Var}(f) \in \{0, 1, \dots, (n-r) = \dim(B)\}$, where $\text{Var}(f)$ is the generic rank of the Kodaira–Spencer map $KS : T_{B^*} \rightarrow R^1 f_*(T_{X/B})$.

When f is submersive, $B^* = B$, $\mu^* = \mu$, and B is smooth. We can then choose compactifications such that \overline{B} is smooth, and $S := \overline{B} - B$ is of simple normal crossings.

We have the following result of Viehweg and Zuo ([18, Theorem 1.4(iii)], also (i)).

THEOREM 6.1. — *Assume that $f : X \rightarrow B$ is submersive and $\text{Var}(f) = \dim(B)$. There exists a line bundle $\mathcal{L} \subset \text{Sym}^m(\Omega_{\overline{B}}^1(\text{Log}(S)))$ such that $\kappa(\overline{B}, \mathcal{L}) = \text{Var}(f) = \dim(M) = \dim(B)$.*

⁽³⁾ The case when X is compact Kähler and \mathcal{F} is of rank 1 was treated in the first version of [1], but was removed in the final version after a simplification of the proof of its main result.

A refinement of Theorem 6.1 by Jabbusch and Kebekus ([12, Theorem 1.4] states that this \mathcal{L} actually comes from the moduli space: $\mathcal{L} \subset \text{Sym}^m(\mu^*(\Omega_M^1))^{\text{sat}}$ (by abuse of notation, we write μ^* for the image of $d\mu$; cf. Section 3). We call such an \mathcal{L} a *Viehweg–Zuo sheaf*. Combining the argument of 6.1 and [12, Theorem 3.4], one may remove the maximal variation condition. Indeed in [12, Theorem 3.4], the general case is deduced from the maximal variation case for families of canonically polarized manifolds, but the only property used in the proof is the finiteness of the polarized automorphism group of the fibres, that is, the subgroup of the automorphism group which preserve an ample line bundle (the canonical bundle in the context of [12]). This finiteness property also holds for polarized manifolds with semi-ample canonical bundle, and our fibres can be globally polarized, since X is quasi-projective.

In our setting of a fibration defined by a foliation, f is not necessarily submersive. However, by assumption, the singular fibers of f are multiple fibres with smooth reduction. Equivalently, the non-smoothness of the fibration is encoded in the finite, but nontrivial holonomy groups around the leaves of \mathcal{F} . In the next two sections, we deal with this problem, recalling the Reeb stability theorem and providing a simple base-change to eliminate the multiple fibres (those with non-trivial holonomy group). The new base then carries a Viehweg–Zuo sheaf. In Section 9 we descend this sheaf to the orbifold base of the original fibration and derive a contradiction with specialness in the non-isotrivial case.

7. Reeb Stability Theorem

Let again \mathcal{F} be a regular algebraic foliation on a Kähler manifold X . We know that all its holonomy groups are finite. In the C^∞ category, Reeb stability theorem asserts that locally around a fiber F with holonomy group G and a local transverse T , X is the quotient of $\tilde{F} \times T$, where \tilde{F} is the G -covering of F , by the diagonal action of G , and the map f is the projection to T/G . In the holomorphic situation, the complex structure on the neighbouring fibers varies; however there is the following adaptation of Reeb stability (see [10, Theorem 2.4]). let G_b be the (finite) holonomy group of \mathcal{F} along a fibre $F_b = f^{-1}(b)$, $b \in B$. There exist an open neighborhood $b \in U \subset B$ and a finite Galois covering $\beta : U' \rightarrow U = U'/G_b$, such that the normalisation $X_{U'}$ of the fibered product $X_U \times_U U'$, where X_U stands for $f^{-1}(U)$, is a G_b -étale covering of X_U and submersive over U' . The map $\beta : U' \rightarrow U$ is obtained by taking a smooth holomorphic local transverse

to (reduced) F_b ; over a sufficiently small $U \subset B$ containing b it is finite surjective.

Since the second projection $f' : X_{U'} \rightarrow U'$ is submersive, it is C^∞ -equivalent to a product, so in the C^∞ context one finds back the usual Reeb stability theorem. In particular, all fibres of f are, up to finite étale equivalence, isomorphic as C^∞ -manifolds.

8. Elimination of multiple fibres by base-change

Our generalisation is based on a simple trick (already introduced in [1] for fibrations in curves, but the general case is similar) which eliminates multiple fibres.

Let (X, \mathcal{F}) be as above, \mathcal{F} algebraic and everywhere regular. Let $f : X \rightarrow B$ be the associated fibration and $\tilde{f} : X \times_B X \rightarrow X$ the projection of the fibered product to the second factor. Let $f_X : X_X \rightarrow X$ be the fibration deduced from $f : X \rightarrow B$ by the base-change $\beta(= f) : X \rightarrow B$ and subsequent normalisation. We thus have: $f_X = \tilde{f} \circ \nu$, where $\nu : X_X \rightarrow X \times_B X$ is the normalisation map.

LEMMA 8.1. — *In the above situation, the fibration $f_X : X_X \rightarrow X$ is submersive.*

Proof. — The proof is exactly the same as that of the Lemma 2.11 of [1] (in the setting of [1], the foliation is of rank one, but the argument goes through in general).

The fibration $F : X \times_B X \rightarrow X$ has a natural section given by the diagonal of X , and the inverse image of this section on the normalisation X_X of $X \times_B X$ has a unique component lying over X which gives a section of the map $f_X : X_X \rightarrow X$. From Reeb stability one sees that f_X still comes from a foliation on a smooth X_X , but now all holonomy groups are trivial, since the section is a local transverse at every point. The details are checked by a local argument, which runs as follows: for any $x \in X$, there is a neighbourhood of $x \in X$ isomorphic to $U' \times F$, where F is a neighbourhood of x in its leaf and U' is a local transverse. By Reeb stability, a small neighbourhood U of $b = f(x)$ in B is U'/G where G is the holonomy group, and the normalization $X_{U'}$ of $X \times_U U'$ is smooth over U' and étale over $f^{-1}(U)$. Hence X_X , which locally in a neighbourhood of x is isomorphic to the normalization of $X \times_U (U' \times F)$, that is, $X_{U'} \times F$, is also smooth over X : the projection to X , locally around x , is a composition of the projection to $X_{U'}$ with the étale map from $X_{U'}$ to $f^{-1}(U)$. \square

LEMMA 8.2. — *In the above situation, the map $\mu : X \rightarrow \text{Mod}$ defined in Section 6 factors through B .*

Proof. — Let $b \in B$ be any point. Let $b \in U \subset B$ be any sufficiently small neighborhood, and let $\beta : U' \rightarrow U$ be the finite Galois cover of group G defined by a germ of manifold U' transverse to the reduction of the fiber F_b as in Section 7. Base-changing by β and normalising, we obtain $\gamma : X' \rightarrow X$ and $f' : X' \rightarrow U'$, γ being G -Galois and étale, and f' submersive. The map $\mu' : U' \rightarrow \text{Mod}$ is well-defined and coincides with $\mu^* \circ \beta : U' \rightarrow \text{Mod}$, if $\mu^* : B^* \cap U = U^* \rightarrow \text{Mod}$ is defined as in Section 7. Since B is normal and $\beta : U' \rightarrow U$ finite and proper, the map $\mu^* : B^* \rightarrow \text{Mod}$ extends to B as a holomorphic map $\mu : B \rightarrow \text{Mod}$. \square

9. Orbifold geometry

We shall actually prove a more detailed version of Theorem 4.2, namely Theorem 9.18 below. Before this, some notions concerning the geometry of orbifold bases need to be recalled.

9.1. Orbifold bases

We recall the set-up from [3] and [4]. An orbifold pair is a connected normal compact complex-analytic variety Z together with a Weil \mathbb{Q} -divisor $D = \sum_j c_j D_j$ where D_j are the irreducible components and the rational coefficients $c_j \in]0, 1]$. The union $[D] = \cup_j D_j$ is called the *support*, or “round-up” of D . The extreme cases are when $D = 0$, or when $D = [D]$, so $c_j = 1 \ \forall j$.

If $F \subset Z$ is an irreducible Weil divisor *not* contained in $[D]$, we define its coefficient $c_D(F)$ in D to be 0, and we set $c_D(D_j) = c_j$. Thus $D = \sum_F c_D(F) \cdot F$, the sum running over all irreducible Weil divisors F of Z .

We say that the orbifold pair (Z, D) is smooth if Z is smooth and the support of D has only simple normal crossings. If moreover $D = [D]$, we say that we have a *smooth logarithmic pair*.

The purpose of introducing these objects here is to encode (and eliminate in codimension one) the multiple fibres of fibrations by means of “virtual base changes”. The orbifold pair (X, D) , $D = \sum_j c_j D_j$, may indeed be seen as a virtual ramified cover of X ramifying to (rational) order $m_j = (1 - c_j)^{-1} \in]1, +\infty]$ over D_j , at least in codimension 1. The (rational or

$+\infty$) numbers $m_j = m_D(D_j)$ will be called the *multiplicities* of D along the D'_j s. We set $m_D(F) = 1$ if F is not a component of D , so that we still have $m_D(F) = (1 - c_D(F))^{-1}$ in this case.

Conversely, $c_j = (1 - \frac{1}{m_j})$, and $D = \sum_F (1 - \frac{1}{m_D(F)})F$, F running over all irreducible Weil divisors of X .

Alternatively, a pair (X, D) interpolates between the projective case when $D = 0$, and the quasi-projective case when $D = [D]$.

The main example of orbifold pairs (with integral or infinite multiplicities) comes from orbifold bases of fibrations:

DEFINITION 9.1. — *Let $f : Z \rightarrow Y$ be a surjective holomorphic proper map with connected fibres (that is, a fibration) between normal connected complex spaces with \mathbb{Q} -factorial singularities. Fix an orbifold pair structure (Z, D) on Z .*

For each irreducible Weil divisor $E \subset Y$, write $f^(E) = \sum_k t_k F_k + R$, where F_k runs through the irreducible Weil divisors of Z mapped onto E by f , while R consists of the f -exceptional Weil divisors of Z mapped into, but not onto, E .*

Define the multiplicity $m_{f,D}(E)$ relative to D of the generic fibre of f over E by the formula $m_{f,D}(E) = \inf_k \{t_k m_D(F_k)\}$.

The orbifold base $(Y, D_{f,D})$ of f is an orbifold pair where the divisor is defined by the following formula

$$D_{f,D} = \sum_E \left(1 - \frac{1}{m_{f,D}(E)}\right) E$$

where E ranges through the irreducible Weil divisors of Y .

This sum is finite since $m_{f,D}(E) = 1$ unless either $t_k > 1$, or $m_D(F_k) \neq 1$ for all k . If $D = 0$, the multiplicity $m_f(E) = \inf_k \{t_k\}$ is the multiplicity of the fiber over a general point of E as considered in [3].

Sometimes, when the data (f, D) is clear from the context, we shall write simply D_Y rather than $D_{f,D}$.

9.2. Orbifold morphisms

DEFINITION 9.2 ([4]). — *Let $f : X \rightarrow Z$ be a fibration between connected complex manifolds equipped with smooth orbifold structures (X, D) and (Z, D_Z) . We say that f is an orbifold morphism if, for any irreducible divisors $F \subset Z$ and $E \subset X$ such that $f(E) \subset F$, with $f^*(F) = tE + R$ where the support of R does not contain E , one has $tm_D(E) \geq m_{D_Z}(F)$,*

where $m_D(E)$ (resp. $m_{D_Z}(F)$) is the multiplicity of E in D (resp. of F in D_Z).

We shall say that f is an orbifold birational equivalence if moreover f is birational and $f_*(D) = D_Z$. Here f_* is the cycle-theoretic direct image; for a birational map $f_*(\sum c_j D_j) := \sum c_j f_*(D_j)$, where $f_*(D_j) = f(D_j) \subset Z$ if $f(D_j)$ is a divisor, and $f_*(D_j) = 0$ otherwise.

The following two simple situations provide examples. We leave the easy check to the reader.

Example 9.3. — Let $u : (Z', D') \rightarrow (Z, D)$ be a proper bimeromorphic holomorphic map between connected complex manifolds Z', Z , equipped with orbifold divisors D', D such that both orbifolds (Z', D') and (Z, D) are smooth, and moreover $u_*(D') = D$. Assume that all u -exceptional divisors of Z' are equipped with the multiplicity $+\infty$. Then f is an orbifold birational equivalence. If $Z = Z'$, then u is a birational orbifold equivalence if and only if $D' = D$.

Example 9.4. — Let $f : (X, D) \rightarrow (Z, D_Z)$ be as in Definition 9.2 above. Assume that (Z, D_Z) is the orbifold base of f . This does not imply in general that f is an orbifold morphism. This will, however, be the case as soon as the multiplicities in D of the f -exceptional divisors $E \subset X$ are sufficiently large; in particular when all these multiplicities are equal to $+\infty$.

We shall need good bimeromorphic models of fibrations as in the proposition below. These are obtained using Raynaud's flattening theorem and Hironaka's desingularisation.

PROPOSITION 9.5 ([4, Proposition 4.10]). — Let (X_1, D_1) be a smooth orbifold pair, with X_1 projective⁽⁴⁾ connected. Let $h_1 : X_1 \rightarrow Z_1$ be a fibration (or, more generally, a dominant meromorphic map with connected fibers). There exists a commutative diagram:

$$\begin{array}{ccc} (X, D) & \xrightarrow{u} & (X_1, D_1) \\ \downarrow h & & \downarrow h_1 \\ (Z, D_Z) & \xrightarrow{v} & Z_1 \end{array}$$

where u, v are birational, and moreover the following holds:

- (1) $u : (X, D) \rightarrow (X_1, D_1)$ is a birational orbifold morphism.

⁽⁴⁾ Compact Kähler would be sufficient.

- (2) $(X, D), (Z, D_Z)$ are smooth.
- (3) (Z, D_Z) is the orbifold base of $h : (X, D) \rightarrow Z$
- (4) $h : (X, D) \rightarrow (Z, D_Z)$ is an orbifold morphism.
- (5) Every h -exceptional divisor of X is also u -exceptional.

9.3. Smooth orbifold bases of equidimensional fibrations

The notions of morphisms and birational equivalence for orbifold pairs are defined in the preceding subsections only for smooth orbifold pairs. The appropriate definitions are in general not (presently) available in the singular case, and the notion of a resolution of a (normal, quasi-projective, say) orbifold pair is not available either.

The problem is that one could introduce the notion of a smooth model of an orbifold as soon as the underlying manifold is \mathbb{Q} -factorial (and so it makes sense to talk about the pullback of a Weil divisor), but it is not clear whether any two such models are necessarily birational in the orbifold sense (see [4, p. 832–833]).

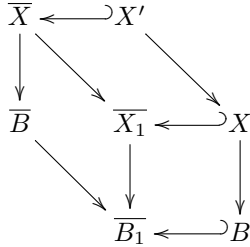
However in the equidimensional case described below, we can construct smooth orbifold pairs $(\overline{B}, \overline{D_{\overline{B}}})$ which are resolutions of compactifications of the quasi-projective pairs (B, D_B) . The important property is that, for a given (B, D_B) , all of these $(\overline{B}, \overline{D_{\overline{B}}})$ are birationally equivalent in the orbifold sense (Corollary 9.11). Roughly speaking, the reason is that we do not need to introduce new exceptional divisors by base change in this particular case. We now give the details.

We consider a *smooth* quasi-projective complex manifold X together with a projective fibration $f : X \rightarrow B$ onto a normal quasi-projective variety B . We assume that f is *equidimensional*, so that its (connected) fibres X_b are all of the same dimension r . In particular, this is the case if f is the family of leaves of an everywhere regular foliation \mathcal{F} on X .

Put the trivial orbifold structure (i.e. the zero divisor) on X and let (B, D_B) be the orbifold base of $f : X \rightarrow B$. Take projective compactifications $\overline{B}_1, \overline{X}_1$ with the following properties: f extends to $\overline{f}_1 : \overline{X}_1 \rightarrow \overline{B}_1$; \overline{X}_1 is smooth; $\overline{D}_1 := \overline{X}_1 - X$ is a simple normal crossing divisor. We call $\overline{f}_1 : \overline{X}_1 \rightarrow \overline{B}_1$ a *compactification* of f .

Next, choose smooth modifications $\overline{X}, \overline{B}$ of $\overline{X}_1, \overline{B}_1$, in such a way that \overline{f}_1 lifts to $\overline{f} : \overline{X} \rightarrow \overline{B}$, and moreover such that $\overline{D}' := \overline{X} - X'$ and $\overline{D}_{\overline{B}} := \overline{B} - B'$ are simple normal crossing divisors, where $X' \subset \overline{X}$ and $B' \subset \overline{B}$ denote the inverse images of X and B in \overline{X} and \overline{B} respectively (for typesetting reasons

B' is not shown on the diagram below):



By further blow-ups of $\overline{X}, \overline{B}$, we may assume that the following two conditions hold.

- (a) Let $E_{\overline{B}}$ be the closure of the exceptional divisor E_B of $\beta : B' \rightarrow B$ and $D_{\overline{B}}$ be the closure of the strict transform D'_B of D_B in \overline{B} , so that $E_{\overline{B}} \cup D_{\overline{B}}$ is the closure of the inverse image of the “old boundary divisor” $D_B \subset B$ in \overline{B} . Then the union of \overline{D}_B and $E_{\overline{B}} \cup D_{\overline{B}}$ is still a simple normal crossing divisor.
- (b) The union of \overline{D}' and of the closure E of the exceptional divisor E' of $\chi : X' \rightarrow X$ is a simple normal crossings divisor.

The simple normal crossing divisors from (a) and (b) are the supports of the orbifold structures we are introducing on \overline{X} and \overline{B} . To do this we need to put multiplicities on their components.

The orbifold structure $(\overline{X}, \overline{D})$ is obtained as follows: we assign the multiplicity $+\infty$ (or equivalently: coefficient 1) to all components of the union in (b), so that \overline{D} is exactly this union.

We equip \overline{B} with the following orbifold divisor $\overline{D}_{\overline{B}}$: its support is the union described in (a), the multiplicities of the exceptional components $E_{\overline{B}}$ and of the border components \overline{D}_B are $+\infty$, while each component of the closure of the strict transform of D_B is assigned its multiplicity in the orbifold base of $f : X \rightarrow B$.

Roughly speaking, the “old” components come with their “old” multiplicities (and so “old” coefficients), whereas the “new” ones acquire infinite multiplicities (and so coefficient one).

The following diagram displays the divisors with finite multiplicities whereas the “logarithmic” part (infinite multiplicities) is implicit:

$$\begin{array}{ccc}
 \overline{X} \supset X' & \xrightarrow{x} & X \\
 \downarrow \bar{f} & & \downarrow f \\
 \overline{B} \supset (B', D_{B'}) & \xrightarrow{\beta} & (B, D_B)
 \end{array}$$

The finite multiplicities are those of D_B and $D_{B'}$, arising from the multiplicities of the fibres of f (in other words, from their finite holonomy groups). All others are infinite: these are the ones on all exceptional divisors E' and E_B for χ, β , as well as on the boundary divisors $\overline{X} - X := \overline{D'}$, and $\overline{B} - B' := \overline{D_B}$.

DEFINITION 9.6. — *Let $f : X \rightarrow (B, D_B)$ be an equidimensional projective fibration with orbifold base (B, D_B) . We suppose that X is smooth and B is normal. Let $\overline{f_1} : \overline{X_1} \rightarrow \overline{B_1}$ be some compactification of f . A compactified resolution of f is $\overline{f} : (\overline{X}, \overline{D}) \rightarrow (\overline{B}, \overline{D_{\overline{B}}})$ where $\overline{f} : \overline{X} \rightarrow \overline{B}$ is a modification of f_1 satisfying the properties (a) and (b), and the orbifold structures \overline{D} and $\overline{D_{\overline{B}}}$ are as we have just described.*

LEMMA 9.7. — *$\overline{f} : (\overline{X}, \overline{D}) \rightarrow (\overline{B}, \overline{D_{\overline{B}}})$ is an orbifold morphism to its orbifold base (smooth by construction).*

Proof. — By the fact that the components of the boundaries $\overline{D}, \overline{D_B}$ of both $\overline{X}, \overline{B}$ are all equipped with infinite multiplicities, it is sufficient to consider only divisors of $\overline{X}, \overline{B}$ which intersect the inverse images of X, B respectively. Because $f : X \rightarrow B$ is equidimensional, the inverse image in \overline{X} of any irreducible divisor $\overline{F} \subset \overline{B}$ which is β -exceptional, where $\beta : B' \rightarrow B$ is the natural birational map, is χ -exceptional, where $\chi : X' \rightarrow X$ is the similar modification. Since all of these exceptional divisors are also equipped with the infinite multiplicity, the inequalities required for \overline{f} to be an orbifold morphism are satisfied for these divisors. The remaining divisors for which these inequalities need to be checked are now the strict transforms in \overline{B} of the components of D_B . But the multiplicities assigned to them being the same ones as in D_B itself, the verification is trivial. \square

Remark 9.8. — Since the closure in \overline{X} of any component C of the exceptional divisor of $X' \rightarrow X$ is, by definition, equipped with the multiplicity $+\infty$, and $f : X \rightarrow B$ has equidimensional fibres, the following properties for a divisor E in \overline{X} which is not contained in $\overline{X} - X'$ are equivalent:

- (1) C is $f \circ \chi$ -exceptional.
- (2) C is equipped with the multiplicity $+\infty$ in \overline{D} .

DEFINITION 9.9. — *Let $f : (X, D) \rightarrow (Z, D_Z)$ and $f' : (X', D') \rightarrow (Z', D_{Z'})$ be fibrations between connected projective manifolds X, Z, X', Z' , equipped with orbifold divisors $D, D_Z, D', D_{Z'}$ respectively. We say that f' dominates f if there exists birational morphisms $u : X' \rightarrow X$, and $v : Z' \rightarrow Z$ such that $v \circ f' = f \circ u$ and $u_*(D') = D, v_*(D_{Z'}) = D_Z$.*

The next lemma is needed to show that all our compactified resolutions are orbifold birationally equivalent.

LEMMA 9.10. — *Let $f : X \rightarrow B$ be an equidimensional fibration with X and B quasi-projective, X smooth and B normal. Let $\bar{f} : (\bar{X}, \bar{D}) \rightarrow (\bar{B}, \bar{D}_{\bar{B}})$ be a compactified resolution of $f : X \rightarrow (B, D_B)$. If $\bar{f}' : (\bar{X}', \bar{D}') \rightarrow (\bar{B}', \bar{D}_{\bar{B}'})$ is another compactified resolution, then:*

- (1) *There exists a third compactified resolution $\bar{f}'' : (\bar{X}'', \bar{D}'') \rightarrow (\bar{B}'', \bar{D}_{\bar{B}''})$ dominating the first two ones.*
- (2) *The domination maps $u : (\bar{X}'', \bar{D}'') \rightarrow (\bar{X}, \bar{D})$ and $v : (\bar{B}'', \bar{D}_{\bar{B}''}) \rightarrow (\bar{B}, \bar{D}_{\bar{B}})$ such that $v \circ \bar{f}'' = \bar{f} \circ u$ are both orbifold birational equivalences, and the same for u', v' .*

Proof. — Let \bar{f} be obtained by a modification of a compactification $\bar{f}_1 : \bar{X}_1 \rightarrow \bar{B}_1$ and \bar{f}' by a modification of a compactification $\bar{f}'_1 : \bar{X}'_1 \rightarrow \bar{B}'_1$. The existence of $\bar{f}'' : (\bar{X}'', \bar{D}'') \rightarrow (\bar{B}'', \bar{D}_{\bar{B}''})$ dominating both \bar{f}, \bar{f}' is obtained by modifying a fibration $\bar{f}''_1 : \bar{X}''_1 \rightarrow \bar{B}''_1$ which compactifies $f : X \rightarrow B$ and dominates both initial compactifications $\bar{f}_1 : \bar{X}_1 \rightarrow \bar{B}_1$ and $\bar{f}'_1 : \bar{X}'_1 \rightarrow \bar{B}'_1$. The fact that u, v, u', v' are orbifold morphisms and thus orbifold birational equivalences now follows from Remark 9.8. \square

COROLLARY 9.11. — *For a given $f : X \rightarrow B$, the smooth pairs $(\bar{B}, \bar{D}_{\bar{B}})$ are all birationally equivalent in the orbifold sense, and may be seen as orbifold resolutions of compactifications of (B, D_B) .*

9.4. Integral parts of orbifold tensors

We recall the construction of orbifold differentials from [4]. Consider a smooth orbifold pair (Z, D) and local analytic coordinates $(z) = (z_1, \dots, z_n)$ near a given point $a \in Z$, centered at a and “adapted” to D , that is such that the support of D is contained in the union of the coordinate hyperplanes in the domain of this chart. Thus D has near a an equation with fractional exponents: $0 = \prod_{j=1}^{j=n} z_j^{c_j}$. This symbolic notation just means that, in the local coordinates (z) , $D = \sum_{j=1}^{j=n} c_j \cdot H_j$, where $H_j, j = 1, \dots, n$ is the coordinate hyperplane of equation $z_j = 0$.

Let $m > 0$ be an integer. We then define $[T^m]\Omega^1(Z, D)$, also written $[\otimes^m]\Omega^1(Z, D)$, as the locally free subsheaf of \mathcal{O}_Z -modules of $\otimes^m \Omega^1_Z(\log([\!|D|]))$ generated by the elements: $z^{-[c_J]} \cdot dz_{j_1} \otimes \dots \otimes dz_{j_m}$. Here J runs over all multi-indices $(j_1, \dots, j_m) \in \{1, \dots, n\}^m$, and $z^{-[c_J]} = z_1^{-[k_1(J)c_1]} \dots z_n^{-[k_n(J)c_n]}$, where, for $j = 1, \dots, n$, $k_j(J)$ is the number of

occurrences of j in J , that is, the number of $k \in \{1, \dots, m\}$ such that $j_k = j$.

So for instance $\frac{dz_1^{\otimes m}}{z_1^{|m \cdot e_1|}}$ is among the generators; if m_1 is the multiplicity of the corresponding component of D this is rewritten as $z_1^{\lceil m/m_1 \rceil} (\frac{dz_1}{z_1})^{\otimes m}$. When the multiplicities are infinite, we obtain the usual logarithmic differentials.

One can easily check that this sheaf is independent from the chosen adapted coordinates, and so well-defined globally. Although we do not use this fact here, let us mention that it is also equal to the G -invariant part $[\pi_*(\otimes^m \pi^* \Omega^1(Z, D))]^G$ of $\pi_*(\otimes^m \pi^* \Omega^1(Z, D))$, where $\pi : Z_D \rightarrow Z$ is any G -Galois Kawamata cover adapted to (Z, D) in the sense of [6]. Here $\pi^* \Omega^1(Z, D)$ is the orbifold differential sheaf of [6]; some sources use a different notation, for instance $\Omega(\pi, D)$ in [7].

The sheaves $[S^m](\Omega^1(Z, D))$ of symmetric orbifold differentials are defined as the (locally free, saturated) subsheaves of $[T^m](\Omega^1(Z, D))$ defined similarly by the obvious symmetrisation conditions. See [4] for an explicit description.

These tensors satisfy, just as in the case $D = 0$, a bimeromorphic invariance property:

PROPOSITION 9.12 ([4, Theorem 3.5]). — *Let $u : (Z', D') \rightarrow (Z, D)$ be a bimeromorphic orbifold morphism.*

Then $u^ : H^0(Z, [T^m](\Omega^1(Z, D))) \rightarrow H^0(Z', [T^m](\Omega^1(Z', D')))$ is an isomorphism, for each $m > 0$.*

Although the proof (which is a simple application of Hartogs theorem) is given there for rank one subsheaves of the orbifold differential sheaves, it immediately implies the version given here.

DEFINITION 9.13 ([4]). — *Let (X, D) be a smooth orbifold pair with X connected complex projective, of dimension n . Let $m > 0$, and $\mathcal{L} \subset [\otimes^m] \Omega^1(X, D)$ be a rank-one coherent subsheaf. For each integer $k \geq 0$, let $\mathcal{L}^{\otimes k, \text{sat}} \subset [\otimes^{mk}] \Omega^1(X, D)$ be the saturation of $\mathcal{L}^{\otimes k}$. We then define:*

$$\kappa_D^{\text{sat}}(X, \mathcal{L}) := \limsup_{k \rightarrow +\infty} \left\{ \frac{\text{Log}(h^0(X, \mathcal{L}^{\otimes k, \text{sat}}))}{\text{Log}(k)} \right\} \in \{-\infty, 1, \dots, n\}.$$

As actually stated in [4, Theorem 3.5], we have the following birational invariance property for rank-one subsheaves:

PROPOSITION 9.14. — *Let $u : (X', D') \rightarrow (X, D)$ be a morphism which is an orbifold birational equivalence between two smooth projective orbifolds. Let $\mathcal{L} \subset [\otimes^m]\Omega^1(X, D)$ and $\mathcal{L}' \subset [\otimes^m]\Omega^1(X', D')$ be rank-one coherent subsheaves. Assume that either $\mathcal{L}' := u^*(\mathcal{L})$, or that $\mathcal{L} = u_*(\mathcal{L}')$. Then:*

$$\kappa_D^{\text{sat}}(X, \mathcal{L}) = \kappa_{D'}^{\text{sat}}(X', \mathcal{L}').$$

9.5. Lifting and descent of integral parts of orbifold tensors

The following theorem will be proved in the Appendix.

THEOREM 9.15. — *Let $h : (X, D) \rightarrow (Z, D_Z)$ be a fibration between smooth orbifolds such that: h is an orbifold morphism, and (Z, D_Z) is its orbifold base. Let $m \geq 0$ be a fixed integer. To shorten the notations, write $E_X^m := [T^m](\Omega^1(X, D))$, and $E_Z^m := [T^m](\Omega^1(Z, D_Z))$. Then, for any $m \geq 0$:*

- (1) $h^*(E_Z^m) \subset E_X^m$.
- (2) Let $h^*(E_Z^m)^{\text{sat}}$ stand for the saturation of $h^*(E_Z^m)$ in E_X^m . Then $h_*(h^*(E_Z^m)^{\text{sat}}) = E_Z^m$.

COROLLARY 9.16. — *In the situation of Theorem 9.15, for some $m > 0$, let $\mathcal{L}_U \subset \otimes^m \Omega_U^1$ be a rank-one subsheaf, where $U \subset Z$ is a dense Zariski-open subset. Let $\mathcal{L} \subset [\otimes^m](\Omega^1(X, D))$ be such that $\mathcal{L}|_{h^{-1}(U)} = h^*(\mathcal{L}_U)^{\text{sat}}$.*

If $\kappa_D^{\text{sat}}(X, \mathcal{L}) \geq 0$, there exists a saturated rank-one subsheaf $\mathcal{L}_Z \subset [\otimes^m](\Omega^1(Z, D_Z))$ such that $h^(\mathcal{L}_Z) \subset \mathcal{L}^{\text{sat}}$, and $\kappa_{D_Z}^{\text{sat}}(Z, \mathcal{L}_Z) = \kappa_D^{\text{sat}}(X, \mathcal{L})$.*

In particular, if $\kappa_D^{\text{sat}}(X, \mathcal{L}) = p = \dim(Z)$, then $\kappa(Z, \mathcal{L}_Z) = p$, with $\mathcal{L}_Z \subset [\otimes^m](\Omega^1(Z, D_Z))$.

Proof. — Indeed, one sets $\mathcal{L}_Z = h_*(\mathcal{L})^{\text{sat}}$. □

In order to prove our isotriviality results, we need this corollary only in the special case when the multiplicities of D are integral or infinite: indeed our orbifold structure arising from a foliation assigns integral multiplicities to the components parameterizing the multiple fibers, and infinite multiplicities to the compactifying components. By construction it is clear that passing to a smooth model we remain in the same special case. This particular case of Theorem 9.15 and its corollary is proved in [11, Theorem 5.8], and our method here is similar; we postpone the proof to the Appendix and refer to [11] for the time being. The main new ingredient of the proof is Lemma A.1 permitting to deal with rational multiplicities.

9.6. Special smooth orbifolds, proof of the isotriviality criteria.

DEFINITION 9.17 ([4, Definition 8.1, Théorème 9.9]). — Let (X, D) be a smooth connected projective⁽⁵⁾ orbifold. We say that (X, D) is “special” if, for any $p > 0$, and any rank-one subsheaf $\mathcal{L} \subset \Omega_X^p$, one has: $\kappa_D^{\text{sat}}(X, \mathcal{L}) < p$.

Let now (X, D) be as in the preceding definition, and let $g : X \dashrightarrow Z$ be a rational dominant fibration onto a variety of dimension $p > 0$ (which one may suppose smooth and projective). We shall always implicitly replace $g : (X, D) \dashrightarrow Z$ by a birational smooth model $g' : (X', D') \rightarrow (Z', D_{Z'})$ enjoying the properties 1-5 listed in Proposition 9.5. In order to simplify notations, we shall also denote $g : (X, D) \rightarrow (Z, D_Z)$ this new “neat” birational model.

THEOREM 9.18. — Let (X, D) be smooth projective and connected. The following properties are equivalent, if $f : X \dashrightarrow Z$ is a rational fibration onto some projective smooth manifold Z of dimension $p > 0$:

- (1) (X, D) is special.
- (2) For any $p > 0$ and any $g : X \dashrightarrow Z$, $\kappa(Z, K_Z + D_Z) < p$.
- (3) For any $p > 0$ and any $g : X \dashrightarrow Z$, $\kappa_D^{\text{sat}}(X, g^*(K_Z)) < p$.
- (4) For any $p > 0$, for any $m > 0$, for any $g : X \dashrightarrow Z$, and for any coherent rank-one $\mathcal{L}_Z \subset \otimes^m \Omega_Z^1$, one has $\kappa_D^{\text{sat}}(X, g^*(\mathcal{L}_Z)) < p$.
- (5) For any $p > 0$, for any $m > 0$, for any $g : X \dashrightarrow Z$ as above, and for any rank-one coherent $\mathcal{L} \subset [\otimes^m] \Omega^1(X, D)$ such that $\mathcal{L}|_{g^{-1}(U)} = g^*(\mathcal{L}_U)$ for some Zariski open subset $U \subset Z$ and some $\mathcal{L}_U \subset g^*(\otimes^m \Omega_U^1)$, one has $\kappa_D^{\text{sat}}(X, g^*(\mathcal{L}_Z)) < p$ for \mathcal{L}_Z defined as in Corollary 9.16.

Proof. — The equivalence between properties (1), (2), (3) is established in [4, Théorèmes 9.9 and 5.3]. The implication 4 \Rightarrow 3 is immediate. The reverse implication follows from [6, Theorem 7.11] by a contradiction argument applied to (Z, D_Z) , together with Corollary 9.16, last assertion. The equivalence between properties (4) and (5) follows from Corollary 9.16. \square

An important example of special smooth orbifold is given by the following:

THEOREM 9.19 ([4, Théorème 7.7]). — Let (X, D) be a smooth projective connected orbifold such that $\kappa(X, K_X + D) = 0$. Then (X, D) is special.

(5) The definition makes sense in the compact Kähler, or even class \mathcal{C} case.

COROLLARY 9.20. — *Let $f : X \rightarrow B$ be a projective fibration between two connected quasi-projective varieties, X smooth, B normal. Assume that f has equidimensional connected fibres. Let $\bar{f} : (\bar{X}, \bar{D}) \rightarrow (\bar{B}, \bar{D}_{\bar{B}})$ be any resolution of $f : X \rightarrow (B, D_B)$ (see Definition 9.6 above). We shall say that (B, D_B) is special if so is $(\bar{B}, \bar{D}_{\bar{B}})$. This does not depend on the choice of $\bar{f} : (\bar{X}, \bar{D}) \rightarrow (\bar{B}, \bar{D}_{\bar{B}})$.*

We then have, for $(\bar{B}, \bar{D}_{\bar{B}})$, the equivalence between the 5 properties listed in Theorem 9.18.

Assume in particular that $(\bar{B}, \bar{D}_{\bar{B}})$ is special. Let $g : \bar{B} \dashrightarrow Z$ be a fibration with $\dim(Z) = p$, and assume the existence of $\mathcal{L} \subset [\otimes^m] \Omega^1(\bar{X}, \bar{D})$ with $\kappa_{\bar{D}}^{\text{sat}}(\bar{X}, \mathcal{L}) = p$. If there is a $\mathcal{L}_U \subset \otimes^m \Omega_U^1$ for some Zariski dense open subset $U \subset Z$ such that $\mathcal{L}_{|(g \circ \bar{f})^{-1}(U)} = (g \circ \bar{f})^*(\mathcal{L}_U)^{\text{sat}}$, then $p = 0$.

Notice that Lemma 3.4 implies that the specialness of (B, D_B) in the sense of the last corollary is the same as the specialness of the orbifold base defined in Section 4.

COROLLARY 9.21. — *Let $f : X \rightarrow B$ be the fibration associated to an everywhere regular and algebraic foliation \mathcal{F} on a connected quasi-projective manifold X . Assume that the fibres of f have semiample canonical bundle. If the orbifold base (B, D_B) of f is special, then f is isotrivial.*

In particular if $\kappa(\bar{X}, \det(\Omega_{\bar{X}/\bar{\mathcal{F}}}^1)) = 0$, then f is isotrivial.

Proof. — Indeed, consider the smooth base-changed family over X as in Section 8. There is a Viehweg–Zuo sheaf $\mathcal{L} \subset [\text{Sym}^m] \Omega_{\bar{X}}^1(\text{Log}(\bar{D}))$ associated to this smooth family. By [12, Theorem 1.8], this sheaf possesses the property of being generically lifted from a subsheaf of $\text{Sym}^m(\Omega_Z^1)$, where Z is the (eventually compactified and modified) image of the moduli map $\mu : X \rightarrow \text{Mod}$ described in Section 6.1, and its Kodaira dimension is equal to the dimension of Z . But by Lemma 8.2, the map μ factors through B , and so generically on B there is another subsheaf \mathcal{L}_U of the symmetric differentials which lifts to \mathcal{L} over an open subset. Now apply Corollary 9.16 to extend it to the sheaf $\mathcal{L}_{\bar{B}}$ of saturated Kodaira dimension equal to $\dim(Z)$. The speciality of B implies $\dim(Z) = 0$. This establishes the first claim. The second one then follows from Theorem 9.19. \square

10. Two examples

10.1. Coisotropic submanifolds

Let $X \subset Y$ be a compact complex submanifold of a compact connected Kähler manifold Y of dimension $n = 2m$ carrying a holomorphic symplectic

2-form s . We say that X is *coisotropic* (relatively to s) if, for any $x \in X$, the complex tangent space $T_x X$ to X at x contains its s -orthogonal. This defines an everywhere regular rank r foliation \mathcal{F} on X , where r is the codimension of X in Y . This foliation is often called *characteristic foliation*.

Every smooth divisor $X \subset Y$ is coisotropic, with $r = 1$, so that it carries the characteristic foliation of rank one. This was the case studied in [1].

If X is coisotropic, we have: $2m - 2r \geq 0$, and $\dim(X) = 2m - r \geq r = \text{codim}_Y(X)$. If $r = m$, X is said to be Lagrangian. A somehow “dual” case is when X is isotropic (that is, when s vanishes on $T_x X \ \forall x \in X$). Thus Lagrangian means both isotropic and coisotropic.

Let $X \subset Y$ and s be as above, with X coisotropic. We say that X is “algebraically coisotropic” if the characteristic foliation \mathcal{F} is algebraic. Such subvarieties appear in the study of “subvarieties of constant cycles” on holomorphically symplectic varieties, but one has to drop the smoothness assumption (see [17]).

One of our main motivations for this paper was to generalize the results of [1], where we have proved that the fibration associated to the characteristic foliation on an algebraically coisotropic smooth divisor is always isotrivial in the projective case, and deduced from this that on an irreducible holomorphically symplectic projective manifold Y , there are no non-uniruled smooth algebraically coisotropic divisors X except in the trivial case when Y is a K3 surface and X is a curve.

The natural question for higher codimension is as follows: let Y be an irreducible holomorphically symplectic manifold and $X \subset Y$ a non-uniruled algebraically coisotropic submanifold. Can one conclude that X is lagrangian?

Our study provides some evidence for the affirmative answer, however the results are still extremely partial. For instance, one has the following.

COROLLARY 10.1. — *Let X be a projective manifold of dimension d with an everywhere regular algebraic foliation \mathcal{F} of rank r whose leaves are canonically polarised (or have trivial canonical bundle). If $\mathcal{F} = \text{Ker}(s)$, where s is a section s of $\Omega_X^{d-r} \otimes L$, with $L \in \text{Pic}(X)$ and $c_1(L) = 0$, then \mathcal{F} is isotrivial. Moreover, $\kappa(X) = r$ in the canonically polarized case and 0 in the trivial canonical bundle case.*

Proof. — Indeed, $\det(\Omega_{X/\mathcal{F}}^1) = \det(\Omega_{X/\mathcal{F}}^1)$ is then numerically trivial, since generated by s , and Theorem 4.3 applies. \square

A more specific example is the following (the case $r = 1$ has been established in [1]). However in this situation one can show, in the same way as

in [1], that the fibration associated to \mathcal{F} does not have multiple fibers in codimension one, so that a simpler proof of isotriviality can be given.

Example 10.2. — Let $X \subset Y$ be a connected projective coisotropic submanifold of codimension r in a smooth projective manifold Y equipped with a holomorphic symplectic 2-form s . Let \mathcal{F} be the characteristic foliation on X defined as $\text{Ker}(s^r)$. Assume that the leaves of \mathcal{F} are compact and canonically polarised. Then \mathcal{F} is isotrivial and $\kappa(X) = r$.

To answer the question raised above, one would need, e.g. in the case when Y is irreducible hyperkähler, a lower bound for Kodaira dimension of X : for instance $\kappa(X) \geq m$ would be sufficient to derive that X is lagrangian. This is the approach from [1], but we do not know whether it might work for higher-codimensional coisotropic subvarieties.

At this point we can obtain the answer only in some very particular cases.

Example 10.3. — In the situation of Example 10.2, assume that X is of general type and K_X is ample in restriction to the leaves of \mathcal{F} (this is the case for instance when the normal bundle $N_{X/Y}$ is ample). Then X is Lagrangian. Indeed: $\kappa(X) = \dim(X) \geq m$.

Example 10.4. — In the above situation of Example 10.2, assume that Y is a *simple* torus (rather than irreducible hyperkähler). Then X is Lagrangian. Indeed: $\kappa(X) = \dim(X)$ since Y is simple.

10.2. Boundary of codimension at least 2

We consider the following situation: Let X^+ be an irreducible (not necessarily normal) complex projective variety of dimension n , let X be the smooth locus of X^+ . Assume that there exists on X an everywhere non-zero d -closed holomorphic form w of degree $m := (n - r)$ defining an everywhere regular foliation $\mathcal{F} := \text{Ker}(u)$ with canonically polarised compact leaves of dimension r on X , or with compact leaves with trivial canonical bundle. The m -form w thus descends to a nowhere vanishing m -form v on the smooth locus of B . Thus v is a nowhere vanishing section of a suitable power N of K_B , if $f : X \rightarrow B$ is the fibration associated to \mathcal{F} , so that B has only quotient singularities, and its canonical bundle is \mathbb{Q} -Cartier. Thus: $w = (f^*(v))^{\otimes N}$ is a generator of $(\det(\Omega_{X/\mathcal{F}}^1))^{\otimes N}$.

We shall assume also that $X^+ \subset M$, where M is a complex space such that $M^{\text{reg}} \cap X^+ = X$, and that w is the restriction to X of a holomorphic m -form \widehat{w} on M^{reg} , which extends holomorphically on some (or any) resolution

of the singularities of M . It follows that if $\delta : \overline{X} \rightarrow X^+$ is an arbitrary desingularisation, then w extends to a holomorphic m -form \overline{w} on \overline{X} (by taking first an embedded resolution of the singularities of X^+ , lifting \widehat{w} , and then observing that the existence of \overline{w} is independent of the resolution of X^+). It is actually sufficient for the existence of \overline{w} that w be induced in local embeddings of X^+ , instead of a global one $X^+ \subset M$).

PROPOSITION 10.5. — *Assume that X^+, X, M, w are as in the above situation, and that $X = X^{+,reg}$ has complement in X^+ of codimension 2 or more. If the leaves of \mathcal{F} on X are compact and canonically polarised (or have trivial canonical bundle), then the family of leaves is isotrivial.*

Proof. — Let $f : X \rightarrow B$ be the proper connected fibration associated to \mathcal{F} on X . This fibration extends naturally to a fibration $\overline{f} : \overline{X} \rightarrow \overline{B}$ where \overline{B} is the normalisation of the (projective) closure in the Chow–Barlet space of X^+ of $f(X)$. Theorem 5.1 shows that we only need to show that $\kappa := \kappa(\overline{X}, \det(\Omega_{\overline{X}/\overline{B}}^1)) = 0$ to prove the claim. But the restriction to X of $\det(\Omega_{\overline{X}/\overline{B}}^1)$ is $\det(\Omega_{X/\mathcal{F}}^1)$, which is generated by w , and hence trivial. Because w extends to \overline{w} , we have $\kappa \geq 0$. Let now \overline{s} be a section of $\det(\Omega_{\overline{X}/\overline{B}}^1)^{\otimes m}$, for some $m > 0$. Let s be its restriction to X . The quotient $\varphi := \frac{s}{w^m}$ thus defines a holomorphic function on X . Because $\text{codim}_{X^+}(X^+ - X) \geq 2$, φ extends as a holomorphic function on the normalisation of X^+ , and is thus constant by compactness of X^+ . Thus $\overline{s} = \varphi \cdot \overline{w}^m$, and $\kappa = 0$, as claimed. \square

Example 10.6. — Let X^+ be a divisor in a connected complex projective variety M of dimension $2d = n + 1$ equipped with a symplectic two-form s on some of its resolutions. The form $u := s^{d-1}$ satisfies the non-vanishing condition and defines an everywhere regular rank-one foliation \mathcal{F} on X . We can also, more generally, consider X^+ of codimension r and coisotropic in the previous pair (M, s) , taking then $u = s^{d-r}$. The coisotropy condition means that s has rank r on X .

COROLLARY 10.7. — *Let $X^+ \subset M$ be complex projective, irreducible, with M^{2d} equipped with a holomorphic symplectic form s as in Example 10.6 above. Let $w := s^{d-r}$. If $X = X^{+,reg}$ is coisotropic of codimension r , if $\text{codim}_{X^+}(X^+ - X^{+,reg}) \geq 2$, and if the foliation $\mathcal{F} = \text{Ker}(w)$ has compact canonically polarised leaves on X (or compact leaves with trivial canonical bundle), then f is isotrivial.*

Example 10.8. — Let S be a K3-surface, $C \subset S$ a smooth connected projective curve of genus $g > 1$, and $k \geq 2$ an integer. Let $q : S^k \rightarrow M :=$

S^k/S_k , where S_k is the permutation group acting on the factors be the quotient map. Let $Q := q \circ j : C \times S^{k-1} \rightarrow M$ be the natural composition map, where $j : C \times S^k$ is the injection. Let $X^+ := Q(C \times S^{k-1}) \subset M$ be its image. Let $\rho : S^{[k]} \rightarrow M$ be the Fogarty resolution by the Hilbert scheme. The preceding result applies to X^+ , $X, w = s^{k-1}$, if s is a symplectic form on $S^{[k]}$. Here the isotriviality is obvious by construction, but this shows that examples which satisfy our quite restrictive conditions do exist.

Appendix A. Proof of Theorem 9.15

Proof. — The first claim of Theorem 9.15 is proved in [4, Proposition 2.11]. We thus check now the second claim. Notice first that we need to check this claim only over the complement of a codimension 2 subset S of Z , because E_Z^m is locally free. In particular, we can assume that h has equidimensional fibres over this complement. Finally, the h -horizontal part of D (that is, the components of D dominating Z) does not play any rôle here, since $f^*(E_Z^m)$ is saturated in E_X^m over the locus $Z - S$, $S := \text{Supp}(D_Z)$. Indeed, up to a Zariski-closed subset of codimension at least 2 in $Z - S$, if $D^+ = \text{Supp}(D) \subset X$, the fibration $h : (X, D^+) \rightarrow Z$ is smooth over $Z - S$ in the logarithmic sense, leading to an exact sequence (over $Z - S$): $0 \rightarrow h^*(\Omega_Z^1) \rightarrow \Omega^1(X, \text{Log } D^+) \rightarrow \Omega_{X/Z}^1(D^+) \rightarrow 0$ with torsionfree cokernel, implying the same property at the level of tensor powers, and a fortiori for $[T^m]\Omega^1(X, D) \subset [T^m]\Omega^1(X, D^+) = \otimes^m(\Omega_X^1(\text{Log } D^+))$.

We may thus choose local coordinates (z_1, z') , $z' := (z_2, \dots, z_p)$ on Z , adapted to D_Z , and such that, locally on Z , D_Z is supported on Z_1 , the divisor of Z of equation $z_1 = 0$, with D_Z -coefficient $c' = (1 - \frac{1}{m.t})$, and such that for suitable local coordinates $x = (x_1, x' = (x_2, \dots, x_n))$ adapted to D on a generic point of a component D_1 of D such that $h^*(Z_1) = t_1.D_1 + \dots$ in the local chart of X considered, we have: $h(x) = (z_1 = x_1^{t_1}, z_2 = x_2, \dots, z_p = x_p)$. By the definition of the orbifold base of h , we also have:

- (1) For some component D' of $h^{-1}(Z_1)$, if $c = (1 - \frac{1}{m})$ is the coefficient of D' in D , and if $h^*(Z_1) = t.D' + \dots$, we have: the coefficient c' of Z_1 in D_Z is $c' := (1 - \frac{1}{m.t})$, introduced above. Moreover:
- (2) $m''.t'' \geq m.t$ for any other component D'' of $h^{-1}(Z_1)$, if m'', t'' are defined as for D' . This inequality holds in particular for D_1, m_1, t_1 , with m_1, t_1 being the above invariants m'', t'' when $D'' := D_1$.

Let now $w := \frac{dz_1^{\otimes K}}{z_1^{[k.c']}} \otimes (dz')^{\otimes(m-K)}$ be any one of the generators of $[T^m](Z, D_Z)$, for some $0 \leq k \leq m$. Here $K \subset \{1, \dots, m\}$ is a subset

of cardinality $0 \leq k \leq m$, $m - K$ its complement there, and $dz_1^{\otimes K} \otimes (dz')^{\otimes(m-K)}$ means the tensor product $dz_{j_1} \otimes \cdots \otimes dz_{j_m}$, where $j_h = 1$ if and only if $h \in K$, while $j_h \in \{2, \dots, n\}$ otherwise.

Computing, we get (up to a nonzero constant):

$$h^*(w) = x_1^{t_1 \cdot (\lceil \frac{k}{m.t} \rceil)} \cdot \left(\frac{dx_1}{x_1}\right)^{\otimes k} \cdot (dx')^{\otimes(m-k)}.$$

But $[T^m](X, D)$ contains the \mathcal{O}_X -module W_X generated by:

$$w_X := \frac{dx_1^{\otimes k}}{x_1^{\lceil k.c_1 \rceil}} \cdot (dx')^{\otimes(m-k)} = x_1^{\lceil \frac{k}{m.t} \rceil} \cdot \left(\frac{dx_1}{x_1}\right)^{\otimes k} \cdot (dx')^{\otimes(m-k)}.$$

The argument now mainly relies on the following elementary lemma, where $\lceil x \rceil$, $x \in \mathbb{R}$, denotes the “round-up” of x , that is the smallest integer greater or equal to x . One also has: $\lceil x \rceil = -\lfloor -x \rfloor$, where $\lfloor x \rfloor$ is the usual integral part:

LEMMA A.1. — *Let $t > 0$ be an integer, and $x \in \mathbb{R}$. Then:*

$$(1) \ t \cdot \lceil \frac{x}{t} \rceil - \lceil x \rceil \in \{0, 1, \dots, (t-1)\}.$$

Let t, t', m, m', x be positive real numbers, with t, t' integers. Then:

$$(2) \ N := t' \cdot \lceil \frac{x}{m.t} \rceil - \lceil \frac{x}{m'} \rceil \geq 0 \text{ if } m'.t' \geq m.t, \text{ and:}$$

$$(2') \ N \in \{0, \dots, (t'-1)\} \text{ if } m.t = m'.t'.$$

Proof.

(1). — $\lceil \frac{x}{t} \rceil = \frac{x}{t} + \vartheta$, $\vartheta \in [0, 1[$, thus $t \cdot \lceil \frac{x}{t} \rceil = x + t \cdot \vartheta$. Also: $\lceil x \rceil = x + \vartheta'$, $\vartheta' \in [0, 1[$. Thus: $t \cdot \lceil \frac{x}{t} \rceil - \lceil x \rceil = t \cdot \vartheta - \vartheta' \in]t, -1[$ being an integer, we get the first claim.

(2). — $t' \cdot \lceil \frac{x}{m.t} \rceil = t' \cdot \frac{x}{m.t} + \vartheta = \frac{m'.t'}{m.t} \cdot \frac{x}{m'} + \vartheta$, $\vartheta \in [0, 1[$. Moreover: $\lceil \frac{x}{m'} \rceil = \frac{x}{m'} + \vartheta'$, $\vartheta' \in [0, 1[$. Since $N := t' \cdot \lceil \frac{x}{m.t} \rceil - \lceil \frac{x}{m'} \rceil = (\frac{m'.t'}{m.t} - 1) \cdot \frac{x}{m'} + t' \cdot \vartheta - \vartheta' \geq t' \cdot \vartheta - \vartheta' > -1$ is an integer, it is non-negative, as asserted.

(2') follows from (1), applied to $t', x' := \frac{x}{m'}$, in place of t, x , since: $\frac{x}{m.t} = \frac{x'}{m'.t'} = \frac{x'}{t'}$. \square

From Lemma A.1, and since $m_1.t_1 \geq m.t$, we get that $h^*(w) \in W_X$, and that $h^*(w) = x_1^\tau \cdot w_X$, with $\tau \in \{0, \dots, (t_1 - 1)\}$ if $m_1.t_1 = m.t$.

The support of $h_*(h^*(E_Z^m)^{\text{sat}})/E_Z^m$ must then have support of codimension one contained in D_Z . Assume that Z_1 for example is contained in this support. Then $h_*(h^*(E_Z^m)^{\text{sat}})|_{Z_1} \subset E_Z^m(k, Z_1)$ for some minimal integer $k \geq 0$. We will show that $k = 0$, implying the claim. Assume $k \geq 1$, then $h^*(E_Z^m)$ vanishes at order $\tau \geq t_1$ on the component D' of $h^{-1}(Z_1)$ introduced above, contradicting the inequality $\tau \leq (t_1 - 1)$ established in the previous lines. \square

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