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# CIRCUMCENTER EXTENSION OF MOEBIUS MAPS TO $\text{CAT}(-1)$ SPACES

by Kingshook BISWAS

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**ABSTRACT.** — Given a Moebius homeomorphism  $f : \partial X \rightarrow \partial Y$  between boundaries of proper, geodesically complete  $\text{CAT}(-1)$  spaces  $X, Y$ , we describe an extension  $\hat{f} : X \rightarrow Y$  of  $f$ , called the circumcenter map of  $f$ , which is constructed using circumcenters of expanding sets. The extension  $\hat{f}$  is shown to coincide with the  $(1, \log 2)$ -quasi-isometric extension constructed in a previous paper of the author, and is locally  $1/2$ -Holder continuous. When  $X, Y$  are complete, simply connected manifolds with sectional curvatures  $K$  satisfying  $-b^2 \leq K \leq -1$  for some  $b \geq 1$  then the extension  $\hat{f} : X \rightarrow Y$  is a  $(1, (1 - \frac{1}{b}) \log 2)$ -quasi-isometry, and is surjective. Circumcenter extension of Moebius maps is natural with respect to composition with isometries.

**RÉSUMÉ.** — Soit  $X, Y$  des  $\text{CAT}(-1)$  espaces propres et géométriquement complètes, et soit  $f : \partial X \rightarrow \partial Y$  un homéomorphisme qui préserve le birapport. Nous décrivons une extension  $\hat{f} : X \rightarrow Y$  de  $f$ , que nous appelons l'extension circoncentre de  $f$ , et qu'on construit à l'aide des circoncentres des ensembles agrandissants. On montre que l'extension  $\hat{f}$  coïncide avec l'extension  $(1, \log 2)$ -quasi-isométrique déjà construit par l'auteur dans un article précédent. Quand  $X, Y$  sont des variétés riemanniennes simplement connexes à courbures dans l'intervalle  $[-b^2, -1]$ , l'extension  $\hat{f} : X \rightarrow Y$  est un  $(1, (1 - \frac{1}{b}) \log 2)$ -quasi-isométrie surjectif. L'extension circoncentre est naturelle par rapport à composition avec des isométries.

## 1. Introduction

Let  $X$  be a  $\text{CAT}(-1)$  space. There is a positive function called the *cross-ratio* on the space of quadruples of distinct points of the boundary at infinity  $\partial X$  of  $X$ , defined for  $\xi, \xi', \eta, \eta' \in \partial X$  by

$$[\xi, \xi', \eta, \eta'] = \lim_{a \rightarrow \xi, a' \rightarrow \xi', b \rightarrow \eta, b' \rightarrow \eta'} \exp \left( \frac{1}{2} (d(a, b) + d(a', b') - d(a, b') - d(a', b)) \right)$$

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(where  $a, a', b, b' \in X$  converge radially towards  $\xi, \xi', \eta, \eta'$ ). A map between boundaries of  $\text{CAT}(-1)$  spaces is called *Moebius* if it preserves cross-ratios. Any isometry between  $\text{CAT}(-1)$  spaces extends to a Moebius homeomorphism between their boundaries. A classical fact which turns out to be crucial in many rigidity results including the Mostow Rigidity theorem is that a Moebius map from the boundary of real hyperbolic space to itself extends to an isometry. More generally Bourdon showed ([5]) that if  $X$  is a rank one symmetric space of noncompact type with maximum of sectional curvatures equal to  $-1$  and  $Y$  a  $\text{CAT}(-1)$  space then any Moebius embedding  $f : \partial X \rightarrow \partial Y$  extends to an isometric embedding  $F : X \rightarrow Y$ . In [2] the problem of extending Moebius maps was considered for general  $\text{CAT}(-1)$  spaces, where it was shown that any Moebius homeomorphism  $f : \partial X \rightarrow \partial Y$  between boundaries of proper, geodesically complete  $\text{CAT}(-1)$  spaces  $X, Y$  extends to a  $(1, \log 2)$ -quasi-isometry  $F : X \rightarrow Y$ . The proof of this theorem uses an isometric embedding of a proper, geodesically complete  $\text{CAT}(-1)$  space into a certain space of Moebius metrics on the boundary of the space. A nearest point projection to the subspace of visual metrics is used to construct the extension. We show that this nearest point is unique, and can be constructed as a limit of circumcenters of certain expanding sets. The extension constructed in [2] is thus uniquely determined. We call the extension the circumcenter map of  $f$ . It is readily seen to satisfy naturality properties with respect to composition with isometries. We have:

**THEOREM 1.1.** — *Let  $X, Y$  be proper, geodesically complete  $\text{CAT}(-1)$  spaces, and  $f : \partial X \rightarrow \partial Y$  a Moebius homeomorphism. Then the circumcenter extension  $\hat{f} : X \rightarrow Y$  of  $f$  is a  $(1, \log 2)$ -quasi-isometry which is locally  $1/2$ -Holder continuous:*

$$d(\hat{f}(x), \hat{f}(y)) \leq 2d(x, y)^{1/2}$$

for all  $x, y \in X$  such that  $d(x, y) \leq 1$ .

When the spaces  $X, Y$  are also assumed to be manifolds with curvature bounded below we have the following improvement on the main result of [2]:

**THEOREM 1.2.** — *Let  $X, Y$  be complete, simply connected Riemannian  $n$ -manifolds with sectional curvatures satisfying  $-b^2 \leq K \leq -1$  for some constant  $b \geq 1$ . For any Moebius homeomorphism  $f : \partial X \rightarrow \partial Y$ , the circumcenter extension  $\hat{f}$  of  $f$  is a  $(1, (1 - \frac{1}{b}) \log 2)$ -quasi-isometry  $\hat{f} : X \rightarrow Y$ . Moreover the circumcenter extension  $\hat{f} : X \rightarrow Y$  is surjective.*

We mention that one of the motivations for considering the problem of extending Moebius maps is the *marked length spectrum rigidity* problem. This asks whether an isomorphism  $\phi : \pi_1(X) \rightarrow \pi_1(Y)$  between fundamental groups of closed negatively curved manifolds which preserves lengths of closed geodesics (recall that in negative curvature each homotopy class of closed curves contains a unique closed geodesic) is necessarily induced by an isometry  $F : X \rightarrow Y$ . Otal ([7]) proved that this is indeed the case in dimension two. The problem remains open in higher dimensions. It is known however to be equivalent to the *geodesic conjugacy* problem, which asks whether the existence of a homeomorphism between the unit tangent bundles  $\phi : T^1X \rightarrow T^1Y$  conjugating the geodesic flows implies isometry of the manifolds. Hamenstadt ([6]) proved that equality of marked length spectra is equivalent to existence of a geodesic conjugacy.

Bourdon showed in [4], that for a Gromov-hyperbolic group  $\Gamma$  with two quasi-convex actions on  $\text{CAT}(-1)$  spaces  $X, Y$ , the natural  $\Gamma$ -equivariant homeomorphism  $f$  between the limit sets  $\Lambda X, \Lambda Y$  is Moebius if and only if there is a  $\Gamma$ -equivariant conjugacy of the abstract geodesic flows  $\mathcal{G}\Lambda X$  and  $\mathcal{G}\Lambda Y$  compatible with  $f$ . In particular for  $\tilde{X}, \tilde{Y}$  the universal covers of two closed negatively curved manifolds  $X, Y$  (with sectional curvatures bounded above by  $-1$ ), the geodesic flows of  $X, Y$  are topologically conjugate if and only if the induced equivariant boundary map  $f : \partial X \rightarrow \partial Y$  is Moebius (here topologically conjugate means that the conjugacy is a homeomorphism). Thus an affirmative answer to the problem of extending Moebius maps to isometries would also yield a solution to the equivalent problems of marked length spectrum rigidity and geodesic conjugacy.

Finally we remark that in [3] it is proved that in certain cases Moebius maps between boundaries of simply connected negatively curved manifolds do extend to isometries (more precisely, local and infinitesimal rigidity results are proved for deformations of the metric on a compact set).

## 2. Spaces of Moebius metrics

We recall in this section the definitions and facts from [2] which we will be needing.

Let  $(Z, \rho_0)$  be a compact metric space with at least four points. For a metric  $\rho$  on  $Z$  we define the metric cross-ratio with respect to  $\rho$  of a quadruple of distinct points  $(\xi, \xi', \eta, \eta')$  of  $Z$  by

$$[\xi\xi'\eta\eta']_\rho := \frac{\rho(\xi, \eta)\rho(\xi', \eta')}{\rho(\xi, \eta')\rho(\xi', \eta)}$$

(cf. the comment after Lemma 3.4, where it is clarified that this definition of cross-ratio reduces to the definition given in the Introduction of cross-ratio on the boundary  $\partial X$  of CAT( $-1$ ) space  $X$ , for a particular choice of metric on  $\partial X$  called a visual metric). We say that a diameter one metric  $\rho$  on  $Z$  is *antipodal* if for any  $\xi \in Z$  there exists  $\eta \in Z$  such that  $\rho(\xi, \eta) = 1$ . We assume that  $\rho_0$  is diameter one and antipodal. We say two metrics  $\rho_1, \rho_2$  on  $Z$  are *Moebius equivalent* if their metric cross-ratios agree:

$$[\xi\xi'\eta\eta']_{\rho_1} = [\xi\xi'\eta\eta']_{\rho_2}$$

for all  $(\xi, \xi', \eta, \eta')$ . The space of Moebius metrics on  $Z$  is defined to be

$$\mathcal{M}(Z, \rho_0) := \left\{ \rho : \begin{array}{l} \rho \text{ is an antipodal, diameter one metric} \\ \text{on } Z \text{ Moebius equivalent to } \rho_0 \end{array} \right\}.$$

We will write  $\mathcal{M}(Z, \rho_0) = \mathcal{M}$ . We have the following from [2]:

**THEOREM 2.1.** — *For any  $\rho_1, \rho_2 \in \mathcal{M}$ , there is a positive continuous function  $\frac{d\rho_2}{d\rho_1}$  on  $Z$ , called the derivative of  $\rho_2$  with respect to  $\rho_1$ , such that the following holds (the “Geometric Mean Value Theorem”):*

$$\rho_2(\xi, \eta)^2 = \frac{d\rho_2}{d\rho_1}(\xi) \frac{d\rho_2}{d\rho_1}(\eta) \rho_1(\xi, \eta)^2$$

for all  $\xi, \eta \in Z$ .

Moreover for  $\rho_1, \rho_2, \rho_3 \in \mathcal{M}$  we have

$$\frac{d\rho_3}{d\rho_1} = \frac{d\rho_3}{d\rho_2} \frac{d\rho_2}{d\rho_1}$$

and

$$\frac{d\rho_2}{d\rho_1} = 1 / \left( \frac{d\rho_1}{d\rho_2} \right).$$

The reason for the terminology “derivative” for the function  $\frac{d\rho_2}{d\rho_1}$  and the name “Geometric Mean Value Theorem” for the identity in the above Theorem is that the function  $\frac{d\rho_2}{d\rho_1}$  satisfies

$$\frac{d\rho_2}{d\rho_1}(\xi) = \lim_{\eta \rightarrow \xi} \frac{\rho_2(\xi, \eta)}{\rho_1(\xi, \eta)}$$

for all non-isolated points  $\xi$  of  $Z$  (we remark however that the function  $\frac{d\rho_2}{d\rho_1}$  is also defined at the isolated points of  $Z$ , but not by the above formula at such points).

**LEMMA 2.2.**

$$\max_{\xi \in Z} \frac{d\rho_2}{d\rho_1}(\xi) \cdot \min_{\xi \in Z} \frac{d\rho_2}{d\rho_1}(\xi) = 1.$$

Moreover if  $\frac{d\rho_2}{d\rho_1}$  attains its maximum at  $\xi$  and  $\rho_1(\xi, \eta) = 1$  then  $\frac{d\rho_2}{d\rho_1}$  attains its minimum at  $\eta$ , and  $\rho_2(\xi, \eta) = \rho_1(\xi, \eta) = 1$ .

*Proof.* — Let  $\lambda, \mu$  denote the maximum and minimum values of  $\frac{d\rho_2}{d\rho_1}$  respectively, and let  $\xi, \xi' \in Z$  denote points where the maximum and minimum values are attained respectively. Given  $\eta \in Z$  such that  $\rho_1(\xi, \eta) = 1$ , we have, using the Geometric Mean-Value Theorem,

$$1 \geq \rho_2(\xi, \eta)^2 = \frac{d\rho_2}{d\rho_1}(\xi) \frac{d\rho_2}{d\rho_1}(\eta) \geq \lambda \cdot \mu$$

while choosing  $\eta' \in Z$  such that  $\rho_2(\xi', \eta') = 1$ , we have

$$1 \geq \rho_1(\xi', \eta')^2 = 1 / \left( \frac{d\rho_2}{d\rho_1}(\xi') \frac{d\rho_2}{d\rho_1}(\eta') \right) \geq 1 / (\lambda \mu)$$

hence  $\lambda \cdot \mu = 1$ .

By the above we have

$$\frac{d\rho_2}{d\rho_1}(\eta) \leq 1 / \frac{d\rho_2}{d\rho_1}(\xi) = 1 / \lambda = \mu$$

hence  $\frac{d\rho_2}{d\rho_1}(\eta) = \mu$ . By the Geometric Mean Value Theorem this gives

$$\rho_2(\xi, \eta)^2 = \rho_1(\xi, \eta)^2 \frac{d\rho_2}{d\rho_1}(\xi) \frac{d\rho_2}{d\rho_1}(\eta) = 1 \cdot \lambda \cdot \mu = 1 . \quad \square$$

For  $\rho_1, \rho_2 \in \mathcal{M}$ , we define

$$d_{\mathcal{M}}(\rho_1, \rho_2) := \max_{\xi \in Z} \log \frac{d\rho_2}{d\rho_1}(\xi) .$$

From [2] we have:

LEMMA 2.3. — *The function  $d_{\mathcal{M}}$  defines a metric on  $\mathcal{M}$ . The metric space  $(\mathcal{M}, d_{\mathcal{M}})$  is proper, i.e. closed and bounded balls are compact.*

### 3. Visual metrics on the boundary of a CAT(−1) space

Let  $X$  be a proper CAT(−1) space such that  $\partial X$  has at least four points.

We recall below the definitions and some elementary properties of visual metrics and Busemann functions; for proofs we refer to [4]:

Let  $x \in X$  be a basepoint. The *Gromov product* of two points  $\xi, \xi' \in \partial X$  with respect to  $x$  is defined by

$$(\xi \mid \xi')_x = \lim_{(a, a') \rightarrow (\xi, \xi')} \frac{1}{2} (d(x, a) + d(x, a') - d(a, a'))$$

where  $a, a'$  are points of  $X$  which converge radially towards  $\xi$  and  $\xi'$  respectively. The *visual metric* on  $\partial X$  based at the point  $x$  is defined by

$$\rho_x(\xi, \xi') := e^{-(\xi|\xi')_x}.$$

We remark that the triangle inequality for the metric  $\rho_x$  is a consequence of the upper bound  $-1$  on the curvature, i.e. the CAT( $-1$ ) condition. The distance  $\rho_x(\xi, \xi')$  is less than or equal to one, with equality iff  $x$  belongs to the geodesic  $(\xi\xi')$ .

LEMMA 3.1. — *If  $X$  is geodesically complete then  $\rho_x$  is a diameter one antipodal metric.*

The Busemann function  $B : X \times X \times \partial X \rightarrow \mathbb{R}$  is defined by

$$B(x, y, \xi) := \lim_{a \rightarrow \xi} d(x, a) - d(y, a)$$

where  $a \in X$  converges radially towards  $\xi$ .

LEMMA 3.2. — *We have  $|B(x, y, \xi)| \leq d(x, y)$  for all  $\xi \in \partial X, x, y \in X$ . Moreover  $B(x, y, \xi) = d(x, y)$  iff  $y$  lies on the geodesic ray  $[x, \xi)$  while  $B(x, y, \xi) = -d(x, y)$  iff  $x$  lies on the geodesic ray  $[y, \xi)$ .*

We recall the following Lemma from [4]:

LEMMA 3.3. — *For  $x, y \in X, \xi, \eta \in \partial X$  we have*

$$\rho_y(\xi, \eta)^2 = \rho_x(\xi, \eta)^2 e^{B(x, y, \xi)} e^{B(x, y, \eta)}.$$

An immediate corollary of the above Lemma is the following:

LEMMA 3.4. — *The visual metrics  $\rho_x, x \in X$  are Moebius equivalent to each other and*

$$\frac{d\rho_y}{d\rho_x}(\xi) = e^{B(x, y, \xi)}.$$

It follows that the metric cross-ratio  $[\xi\xi'\eta\eta']_{\rho_x}$  of a quadruple  $(\xi, \xi', \eta, \eta')$  is independent of the choice of  $x \in X$ . Denoting this common value by  $[\xi\xi'\eta\eta']$ , it is shown in [5] that the cross-ratio is given by

$$[\xi\xi'\eta\eta'] = \lim_{(a, a', b, b') \rightarrow (\xi, \xi', \eta, \eta')} \exp\left(\frac{1}{2}(d(a, b) + d(a', b') - d(a, b') - d(a', b))\right)$$

where the points  $a, a', b, b' \in X$  converge radially towards  $\xi, \xi', \eta, \eta' \in \partial X$  (this can also be easily verified using the definitions of the visual metrics and the Gromov inner product).

We assume henceforth that  $X$  is a proper, geodesically complete CAT( $-1$ ) space. We let  $\mathcal{M} = \mathcal{M}(\partial X, \rho_x)$  (this space is independent of the choice of  $x \in X$ ). From [2] we have:

LEMMA 3.5. — *The map*

$$\begin{aligned} i_X : X &\rightarrow \mathcal{M} \\ x &\mapsto \rho_x \end{aligned}$$

*is an isometric embedding and the image is closed in  $\mathcal{M}$ .*

For  $k > 0$  and  $y, z \in X$  distinct from  $x \in X$  let  $\angle^{(-k^2)} yxz \in [0, \pi]$  denote the angle at the vertex  $\bar{x}$  in a comparison triangle  $\overline{xyz}$  in the model space  $\mathbb{H}_{-k^2}$  of constant curvature  $-k^2$ .

LEMMA 3.6. — *For  $\eta, \zeta \in \partial X$ , the limit of the comparison angles  $\angle^{(-k^2)} yxz$  exists as  $y, z$  converge to  $\eta, \zeta$  along the geodesic rays  $[x, \eta), [x, \zeta)$  respectively. Denoting this limit by  $\angle^{(-k^2)} \eta x \zeta$ , it satisfies*

$$\sin \left( \frac{\angle^{(-k^2)} \eta x \zeta}{2} \right) = \rho_x(\eta, \zeta)^k .$$

*Proof.* — A comparison triangle in  $\mathbb{H}_{-k^2}$  with side lengths  $a = d(x, y), b = d(x, z), c = d(y, z)$  and angle  $\theta = \angle^{(-k^2)} yxz$  at the vertex corresponding to  $x$  corresponds to a triangle in  $\mathbb{H}_{-1}$  with side lengths  $ka, kb, kc$  and angle  $\theta$  at the vertex opposite the side with length  $kc$ . By the hyperbolic law of cosine we have

$$\cosh kc = \cosh ka \cosh kb - \sinh ka \sinh kb \cos \theta .$$

As  $y \rightarrow \eta, z \rightarrow \zeta$ , we have  $a, b, c \rightarrow \infty$ , and  $a + b - c \rightarrow 2(\eta|\zeta)_x$ , thus

$$\begin{aligned} \cos \theta &= \frac{\cosh ka \cosh kb}{\sinh ka \sinh kb} - \frac{\cosh kc}{\sinh ka \sinh kb} \\ &\rightarrow 1 - 2e^{-2k(\eta|\zeta)_x} \end{aligned}$$

hence the angle  $\theta$  converges to a limit. Denoting this limit by  $\angle^{(-k^2)} \eta x \zeta$ , by the above it satisfies

$$\cos(\angle^{(-k^2)} \eta x \zeta) = 1 - 2\rho_x(\eta, \zeta)^{2k}$$

and hence

$$\sin \left( \frac{\angle^{(-k^2)} \eta x \zeta}{2} \right) = \rho_x(\eta, \zeta)^k . \quad \square$$

LEMMA 3.7. — *For  $x, y \in X, \xi \in \partial X$  and  $k > 0$ , the limit of the comparison angles  $\angle^{(-k^2)} yxz$  exists as  $z$  converges to  $\xi$  along the geodesic ray  $[x, \xi)$ . Denoting this limit by  $\angle^{(-k^2)} yx\xi$ , it satisfies*

$$e^{kB(y, x, \xi)} = \cosh(kd(x, y)) - \sinh(kd(x, y)) \cos(\angle^{(-k^2)} yx\xi) .$$



*Proof.* — A comparison triangle in  $\mathbb{H}_{-k^2}$  with side lengths  $a = d(x, y)$ ,  $b = d(x, z)$ ,  $c = d(y, z)$  and angle  $\theta = \angle^{(-k^2)} yxz$  at the vertex corresponding to  $x$  corresponds to a triangle in  $\mathbb{H}_{-1}$  with side lengths  $ka, kb, kc$  and angle  $\theta$  at the vertex opposite the side with length  $kc$ . By the hyperbolic law of cosine we have

$$\cosh kc = \cosh ka \cosh kb - \sinh ka \sinh kb \cos \theta .$$

As  $z \rightarrow \xi$ , we have  $b, c \rightarrow \infty$ , and  $c - b \rightarrow B(y, x, \xi)$ , thus

$$\begin{aligned} \cos \theta &= \frac{\cosh ka \cosh kb}{\sinh ka \sinh kb} - \frac{\cosh kc}{\sinh ka \sinh kb} \\ &\rightarrow \frac{\cosh ka}{\sinh ka} - \frac{e^{kB(y, x, \xi)}}{\sinh ka} \end{aligned}$$

hence the angle  $\theta$  converges to a limit. Denoting this limit by  $\angle^{(-k^2)} \xi x \eta$ , by the above it satisfies

$$e^{kB(y, x, \xi)} = \cosh(kd(x, y)) - \sinh(kd(x, y)) \cos(\angle^{(-k^2)} yx\xi) . \quad \square$$

We now consider the behaviour of the derivatives  $\frac{d\rho_y}{d\rho_x}$  as  $t = d(x, y) \rightarrow 0$  and the point  $y$  converges radially towards  $x$  along a geodesic. For functions  $F_t$  on  $\partial X$  we write  $F_t = o(t)$  if  $\|F_t\|_\infty = o(t)$ . We have the following formula from [2], which may be thought of as a formula for the derivative of the map  $i_X$  along a geodesic:

LEMMA 3.8. — *For  $x \in X$  and  $\xi \in \partial X$ , as  $y \in X$  converges to  $x$  along a geodesic we have*

$$\log \frac{d\rho_y}{d\rho_x}(\xi) = t \cos(\angle^{(-1)} yx\xi) + o(t)$$

where  $t = d(x, y) \rightarrow 0$ .

#### 4. Conformal maps, Moebius maps and geodesic conjugacies

We start by recalling the definitions of conformal maps, Moebius maps, and the abstract geodesic flow of a  $\text{CAT}(-1)$  space.

DEFINITION 4.1. — *A homeomorphism between metric spaces  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  with no isolated points is said to be conformal if for all  $\xi \in Z_1$ , the limit*

$$df_{\rho_1, \rho_2}(\xi) := \lim_{\eta \rightarrow \xi} \frac{\rho_2(f(\xi), f(\eta))}{\rho_1(\xi, \eta)}$$

exists and is positive. The positive function  $df_{\rho_1, \rho_2}$  is called the derivative of  $f$  with respect to  $\rho_1, \rho_2$ . We say  $f$  is  $C^1$  conformal if its derivative is continuous.

Two metrics  $\rho_1, \rho_2$  inducing the same topology on a set  $Z$ , such that  $Z$  has no isolated points, are said to be conformal (respectively  $C^1$  conformal) if the map  $\text{id}_Z : (Z, \rho_1) \rightarrow (Z, \rho_2)$  is conformal (respectively  $C^1$  conformal). In this case we denote the derivative of the identity map by  $\frac{d\rho_2}{d\rho_1}$ .

DEFINITION 4.2. — A homeomorphism between metric spaces  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  (where  $Z_1$  has at least four points) is said to be Moebius if it preserves metric cross-ratios with respect to  $\rho_1, \rho_2$ . The derivative of  $f$  is defined to be the derivative  $\frac{df^* \rho_2}{\rho_1}$  of the Moebius equivalent metrics  $f^* \rho_2, \rho_1$  as defined in section 2 (where  $f^* \rho_2$  is the pull-back of  $\rho_2$  under  $f$ ).

From the results of section 2 it follows that any Moebius map between compact metric spaces with no isolated points is  $C^1$  conformal, and the two definitions of the derivative of  $f$  given above coincide. Moreover any Moebius map  $f$  satisfies the geometric mean-value theorem,

$$\rho_2(f(\xi), f(\eta))^2 = \rho_1(\xi, \eta)^2 df_{\rho_1, \rho_2}(\xi) df_{\rho_1, \rho_2}(\xi) .$$

DEFINITION 4.3. — Let  $(X, d)$  be a CAT( $-1$ ) space. The abstract geodesic flow space of  $X$  is defined to be the space of bi-infinite geodesics in  $X$ ,

$$\mathcal{G}X := \{\gamma : (-\infty, +\infty) \rightarrow X \mid \gamma \text{ is an isometric embedding}\}$$

endowed with the topology of uniform convergence on compact subsets. This topology is metrizable with a distance defined by

$$d_{\mathcal{G}X}(\gamma_1, \gamma_2) := \int_{-\infty}^{\infty} d(\gamma_1(t), \gamma_2(t)) \frac{e^{-|t|}}{2} dt .$$

We define also two projections

$$\begin{aligned} p : \mathcal{G}X &\rightarrow X \\ \gamma &\mapsto \gamma(0) \end{aligned}$$

and

$$\begin{aligned} \pi : \mathcal{G}X &\rightarrow \partial X \\ \gamma &\mapsto \gamma(+\infty) . \end{aligned}$$

It is shown in Bourdon [4] that  $p$  is 1-Lipschitz.

For  $x \in X$ , the unit tangent sphere  $T_x^1 X \subset \mathcal{G}X$  is defined to be

$$T_x^1 X := p^{-1}(x) .$$

The abstract geodesic flow of  $X$  is defined to be the one-parameter group of homeomorphisms

$$\begin{aligned} \phi_t : \mathcal{G}X &\rightarrow \mathcal{G}X \\ \gamma &\mapsto \gamma_t \end{aligned}$$

for  $t \in \mathbb{R}$ , where  $\gamma_t$  is the geodesic  $s \mapsto \gamma(s+t)$ .

The flip is defined to be the map

$$\begin{aligned} \mathcal{F} : \mathcal{G}X &\rightarrow \mathcal{G}X \\ \gamma &\mapsto \bar{\gamma} \end{aligned}$$

where  $\bar{\gamma}$  is the geodesic  $s \mapsto \gamma(-s)$ .

We observe that for a simply connected complete Riemannian manifold  $X$  with sectional curvatures bounded above by  $-1$ , the map

$$\begin{aligned} \mathcal{G}X &\rightarrow T^1 X \\ \gamma &\mapsto \gamma'(0) \end{aligned}$$

is a homeomorphism conjugating the abstract geodesic flow of  $X$  to the usual geodesic flow of  $X$  and the flip  $\mathcal{F}$  to the usual flip on  $T^1 X$ .

Let  $f : \partial X \rightarrow \partial Y$  be a conformal map between the boundaries of  $\text{CAT}(-1)$  spaces  $X, Y$  equipped with visual metrics. Then  $f$  induces a bijection  $\phi_f : \mathcal{G}X \rightarrow \mathcal{G}Y$  conjugating the geodesic flows, which is defined as follows:

Given  $\gamma \in \mathcal{G}X$ , let  $\gamma(-\infty) = \xi, \gamma(+\infty) = \eta, x = \gamma(0)$ , then there is a unique point  $y$  on the bi-infinite geodesic  $(f(\xi), f(\eta))$  such that  $df_{\rho_x, \rho_y}(\eta) = 1$ . Define  $\phi_f(\gamma) = \gamma^*$  where  $\gamma^*$  is the unique geodesic in  $Y$  satisfying  $\gamma^*(-\infty) = f(\xi), \gamma^*(+\infty) = f(\eta), \gamma^*(0) = y$ . Then  $\phi_f : \mathcal{G}X \rightarrow \mathcal{G}Y$  is a bijection conjugating the geodesic flows. From [2] we have:

**PROPOSITION 4.4.** — *The map  $\phi_f$  is a homeomorphism if  $f$  is  $C^1$  conformal. If  $f$  is Moebius then  $\phi_f$  is flip-equivariant.*

## 5. Circumcenters of expanding sets and $\mathcal{FK}$ -convex functions

Let  $X$  be a proper, geodesically complete  $\text{CAT}(-1)$  space. Recall that for any bounded subset  $B$  of  $X$ , there is a unique point  $x$  which minimizes

the function

$$z \mapsto \sup_{y \in B} d(z, y) .$$

The point  $x$  is called the *circumcenter* of  $B$ , and the number  $\sup_{y \in B} d(x, y)$  is called the *circumradius* of  $B$ . We will denote these by  $c(B)$  and  $r(B)$  respectively.

Given  $K \leq 0$ , a function  $f : X \rightarrow \mathbb{R}$  is said to be  $\mathcal{F}K$ -convex if it is continuous and its restriction to any geodesic satisfies  $f'' + Kf \geq 0$  in the barrier sense. This means that  $f \leq g$  if  $g$  coincides with  $f$  at the endpoints of a subsegment and satisfies  $g'' + Kg = 0$  (where the parametrization of the geodesic segment is unit speed). If  $f$  restricted to the geodesic segment is smooth, this just means  $f'' + Kf \geq 0$  on the geodesic in the classical sense. We have the following from [1]:

PROPOSITION 5.1. — *Let  $y \in X, \xi \in \partial X$ . Then:*

- (1) *The function  $x \mapsto \cosh(d(x, y))$  is  $\mathcal{F}(-1)$ -convex.*
- (2) *The function  $x \mapsto \exp(B(x, y, \xi))$  is  $\mathcal{F}(-1)$ -convex.*

PROPOSITION 5.2. — *Let  $f$  be a positive, proper,  $\mathcal{F}(-1)$ -convex function on  $X$ . Then  $f$  attains its minimum at a unique point  $x \in X$ .*

*Proof.* — Since  $f$  is continuous, bounded below, and proper,  $f$  attains its minimum at some  $x \in X$ . If  $x' \neq x$  is another point where  $f$  attains its minimum, let  $\gamma : [-d, d] \rightarrow X$  be the geodesic joining  $x$  to  $x'$  where  $d = d(x, x')/2 > 0$ . Then  $g(t) = f(x) \cosh t / \cosh d$  satisfies  $g'' - g = 0$ , and agrees with  $f$  at the endpoints of  $\gamma$ , hence  $f(\gamma(0)) \leq g(0) = f(x) / \cosh d < f(x)$ , a contradiction.  $\square$

PROPOSITION 5.3. — *Let  $f_n, f$  be positive, proper,  $\mathcal{F}(-1)$ -convex functions on  $X$  such that  $f_n \rightarrow f$  uniformly on compacts. If  $x_n, x$  denote the points where  $f_n, f$  attain their minima, then  $x_n \rightarrow x$ .*

*Proof.* — We first show that  $\{x_n\}$  is bounded. If not, passing to a subsequence we may assume  $d(x, x_n) \rightarrow +\infty$ . For  $n$  sufficiently large we have  $f_n(x) \leq 2f(x)$ . Thus  $f_n(x_n) \leq f_n(x) \leq 2f(x)$  as well. Let  $\gamma_n : [-d_n, d_n] \rightarrow X$  be the unique geodesic joining  $x$  to  $x_n$ , where  $d_n = d(x, x_n)/2$ . Then the function

$$g(t) = \frac{1}{\sinh(2d_n)} \times [(\sinh d_n)(f_n(x) + f_n(x_n)) \cosh t + (\cosh d_n)(f_n(x_n) - f_n(x)) \sinh t]$$

satisfies  $g'' - g = 0$ , and agrees with  $f_n$  at the endpoints of  $\gamma_n$ . Thus for any  $s > 0$ , for  $n$  large such that  $s < d_n$ , letting  $y_n = \gamma_n(-d_n + s)$ , we have

$$\begin{aligned} f_n(y_n) &\leq g(-d_n + s) \\ &\leq \frac{1}{2 \sinh d_n \cosh d_n} \\ &\quad \times [(\sinh d_n \cosh(d_n - s))(4f(x)) + \cosh d_n \sinh(d_n - s)(4f(x))] . \end{aligned}$$

Since  $d_n \rightarrow +\infty$  this implies that for  $n$  sufficiently large we have

$$f_n(y_n) \leq \frac{1}{2} e^{-s}(1 + o(1))(8f(x)) < f(x)/2$$

for  $s > 0$  large enough. Fixing such an  $s$ , the points  $y_n = \gamma_n(-d_n + s)$  lie in the closed ball  $B$  of radius  $s$  around  $x$ , so passing to a subsequence we may assume that  $y_n \rightarrow y \in B$ . Since  $f_n \rightarrow f$  uniformly on  $B$ ,  $f_n(y_n) \rightarrow f(y)$ , hence  $f(y) \leq f(x)/2 < f(x)$ , a contradiction.

Thus the sequence  $\{x_n\}$  is bounded. To show  $x_n \rightarrow x$ , it suffices to show that the only limit point of  $\{x_n\}$  is  $x$ . Let  $K$  be a compact containing  $\{x_n\}$ . Suppose  $x_{n_k} \rightarrow y$ . Then  $f_{n_k}(x_{n_k}) \leq f_{n_k}(x)$  for all  $k$ . Since  $f_{n_k} \rightarrow f$  uniformly on  $K$ , letting  $k$  tend to infinity gives  $f(y) \leq f(x)$ . By the previous proposition this implies  $y = x$ .  $\square$

Let  $K$  be a compact subset of  $\mathcal{G}X$ . Define the function  $u_K : X \rightarrow \mathbb{R}$  by

$$u_K(z) = \sup_{\gamma \in K} \exp(B(z, p(\gamma), \pi(\gamma))) .$$

PROPOSITION 5.4. — *The function  $u_K$  is a positive,  $\mathcal{F}(-1)$ -convex function. It is proper if  $\pi(K) \subset \partial X$  is not a singleton.*

*Proof.* — For each  $\gamma \in K$ , the function  $z \mapsto \exp(B(z, p(\gamma), \pi(\gamma)))$  is  $\mathcal{F}(-1)$ -convex. Thus  $u_K$ , being the supremum of a family of  $\mathcal{F}(-1)$ -convex functions, satisfies the  $\mathcal{F}(-1)$ -convexity inequality. It remains to show that  $u_K$  is continuous.

Let  $z_n \rightarrow z$  in  $X$ . Define functions  $h_n, h : K \rightarrow \mathbb{R}$  by

$$h_n(\gamma) := B(z_n, p(\gamma), \pi(\gamma)), h(\gamma) := B(z, p(\gamma), \pi(\gamma)) .$$

Then  $|h_n(\gamma) - h(\gamma)| = |B(z_n, z, \pi(\gamma))| \leq d(z_n, z)$ , so  $h_n \rightarrow h$  uniformly on  $K$ . It follows that

$$u_K(z_n) = \|e^{h_n}\|_\infty \rightarrow \|e^h\|_\infty = u_K(z) .$$

Thus  $u_K$  is continuous.

Now suppose  $\pi(K)$  is not a singleton, so there exist  $\gamma_1, \gamma_2 \in K$  such that the endpoints  $\xi_i = \pi(\gamma_i), i = 1, 2$  are distinct. Let  $x_n$  be a sequence in  $X$  tending to infinity. Suppose  $u_K(x_n)$  does not tend to  $+\infty$ . Passing

to a subsequence we may assume  $u_K(x_n) \leq M$  for all  $n$  for some  $M > 0$ . Passing to a further subsequence we may assume  $x_n \rightarrow \xi \in \partial X$ . We can choose a  $\xi_i \neq \xi$ . Let  $x = p(\gamma_i)$ , then by Lemma 3.7 we have

$$\begin{aligned} \exp(B(x_n, x, \xi_i)) &= \cosh(d(x_n, x)) - \sinh(d(x_n, x)) \cos(\angle^{(-1)} x_n x \xi_i) \\ &= e^{-d(x_n, x)} + 2 \sinh(d(x_n, x)) \sin^2 \left( \frac{\angle^{(-1)} x_n x \xi_i}{2} \right) \\ &\rightarrow +\infty \end{aligned}$$

since  $\angle^{(-1)} x_n x \xi_i \rightarrow \angle^{(-1)} \xi x \xi_i > 0$ . Hence  $u_K(x_n) \geq \exp(B(x_n, x, \xi_i)) \rightarrow +\infty$ , a contradiction. This shows that  $u_K$  is proper.  $\square$

We remark that if  $\pi(K) \subset \partial X$  is a singleton  $\{\xi\}$ , then the properness of the function  $u_K$  does not hold, in fact  $u_K(x) \rightarrow 0$  as  $x$  converges to  $\xi$  along a geodesic.

**DEFINITION 5.5.** — *Let  $K$  be a compact subset of  $\mathcal{G}X$  such that  $\pi(K) \subset \partial X$  is not a singleton. The asymptotic circumcenter of  $K$  is defined to be the unique  $x$  in  $X$  where the function  $u_K$  attains its minimum. We denote the asymptotic circumcenter by  $x = c_\infty(K)$ .*

The reason for the name ‘asymptotic circumcenter’ is explained by the following proposition:

**PROPOSITION 5.6.** — *Let  $K$  be a compact subset of  $\mathcal{G}X$  such that  $\pi(K)$  is not a singleton. Define for  $t > 0$  bounded subsets  $A_t$  of  $X$  by  $A_t = p(\phi_t(K))$ , where  $\phi_t$  denotes the geodesic flow on  $\mathcal{G}X$ . Then*

$$c(A_t) \rightarrow c_\infty(K)$$

as  $t \rightarrow +\infty$ , i.e. the circumcenters of the sets  $A_t$  converge to the asymptotic circumcenter of  $K$ .

*Proof.* — Let  $u = u_K$ , and for  $t > 0$  define  $u_t : X \rightarrow \mathbb{R}$  by

$$u_t(z) = \left( \sup_{y \in A_t} \cosh(d(z, y)) \right) \cdot 2e^{-t}.$$

It is easy to see that  $u_t$  is a positive, proper,  $\mathcal{F}(-1)$ -convex function, and that the circumcenter of  $A_t$  is the unique minimizer of the function  $u_t$ . Since  $c_\infty(K)$  is the unique minimizer of  $u$ , by the previous proposition it suffices to show that  $u_t \rightarrow u$  uniformly on compacts as  $t \rightarrow \infty$ .

Note

$$u_t(z) = \left( \sup_{\gamma \in K} \cosh(d(z, \gamma(t))) \right) \cdot 2e^{-t}.$$

Now for  $z$  in a compact ball  $B$  and  $\gamma$  in the compact  $K$ ,

$$d(z, \gamma(t)) - t \rightarrow B(z, p(\gamma), \pi(\gamma))$$

as  $t \rightarrow +\infty$  uniformly in  $z \in B, \gamma \in K$ . It follows that

$$\cosh(d(z, \gamma(t))) \cdot 2e^{-t} \rightarrow \exp(B(z, p(\gamma), \pi(\gamma)))$$

as  $t \rightarrow +\infty$  uniformly in  $z \in B, \gamma \in K$ . Since the convergence in  $z, \gamma$  is uniform, the supremums over  $\gamma \in K$  converge, uniformly for  $z \in B$ :

$$u_t(z) = \left( \sup_{\gamma \in K} \cosh(d(z, \gamma(t))) \right) \cdot 2e^{-t} \rightarrow \sup_{\gamma \in K} \exp(B(z, p(\gamma), \pi(\gamma))) = u(z)$$

uniformly in  $z \in B$ .  $\square$

## 6. Circumcenter extension of Moebius maps and nearest point projections

Let  $f : \partial X \rightarrow \partial Y$  be a Moebius homeomorphism between boundaries of proper, geodesically complete CAT(-1) spaces  $X, Y$ , and let  $\phi_f : \mathcal{GX} \rightarrow \mathcal{GY}$  denote the associated geodesic conjugacy.

DEFINITION 6.1. — *The circumcenter extension of the Moebius map  $f$  is the map  $\hat{f} : X \rightarrow Y$  defined by*

$$\hat{f}(x) := c_\infty(\phi_f(T_x^1 X)) \in Y$$

(note that  $\pi(\phi_f(T_x^1 X)) = \partial Y$  is not a singleton, so the asymptotic circumcenter of  $\phi_f(T_x^1 X) \subset \mathcal{GY}$  exists).

In [2], a  $(1, \log 2)$ -quasi-isometric extension  $F : X \rightarrow Y$  of the Moebius map  $f$  is constructed as follows. Since  $f$  is Moebius, push-forward by  $f$  of metrics on  $\partial X$  to metrics on  $\partial Y$  gives a map between the spaces of Moebius metrics  $f_* : \mathcal{M}(\partial X) \rightarrow \mathcal{M}(\partial Y)$ , which is easily seen to be an isometry. For each  $\rho \in \mathcal{M}(\partial Y)$ , we can choose a nearest point to  $\rho$  in the subspace of visual metrics  $i_Y(Y) \subset \mathcal{M}(\partial Y)$ . This defines a nearest-point projection  $r_Y : \mathcal{M}(\partial Y) \rightarrow Y$ . The extension  $F$  is then defined by

$$F = r_Y \circ f_* \circ i_X.$$

We show below that if  $\rho \in \mathcal{M}(\partial Y)$  is the push-forward of a visual metric on  $\partial X$ ,  $\rho = f_* \rho_x$  for some  $x \in X$ , then in fact there is a unique visual metric  $\rho_y \in \mathcal{M}(\partial Y)$  nearest to  $\rho$ , given by  $y = \hat{f}(x)$ , the asymptotic circumcenter of  $\phi_f(T_x^1 X)$ . It follows that the extension  $F$  defined above is uniquely determined and equals the circumcenter extension  $\hat{f}$ .

PROPOSITION 6.2. — *Let  $x \in X$  and let  $\rho = f_*\rho_x \in \mathcal{M}(\partial Y)$ . Then  $y = \widehat{f}(x)$  is the unique minimizer of the function  $z \in Y \mapsto d_{\mathcal{M}}(\rho, \rho_z)$ . In particular,  $\widehat{f} = F$ , so  $\widehat{f}$  is a  $(1, \log 2)$ -quasi-isometry.*

*Proof.* — Fix a  $z \in Y$ . Given  $\xi \in \partial X$ , let  $\gamma \in T_x^1 X$  be such that  $\gamma(+\infty) = \xi$ . Let  $q = p(\phi_f(\gamma)) \in Y$ . Then by definition of  $\phi_f$ , we have

$$\frac{df_*\rho_x}{d\rho_q}(f(\xi)) = 1 .$$

It follows from the Chain Rule for Moebius metrics that

$$\begin{aligned} \frac{df_*\rho_x}{d\rho_z}(f(\xi)) &= \frac{df_*\rho_x}{d\rho_q}(f(\xi)) \cdot \frac{d\rho_q}{d\rho_z}(f(\xi)) \\ &= \exp(B(z, q, f(\xi))) \\ &= \exp(B(z, p(\phi_f(\gamma)), \pi(\phi_f(\gamma)))) . \end{aligned}$$

Moreover, for any  $\gamma \in T_x^1 X$ , the same argument shows that if  $\xi = \gamma(+\infty)$ , then

$$\exp(B(z, p(\phi_f(\gamma)), \pi(\phi_f(\gamma)))) = \frac{df_*\rho_x}{d\rho_z}(f(\xi)) .$$

Thus

$$\sup_{\xi \in \partial X} \frac{df_*\rho_x}{d\rho_z}(f(\xi)) = \sup_{\gamma \in \phi_f(T_x^1 X)} \exp(B(z, p(\gamma), \pi(\gamma)))$$

which gives, using the definition of the metric  $d_{\mathcal{M}}$ ,

$$\exp(d_{\mathcal{M}}(\rho, \rho_z)) = u_K(z)$$

where  $K = \phi_f(T_x^1 X)$ . Since the unique minimizer of  $u_K$  is given by  $y = \widehat{f}(x)$ , it follows that the function  $z \mapsto d_{\mathcal{M}}(\rho, \rho_z)$  also has a unique minimizer given by  $\widehat{f}(x)$ .  $\square$

The circumcenter extension has the following naturality properties with respect to composition with isometries:

PROPOSITION 6.3. — *Let  $f : \partial X \rightarrow \partial Y$  be a Moebius homeomorphism.*

- (1) *If  $f$  is the boundary map of an isometry  $F : X \rightarrow Y$  then  $\widehat{f} = F$ .*
- (2) *If  $G : X \rightarrow X, H : Y \rightarrow Y$  are isometries with boundary maps  $g, h$ , then*

$$h \circ \widehat{f \circ g} = H \circ \widehat{f} \circ G$$

*Proof.* — Let  $x \in X$ .

- (1). — If  $f$  is the boundary map of an isometry  $F$ , then  $f_*\rho_x = \rho_{F(x)}$ , so the nearest point to  $f_*\rho_x$  is  $\rho_{F(x)}$ , so by the previous proposition  $\widehat{f}(x) = F(x)$ .



(2). — Note  $f_*g_*\rho_x = f_*\rho_{G(x)}$ . Let  $z = \widehat{f}(G(x))$ , so  $\rho_z$  is the nearest point to  $f_*\rho_{G(x)}$ . Since  $h_* : \mathcal{M}(\partial Y) \rightarrow \mathcal{M}(\partial Y)$  is an isometry which preserves the subspace of visual metrics,  $h_*\rho_z = \rho_{H(z)}$  is the nearest point to  $h_*f_*\rho_{G(x)} = (h \circ f \circ g)_*\rho_x$ , hence by the previous proposition  $H(z) = \widehat{h \circ f \circ g}(x)$ , and  $H(z) = H(\widehat{f}(G(x)))$  so we are done.  $\square$

The key to Theorem 1.2 is the following proposition:

PROPOSITION 6.4. — *Let  $X$  be a proper, geodesically complete CAT(-1) space. Given  $\rho \in \mathcal{M}(\partial X)$ , if  $x \in X$  minimizes  $z \in X \mapsto d_{\mathcal{M}}(\rho, \rho_z)$ , then for any  $y \in X \cup \partial X$  distinct from  $x$ , there exists  $\eta \in \partial X$  maximizing  $\zeta \in \partial X \mapsto \frac{d\rho}{d\rho_x}(\zeta)$  such that  $\angle^{(-1)}yx\eta \geq \pi/2$ .*

*Proof.* — Let  $K \subset \partial X$  be the set where  $\frac{d\rho}{d\rho_x}$  attains its maximum value  $e^M$ , where  $M = d_{\mathcal{M}}(\rho, \rho_x)$ , and suppose there is a  $y \in X \cup \partial X$  such that  $\angle^{(-1)}yx\eta < \pi/2$  for all  $\eta \in K$ . Then we can choose  $\epsilon, \delta > 0$  and a neighbourhood  $N$  of  $K$  such that  $\angle^{(-1)}yx\eta \leq \pi/2 - \epsilon$  for all  $\eta \in N$ , and such that  $\log \frac{d\rho}{d\rho_x} \leq M - \delta$  on  $\partial X - N$ .

Let  $z$  be the point on the geodesic ray  $[x, y)$  at a distance  $t > 0$  from  $x$ . As  $t \rightarrow 0$ , for  $\eta \in N$  we have, noting that  $\angle^{(-1)}zx\eta \leq \angle^{(-1)}yx\eta$ , by Lemma 3.8,

$$\begin{aligned} \log \frac{d\rho}{d\rho_z}(\eta) &= \log \frac{d\rho}{d\rho_x}(\eta) - \log \frac{d\rho_z}{d\rho_x}(\eta) \\ &\leq M - t \cos(\angle^{(-1)}zx\eta) + o(t) \\ &\leq M - t \cos(\angle^{(-1)}yx\eta) + o(t) \\ &\leq M - t \sin \epsilon + o(t) \\ &< M \end{aligned}$$

for  $t$  small enough depending only on  $\epsilon$ , while for  $\eta \in \partial X - N$  we have

$$\begin{aligned} \log \frac{d\rho}{d\rho_z}(\eta) &= \log \frac{d\rho}{d\rho_x}(\eta) - \log \frac{d\rho_z}{d\rho_x}(\eta) \\ &\leq (M - \delta) + t \\ &< M \end{aligned}$$

for  $t < \delta$ , thus for  $t > 0$  small enough we have  $d_{\mathcal{M}}(\rho, \rho_z) < M = d_{\mathcal{M}}(\rho, \rho_x)$ , a contradiction.  $\square$

Theorem 1.1 now follows from the following proposition:

PROPOSITION 6.5. — *Let  $f : \partial X \rightarrow \partial Y$  be a Moebius homeomorphism between boundaries of proper, geodesically complete CAT(-1) spaces  $X, Y$ .*

Then the circumcenter extension  $\widehat{f} : X \rightarrow Y$  satisfies

$$\cosh(d(\widehat{f}(x), \widehat{f}(y))) \leq e^{d(x,y)}$$

for all  $x, y \in X$ . In particular  $\widehat{f}$  is locally  $1/2$ -Holder continuous:

$$d(\widehat{f}(x), \widehat{f}(y)) \leq 2d(x, y)^{1/2}$$

for all  $x, y \in X$  such that  $d(x, y) \leq 1$ .

*Proof.* — Given  $x, y \in X$ , let  $x' = \widehat{f}(x), y' = \widehat{f}(y)$ . We may assume  $x' \neq y'$  (otherwise the above inequality holds trivially), and also (interchanging  $x, y$  if necessary) that

$$d_{\mathcal{M}}(f_*\rho_x, \rho_{x'}) \geq d_{\mathcal{M}}(f_*\rho_y, \rho_{y'}) .$$

Let  $\rho = f_*\rho_x \in \mathcal{M}(\partial Y)$ . By Proposition 6.2,  $x'$  minimizes  $z \in Y \mapsto d_{\mathcal{M}}(\rho, \rho_z)$ . Hence by the previous Proposition 6.4, there exists  $\eta \in \partial Y$  maximizing  $\zeta \in \partial Y \mapsto \frac{d\rho}{d\rho_{x'}}(\zeta)$  such that  $\angle^{(-1)}y'x'\eta \geq \pi/2$ . By Lemma 3.7, we have

$$\begin{aligned} e^{B(y', x', \eta)} &= \cosh(d(x', y')) - \sinh(d(x', y')) \cos(\angle^{(-1)}y'x'\eta) \\ &\geq \cosh(d(x', y')) . \end{aligned}$$

Also,

$$\begin{aligned} e^{B(y', x', \eta)} &= \frac{d\rho_{x'}}{d\rho_{y'}}(\eta) \\ &= \frac{d\rho_{x'}}{df_*\rho_x}(\eta) \frac{df_*\rho_x}{df_*\rho_y}(\eta) \frac{df_*\rho_y}{d\rho_{y'}}(\eta) \\ &\leq \exp(-d_{\mathcal{M}}(f_*\rho_x, \rho_{x'})) \frac{d\rho_x}{d\rho_y}(f^{-1}(\eta)) \exp(d_{\mathcal{M}}(f_*\rho_y, \rho_{y'})) \\ &\leq \frac{d\rho_x}{d\rho_y}(f^{-1}(\eta)) \\ &= e^{B(y, x, f^{-1}(\eta))} \\ &\leq e^{d(x, y)} \end{aligned}$$

thus

$$\cosh(d(x', y')) \leq e^{d(x, y)}$$

as required.

It follows easily that  $\widehat{f}$  is locally  $1/2$ -Holder:

Since  $e^t \leq 1 + 2t$  for  $0 \leq t \leq 1$ , for  $x, y \in X$ , if  $d(x, y) \leq 1$  we have

$$\begin{aligned} 1 + \frac{d(\widehat{f}(x), \widehat{f}(y))^2}{2} &\leq \cosh(d(\widehat{f}(x), \widehat{f}(y))) \\ &\leq e^{d(x, y)} \\ &\leq 1 + 2d(x, y) \end{aligned}$$

hence

$$d(\widehat{f}(x), \widehat{f}(y)) \leq 2d(x, y)^{1/2}. \quad \square$$

Let  $X$  be a complete, simply connected Riemannian manifold with sectional curvatures  $K$  satisfying  $-b^2 \leq K \leq -1$  for some  $b \geq 1$ . For  $x \in X$  and  $\xi, \eta \in \partial X$ , let  $\angle \xi x \eta \in [0, \pi]$  denote the Riemannian angle at  $x$  between the geodesic rays  $[x, \xi]$  and  $[x, \eta]$ .

LEMMA 6.6. — We have

$$\rho_x(\xi, \eta)^b \leq \sin\left(\frac{\angle \xi x \eta}{2}\right) \leq \rho_x(\xi, \eta).$$

*Proof.* — Since the sectional curvature of  $X$  is bounded above and below by  $-1$  and  $-b^2$ , we have

$$\angle^{(-b^2)} \xi x \eta \leq \angle \xi x \eta \leq \angle^{(-1)} \xi x \eta$$

hence by Lemma 3.6

$$\begin{aligned} \rho_x(\xi, \eta)^b &= \sin\left(\frac{\angle^{(-b^2)} \xi x \eta}{2}\right) \leq \sin\left(\frac{\angle \xi x \eta}{2}\right) \\ &\leq \sin\left(\frac{\angle^{(-1)} \xi x \eta}{2}\right) = \rho_x(\xi, \eta). \end{aligned} \quad \square$$

We now come to the proof of Theorem 1.2:

*Proof.* — Let  $f : \partial X \rightarrow \partial Y$  be a Moebius homeomorphism between boundaries of complete, simply connected manifolds with sectional curvatures  $K$  satisfying  $-b^2 \leq K \leq -1$ .

Let  $x \in X$ , and let  $y = \widehat{f}(x)$ . Let  $M = d_{\mathcal{M}}(f_*\rho_x, \rho_y)$ . Let  $K \subset \partial Y$  be the set where  $\frac{df_*\rho_x}{d\rho_y}$  attains its maximum value  $e^M$ , and let  $\eta_1 \in K$ . Then by Proposition 6.4, there exists  $\eta_2 \in K$  such that  $\angle^{(-1)} \eta_1 y \eta_2 \geq \pi/2$ , so  $\rho_y(\eta_1, \eta_2) \geq 1/\sqrt{2}$ .

Let  $\xi_i = f^{-1}(\eta_i) \in \partial X$ ,  $i = 1, 2$ . Let  $\eta'_i \in \partial Y$  be the unique point such that  $\rho_y(\eta_i, \eta'_i) = 1$ ,  $i = 1, 2$ . Then by Lemma 2.2,  $\frac{df_*\rho_x}{d\rho_y}$  attains its minimum value  $e^{-M}$  at  $\eta'_1, \eta'_2$ , and the points  $\xi'_i = f^{-1}(\eta'_i)$  satisfy  $\rho_x(\xi_i, \xi'_i) = 1$ ,  $i = 1, 2$ . The Geometric Mean Value Theorem gives

$$\rho_x(\xi_1, \xi_2) = e^M \rho_y(\eta_1, \eta_2), \rho_x(\xi'_1, \xi'_2) = e^{-M} \rho_y(\eta'_1, \eta'_2).$$

Noting that  $\angle \xi_1 x \xi_2 = \angle \xi'_1 x \xi'_2$  and  $\angle \eta_1 y \eta_2 = \angle \eta'_1 y \eta'_2$ , by Lemma 6.6 we have

$$\begin{aligned} \rho_x(\xi'_1, \xi'_2) &\geq \sin \left( \frac{\angle \xi'_1 x \xi'_2}{2} \right) \\ &= \sin \left( \frac{\angle \xi_1 x \xi_2}{2} \right) \\ &\geq \rho_x(\xi_1, \xi_2)^b \end{aligned}$$

and

$$\begin{aligned} \rho_y(\eta'_1, \eta'_2) &\leq \left( \sin \left( \frac{\angle \eta'_1 y \eta'_2}{2} \right) \right)^{1/b} \\ &= \left( \sin \left( \frac{\angle \eta_1 y \eta_2}{2} \right) \right)^{1/b} \\ &\leq \rho_y(\eta_1, \eta_2)^{1/b} . \end{aligned}$$

Using the above two inequalities in the equality

$$\frac{\rho_x(\xi_1, \xi_2)}{\rho_x(\xi'_1, \xi'_2)} = e^{2M} \frac{\rho_y(\eta_1, \eta_2)}{\rho_y(\eta'_1, \eta'_2)}$$

gives

$$\frac{1}{\rho_x(\xi_1, \xi_2)^{b-1}} \geq e^{2M} \rho_y(\eta_1, \eta_2)^{1-1/b} .$$

Thus

$$\begin{aligned} 1 &\geq e^{2M} \rho_x(\xi_1, \xi_2)^{b-1} \rho_y(\eta_1, \eta_2)^{1-1/b} \\ &= e^{2M} e^{(b-1)M} \rho_y(\eta_1, \eta_2)^{(b-1)+(1-1/b)} \\ &\geq \frac{e^{(b+1)M}}{\sqrt{2}^{b-1/b}} \end{aligned}$$

hence

$$M \leq \frac{1}{2} \frac{b-1/b}{b+1} \log 2 = \frac{1}{2} (1-1/b) \log 2 .$$

Thus

$$d_{\mathcal{M}}(f_* \rho_x, \rho_{\widehat{f}(x)}) \leq \frac{1}{2} (1-1/b) \log 2$$

for all  $x \in X$ . Then for any  $x, y \in X$ ,

$$\begin{aligned} |d(\widehat{f}(x), \widehat{f}(y)) - d(x, y)| &= |d_{\mathcal{M}}(\rho_{\widehat{f}(x)}, \rho_{\widehat{f}(y)}) - d_{\mathcal{M}}(f_* \rho_x, f_* \rho_y)| \\ &\leq d_{\mathcal{M}}(f_* \rho_x, \rho_{\widehat{f}(x)}) + d_{\mathcal{M}}(f_* \rho_y, \rho_{\widehat{f}(y)}) \\ &\leq (1-1/b) \log 2 \end{aligned}$$

thus  $\widehat{f}$  is a  $(1, (1 - 1/b) \log 2)$ -quasi-isometry. As in [2] it is straightforward to show that the map of  $\widehat{f} : X \rightarrow Y$  extends to a continuous map  $F : X \cup \partial X \rightarrow Y \cup \partial Y$  such that  $F|_{\partial X} = f : \partial X \rightarrow \partial Y$ , so that  $\widehat{f}$  is an extension of  $f$ .

The surjectivity of  $\widehat{f} : X \rightarrow Y$  then follows from a simple topological argument using the fact that the map  $\widehat{f}$  is continuous (being locally Hölder continuous), and the fact that the boundary map  $f : \partial X \rightarrow \partial Y$  is a homeomorphism. The compactified spaces  $X \cup \partial X, Y \cup \partial Y$  can be identified homeomorphically with the closed unit ball  $B^n \cup S^{n-1}$ , where  $B^n$  is the open unit ball in  $\mathbb{R}^n$  and  $S^{n-1}$  the unit sphere in  $\mathbb{R}^n$ . The map  $F : X \cup \partial X \rightarrow Y \cup \partial Y$  is then identified with a continuous map  $G : B^n \cup S^{n-1} \rightarrow B^n \cup S^{n-1}$  such that  $G|_{S^{n-1}} = g : S^{n-1} \rightarrow S^{n-1}$  is a homeomorphism.

If  $G$  is not surjective, then there is a point  $x \in B^n$  which is not in the image of  $G$ . Now  $(B^n \cup S^{n-1}) - \{x\}$  retracts onto  $S^{n-1}$ , and we can compose  $G$  with this retraction to get a continuous map  $H : B^n \cup S^{n-1} \rightarrow S^{n-1}$  such that  $H|_{S^{n-1}} = G|_{S^{n-1}} = g$ . Since  $g = H|_{S^{n-1}} : S^{n-1} \rightarrow S^{n-1}$  extends to the continuous map  $H : B^n \cup S^{n-1} \rightarrow S^{n-1}$  it follows that  $g$  is homotopic to a constant (the explicit homotopy is  $g_t(z) = H(tz), z \in S^{n-1}, t \in [0, 1]$ ), a contradiction to the fact that  $g : S^{n-1} \rightarrow S^{n-1}$  is a homeomorphism. Thus  $\widehat{f} : X \rightarrow Y$  must be surjective.  $\square$

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