ANNALES DE L'INSTITUT FOURIER

ANNALES

DE

L'INSTITUT FOURIER

Julien MARCHÉ

Distribution of Chern–Simons invariants

Article à paraître, mis en ligne le 5 février 2019, 10 p.

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 5 février 2019.

DISTRIBUTION OF CHERN–SIMONS INVARIANTS

by Julien MARCHÉ (*)

ABSTRACT. — Let M be a closed 3-manifold with a finite set $X(M)$ of conjugacy classes of representations $\rho : \pi_1(M) \to \mathrm{SU}_2$. We study here the distribution of the values of the Chern–Simons function $\mathrm{CS} : X(M) \to \mathbb{R}/2\pi\mathbb{Z}$. We observe in some examples that it resembles the distribution of quadratic residues. In particular for specific sequences of 3-manifolds, the invariants tends to become equidistributed on the circle with white noise fluctuations of order $|X(M)|^{-1/2}$. We prove that for a manifold with toric boundary the Chern–Simons invariants of the Dehn fillings $M_{p/q}$ have the same behaviour when p and q go to infinity and compute fluctuations at first order.

RÉSUMÉ. — Soit M une 3-variété sans bord avec un ensemble fini de classes de conjugaison de représentations $\rho : \pi_1(M) \to \mathrm{SU}_2$. On étudie la répartition des valeurs de la fonction de Chern–Simons $\mathrm{CS} : X(M) \to \mathbb{R}/2\pi\mathbb{Z}$. On observe dans quelques exemples qu'elle ressemble à la répartition des résidus quadratiques. En particulier, pour quelques suites de 3-variétés, ces invariants tendent à se répartir uniformément sur le cercle avec des fluctuations de type bruit blanc d'ordre $|X(M)|^{-1/2}$. On prouve que pour une variété à bord torique, les invariants de Chern–Simons des remplissages de Dehn $M_{p/q}$ ont le même comportement quand p et q tendent vers l'infini et on calcule les fluctuations au premier ordre.

1. Introduction

1.1. Distribution of quadratic residues

Let p be a prime number congruent to 1 modulo 4. We consider the normalised counting measure on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ defined by quadratic residues modulo p, that is:

$$\mu_p = \frac{1}{p}\sum_{k=0}^{p-1} \delta_{\frac{2\pi k^2}{p}}$$

Keywords: Chern–Simons, 3-manifold, equidistribution, Gauss sum.
2010 *Mathematics Subject Classification:* 10X99, 14A12, 11L05.
(*) The author would like to thank Bertrand Deroin, Gregor Masbaum and Pierre Charollois for their interest in this work.

where δ_x denotes the Dirac measure at $x \in \mathbb{T}$. We investigate the limit of μ_p when p goes to infinity and to that purpose, we consider its ℓ-th momentum i.e. $\mu_p^\ell = \int e^{i\ell\theta} d\mu_p(\theta) = \frac{1}{p}\sum_{k=0}^{p-1}\exp(2 i\pi\ell k^2/p)$. We have $\mu_p^\ell = 1$ if $p|\ell$, and else by the Gauss sum formula, $\mu_p^\ell = \left(\frac{\ell}{p}\right)\frac{1}{\sqrt{p}}$ where $\left(\frac{\ell}{p}\right)$ is the Legendre symbol.

This shows that μ_p converges to the uniform measure μ_∞ defined by $\int f d\mu_\infty = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$ whereas the renormalized measure $\sqrt{p}(\mu_p - \mu_\infty)$ (that we call fluctuation) has l-th momentum ± 1 depending on the residue of l modulo p and hence is a kind of "white noise". By this we mean that the modulus of the l-th Fourier coefficient of μ does not depend on l.

On the other hand, such Gauss sums appear naturally in the context of Chern–Simons invariants of 3-manifolds. Consider an oriented and compact 3-manifold M and define its character variety as the set $X(M) = \mathrm{Hom}(\pi_1(M), \mathrm{SU}_2)/\mathrm{SU}_2$. In what follows, we will confuse between representations and their conjugacy classes. The Chern–Simons invariant may be viewed as a locally constant map $\mathrm{CS} : X(M) \to \mathbb{T}$. We refer to [3] for background on Chern–Simons invariants and give here a quick definition for the convenience of the reader.

Definition 1.1. — *Let ν be the Haar measure of SU_2 i.e. the unique Borel measure invariant by translation and normalised by $\nu(\mathrm{SU}_2) = 2\pi$ and let $\pi : \tilde{M} \to M$ be the universal cover of M. There is an equivariant map $F : \tilde{M} \to \mathrm{SU}_2$ in the sense that $F(\gamma x) = \rho(\gamma)F(x)$ for all $\gamma \in \pi_1(M)$ and $x \in \tilde{M}$. The form $F^*\nu$ is invariant hence can be written $F^*\nu = \pi^*\nu_F$. We set*

$$\mathrm{CS}(\rho) = \int_M \nu_F \mod 2\pi\mathbb{Z}$$

and claim that it is independent on the choice of equivariant map F modulo 2π.

Definition 1.2. — *Let M be a 3-manifold whose character variety is finite. We define its* Chern–Simons measure *as $\mu_M = \frac{1}{|X(M)|}\sum_{\rho\in X(M)} \delta_{\mathrm{CS}(\rho)}$.*

The aim of this article is to describe some sequences of 3-manifolds M_n for which the measure μ_{M_n} converges. In all cases we could handle, the limit measure is μ_∞ and the fluctuations have a similar behaviour with the case of the distribution of quadratic residues. In the first section, we present some examples and state our main theorem which concerns the case of the Dehn fillings of a given 3-manifold with toric boundary. In the second section, we place this problem in the more general context of intersection of Legendrian submanifolds and prove the main result. In the last section, we address the same question in the case where all manifolds are coverings of a given one.

1.2. Distribution of Chern–Simons invariants

1.2.1. Lens spaces

For instance, if $M = L(p,q)$ is a lens space, then $\pi_1(M) = \mathbb{Z}/p\mathbb{Z}$ and $X(M) = \{\rho_n, n \in \mathbb{Z}/p\mathbb{Z}\}$ where ρ_n maps the generator of $\mathbb{Z}/p\mathbb{Z}$ to a matrix with eigenvalues $\mathrm{e}^{\pm\frac{2\,\mathrm{i}\,\pi n}{p}}$. We know from [3] that $\mathrm{CS}(\rho_n) = 2\pi\frac{q^* n^2}{p}$ where $qq^* = 1 \bmod p$. Hence, the Chern–Simons invariants of $L(p,q)$ behave exactly like quadratic residues when p goes to infinity.

1.2.2. Brieskorn spheres

To give a more complicated but still manageable example, consider the Brieskorn sphere $M = \Sigma(p_1,p_2,p_3)$ where p_1,p_2,p_3 are distinct primes. This is a homology sphere whose irreducible representations in SU_2 have the form ρ_{n_1,n_2,n_3} where $0 < n_1 < p_1, 0 < n_2 < p_2, 0 < n_3 < p_3$. From [3] we have

$$\mathrm{CS}(\rho_{n_1,n_2,n_3}) = 2\pi\frac{(n_1p_2p_3 + p_1n_2p_3 + p_1p_2n_3)^2}{4p_1p_2p_3}.$$

Setting $n = n_1p_2p_3 + p_1n_2p_3 + p_1p_2n_3$, we observe that, due to Chinese remainder theorem, n describes $(\mathbb{Z}/p_1p_2p_3\mathbb{Z})^\times$ when n_i describes $(\mathbb{Z}/p_i\mathbb{Z})^\times$ for $i = 1,2,3$. Hence, we compute that the following ℓ-th momentum:

$$\mu^\ell_{p_1p_2p_3} = \frac{1}{|X(M)|}\sum_{\rho\in X(M)} \exp(\mathrm{i}\,\ell\,\mathrm{CS}(\rho)) \sim \frac{1}{p_1p_2p_3}\sum_{n=0}^{p_1p_2p_3-1} \mathrm{e}^{\frac{\mathrm{i}\,\pi\ell n^2}{2p_1p_2p_3}}.$$

Assuming ℓ is coprime with $p = p_1p_2p_3$ we get from [1] the following estimates where $\epsilon_n = 1$ is $n = 1 \bmod 4$ and $\epsilon_n = i$ if $n = 3 \bmod 4$:

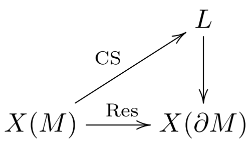
$$\mu^\ell_p \sim \begin{cases} \frac{\epsilon_p}{\sqrt{p}}\left(\frac{\ell/4}{p}\right) & \text{if } \ell = 0 \bmod 4 \\ 0 & \text{if } \ell = 2 \bmod 4 \\ \frac{1+i}{2\sqrt{p}\epsilon_l}\left(\frac{p}{\ell}\right) & \text{otherwise.} \end{cases}$$

Again we obtain that μ_p converges to the uniform measure when p goes to infinity. The renormalised measure $\sqrt{p}(\mu_p - \mu_\infty)$ have ℓ-th momentum with modulus equal to $1, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$ depending on ℓ mod 4.

1.3. Dehn Fillings

The main question we address in this article is the following: fix a manifold M with boundary $\partial M = \mathbb{T} \times \mathbb{T}$. For any $\frac{p}{q} \in \mathbb{P}^1(\mathbb{Q})$, we denote by $\mathbb{T}_{p/q}$ the curve on $\mathbb{T}^2$ parametrised by (pt, qt) for t in $\mathbb{T}$. We define the manifold $M_{p/q}$ by Dehn filling i.e. the result of gluing M with a solid torus such that $\mathbb{T}_{p/q}$ bounds a disc.

We recall from [3] that in the case where M has boundary, there is a principal $\mathbb{T}$-bundle with connection $L \to X(\partial M)$ such that the Chern–Simons invariant is a flat section of $\operatorname{Res}^* L$



where $\operatorname{Res}(\rho) = \rho \circ i_*$ and $i : \partial M \to M$ is the inclusion. This shows that if Res is an immersion, which we will soon assume, $X(M)$ is 1-dimensional and Res is Legendrian.

We will denote by $|\mathrm{d}\theta|$ the natural density on $X(\mathbb{T}) = \mathbb{T}/(\theta \sim -\theta)$.

We also have $X(\mathbb{T}^2) = \mathbb{T}^2/(x, y) \sim (-x, -y)$ and for any p, q the map $\operatorname{Res}_{p/q} : X(\mathbb{T}^2) \to X(\mathbb{T}_{p/q})$ is given by $(x, y) \mapsto px + qy$.

Moreover, for any $\frac{p}{q}$, $\ell > 0$ and $0 \leqslant k \leqslant \ell$, there are natural flat sections $\mathrm{CS}^{k/\ell}_{p/q}$ of L^ℓ over the preimage $\operatorname{Res}^{-1}_{p/q}(\frac{\pi k}{\ell})$. These sections are constructed using coordinates in Definition 2.4. They are called Bohr–Sommerfeld sections and they coincide for $k = 0$ with CS^ℓ. See [3] or [2] for a detailed description.

Theorem 1.3. — *Let M be a 3-manifold with $\partial M = \mathbb{T}^2$ satisfying the hypothesis of Section 2.4. Let p, q, r, s be integers satisfying $ps - qr = 1$ and for any integer n, set $p_n = pn - r$ and $q_n = qn - s$. Then setting*

$$\mu_n^\ell = \frac{1}{n} \sum_{\rho \in X(M_{p_n/q_n})} \mathrm{e}^{\mathrm{i}\,\ell\,\mathrm{CS}(\rho)}$$

we get first

$$\mu_n^0 = \int_{X(M)} \operatorname{Res}^*_{r/s} |\mathrm{d}\theta| + O\left(\frac{1}{n}\right)$$

and for $\ell > 0$

$$\mu_n^\ell = \frac{1}{\sqrt{2n}} \sum_{k=0}^{l} \sum_{\rho, k / \operatorname{Res}_{r/s}(\rho) = \pi \frac{k}{l}} \exp\left(-2\,\mathrm{i}\,\pi n \frac{k^2}{4\ell} + \mathrm{i}\,\ell\,\mathrm{CS}(\rho) - \mathrm{i}\,\mathrm{CS}_{r/s}^{k/l}(\rho) \right) + O\left(\frac{1}{n}\right).$$

Hence, we recover the behaviour that we observed for Lens spaces and Brieskorn spheres. The measure converges to a uniform measure μ_∞ and the renormalised measure $\sqrt{n}(\mu_n - \mu_\infty)$ has an oscillating behaviour controlled by representations in $X(M)$ with rational angle along $\mathbb{T}_{r/s}$.

2. Intersection of Legendrian subvarieties

We will prove Theorem 1.3 in the more general situation of curves immersed in a torus. Indeed, the problem makes sense in an even more general setting that we present here.

2.1. Prequantum bundles

DEFINITION 2.1. — *Let (M, ω) be a symplectic manifold. A prequantum bundle is a principal $\mathbb{T}$-bundle with connection whose curvature is ω.*

It is well-known that the set of isomorphism classes of prequantum bundles is homogeneous under $H^1(M, \mathbb{T})$ and non-empty if and only if ω vanishes in $H^2(M, \mathbb{T})$. Let us give three examples:

(i) Take $\mathbb{R}^2 \times \mathbb{T}$ with coordinates (x, y, θ) and set $\lambda = \mathrm{d}\theta + \frac{1}{2\pi}(x\mathrm{d}y - y\mathrm{d}x)$. This gives a prequantum bundle on $\mathbb{R}^2$. Dividing by the action of $\mathbb{Z}^2$ given by

$$(m, n) \cdot (x, y, \theta) = (x + 2\pi m, y + 2\pi n, \theta + my - nx) \tag{2.1}$$

gives a prequantum bundle $\pi : L \to \mathbb{T}^2$.

(ii) Any complex projective manifold $M \subset \mathbb{P}^n(\mathbb{C})$ has such a structure by restricting the tautological bundle whose curvature is the restriction of the Fubini–Study 2-form.

(iii) The Chern–Simons bundle over the character variety of a surface.

In all these cases, there is a natural subgroup of the group of symplectomorphisms of (M, ω) which acts on the prequantum bundle. The group $\mathrm{SL}_2(\mathbb{Z})$ acts in the first case and the mapping class group in the third case. In the second case, a group acting linearly on $\mathbb{C}^{n+1}$ and preserving M will give an example.

2.2. Legendrian submanifolds and their pairing

Consider a prequantum bundle $\pi : L \to M$ where M has dimension $2n$ and denote by $\lambda \in \Omega^1(L)$ the connection 1-form. By Legendrian immersion we will mean an immersion $i : N \to L$ where N is a manifold of dimension n such that $i^*\lambda = 0$. This condition implies that i is transverse to the fibres of π and hence $\pi \circ i : N \to M$ is a Lagrangian immersion.

Definition 2.2.

(1) *Given $i_1 : N_1 \to L$ and $i_2 : N_2 \to L$ two Legrendrian immersions, we will say that they are transverse if it is the case of $\pi \circ i_1$ and $\pi \circ i_2$.*

(2) *Given such transverse Legendrian immersions and an intersection point, i.e. $x_1 \in N_1$ and $x_2 \in N_2$ such that $\pi(i_1(x_1)) = \pi(i_2(x_2))$ we define their phase $\phi(i_1(x_1), i_2(x_2))$ as the element $\theta \in \mathbb{T}$ such that $i_2(x_2) = i_1(x_1) + \theta$.*

(3) *The phase measure $\phi(i_1, i_2)$ is the measure on the circle defined by*

$$\phi(i_1, i_2) = \sum_{\pi(i_1(x_1))=\pi(i_2(x_2))} \delta_{\phi(i_1(x_1), i_2(x_2))}\,.$$

If M is a 3-manifold obtained as $M = M_1 \cup M_2$ then, assuming transversality, the Chern–Simons measure of M is given by $\mu_M = \phi(\mathrm{CS}_1, \mathrm{CS}_2)$ where $\mathrm{CS}_i : X(M_i) \to L$ is the Chern–Simons invariant with values in the Chern–Simons bundle.

2.3. Immersed curves in the torus

Consider the prequantum bundle $\pi : L \to \mathbb{T}^2$ given in the first item of Example 2.1. We consider a fixed Legendrian immersion $i : [a, b] \to L$ and for any coprime integers p, q the Legendrian immersion

$$i_{p/q} : \mathbb{T} \to L, i_{p/q}(t) = (pt, qt, 0)\,.$$

Our aim here is to study the behaviour of $\phi(i, i_{p/q})$ when $(p, q) \to \infty$.

We first lift i to an immersion $I : [a, b] \to \mathbb{R}^2 \times \mathbb{R}$ of the form $I(t) = (x(t), y(t), \theta(t))$. By assumption we have $\dot\theta = -\frac{1}{2\pi}(x\dot y - y\dot x)$. For instance, lifting $i_{p/q}$ we get simply the map $I_{p/q} : t \mapsto (pt, qt, 0)$.

Let r, s be integers such that $A = \left(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right)$ has determinant 1. Take $F_A : \mathbb{R}^2 \to \mathbb{R}$ the function

$$F_A(x, y) = \frac{1}{2\pi}(sx - ry)(qx - py)\,.$$

A direct computation shows that this function satisfies $(m,n).I_{p/q}(t) = (pt+2\pi m, qt+2\pi n, F(pt+2\pi m, qt+2\pi n))$. We obtain from it the following formula:

$$\phi(i, i_{p/q}) = \sum_{a\leqslant t\leqslant b, qx(t)-py(t)\in 2\pi\mathbb{Z}} \delta_{\theta(t)-F(x(t),y(t))}\,. \tag{2.2}$$

If we put $i = i_{0/1}$ this formula becomes $\phi(i_{0/1}, i_{p/q}) = \sum_{k=0}^{p-1} \delta_{2\pi\frac{rk^2}{p}}$. This measure is related to the usual Gauss sum in the sense that denoting by q^* an inverse of q mod p we have:

$$\int e^{i\theta}\, d\phi(i_{0/1}, i_{p/q})(\theta) = \sum_{k\in\mathbb{Z}/q\mathbb{Z}} \exp\left(2\,i\,\pi\frac{q^*k^2}{p}\right).$$

This formula is the same as the one obtained for the Lens spaces in Subsection 1.2.1. The two Legendrian immersions $i_{0/1}$ and $i_{p/q}$ correspond to the two solid tori which glued together provide $L(p,q)$.

Suppose that $p_n = pn - r$ and $q_n = qn - s$. A Bézout matrix is given by $A_n = \begin{pmatrix} pn-r & p \\ qn-s & q\end{pmatrix}$. Up to the action of $SL_2(\mathbb{Z})$, we can suppose that $p = s = 1$ and $q = r = 0$ in which case $F_{A_n}(x,y) = -\frac{y}{2\pi}(x+ny)$. We get from Equation (2.2) the following formula for $\mu_n^\ell = \frac{1}{n}\int e^{i\,\ell\theta}\, d\phi(i, i_{pn/-1})(\theta)$:

$$\mu_n^\ell = \frac{1}{n} \sum_{\substack{x(t)+ny(t)\in 2\pi\mathbb{Z} \\ a\leqslant t\leqslant b}} \exp\left(i\,\ell\left(\theta(t) + \frac{y(t)}{2\pi}(x(t)+ny(t))\right)\right). \tag{2.3}$$

Taking $\ell = 0$, we are simply counting the number of solutions of $x(t) + ny(t) \in 2\pi\mathbb{Z}$ for $t\in[a,b]$. Assuming that y is monotonic, the number of solutions for $t \in [a,b]$ is asymptotic to $|y(b) - y(a)|$. Hence the asymptotic density of intersection points is $i^*|dy|$ and we get

$$\lim_{n\to\infty} \mu_n^0 = \int_a^b i^*|dy|\,.$$

To treat the case $\ell > 0$, we need the following version of the Poisson formula:

Lemma 2.3. — *If $f, g : [a,b] \to \mathbb{R}$ are respectively C^1 and continuous and f is piecewise monotonic, then if further $f(a), f(b) \notin 2\pi\mathbb{Z}$ we have*

$$\sum_{a\leqslant t\leqslant b, f(t)\in 2\pi\mathbb{Z}} g(t) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \int_a^b e^{-i\,kf(t)}\, |f'(t)| g(t)\, dt\,.$$

Applying it here, we get

$$\mu_n^\ell = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_a^b e^{-i\,k(x+ny)+i\,\ell(\theta+\frac{y}{2\pi}(x+ny))}\left|\frac{\dot{x}}{n}+\dot{y}\right| dt\,.$$

We apply a stationary phase expansion in this integral, the phase being $\Phi = -ky + ly^2/2\pi$ and its derivative being $\dot{\Phi} = (-k + ly/\pi)\dot{y}$. We find two types of critical points: the horizontal tangents $\dot{y} = 0$ and the points of rational height $y = \pi\frac{k}{l}$. We observe that when $\dot{y} = 0$ the amplitude is $O(\frac{1}{n})$ and hence these contributions can be neglected compared with the other ones, where $y = \pi\frac{k}{l}$.

We compute $\ddot{\Phi} = \frac{l}{\pi}\dot{y}^2 + (-k + ly/\pi)\ddot{y} = \frac{l}{\pi}\dot{y}^2$ and $\Phi = -\frac{\pi k^2}{2l}$. As $\ddot{\Phi} > 0$, the stationary phase approximation gives

$$\mu_n^l = \frac{1}{\sqrt{2n}} \sum_{y=\frac{\pi k}{l}} \mathrm{e}^{-\mathrm{i}\, n\frac{k^2\pi}{2l} - \mathrm{i}\,\frac{kx}{2} + \mathrm{i}\, l\theta} + O\left(\frac{1}{n}\right).$$

In order to give the final result, we give the following definition:

Definition 2.4. — *The map $t \mapsto (t, \pi\frac{k}{l}, \frac{kt}{2})$ defines a flat section of L^ℓ that we denote by $i_{1/0}^{k/\ell}$.*

We can sum up the discussion by stating the following proposition.

Proposition 2.5. — *Let $i : \mathbb{T} \to L$ be a Legendrian immersion and suppose that $\pi \circ i$ is transverse to $i_{pn/-1}$ for n large enough and to the circles of equation $y = \pi\xi$ for $\xi \in \mathbb{Q}$.*

Then writing $i(t) = (x(t), y(t), \theta(t))$ and $\mu_n^\ell = \frac{1}{n}\int \mathrm{e}^{\mathrm{i}\,\ell\theta}\, \mathrm{d}\phi(i, i_{pn/-1})(\theta)$ we have for all $\ell > 0$:

$$\mu_n^\ell = \frac{1}{\sqrt{2n}} \sum_{k\in\mathbb{Z}/2\ell\mathbb{Z}} \sum_{t\in\mathbb{T}, y(t)=\pi k/\ell} \mathrm{e}^{-\mathrm{i}\, n\pi\frac{k^2}{2\ell} + \mathrm{i}\,\phi\left(i(t), i_{1/0}^{k/l}(x(t))\right)} + O\left(\frac{1}{n}\right).$$

2.4. Application to Chern–Simons invariants

Let M be a 3-manifold with $\partial M = \mathbb{T} \times \mathbb{T}$. We assume that $X(M)$ is at most 1-dimensional and that the restriction map $\mathrm{Res} : X(M) \to X(\partial M)$ is an immersion on the smooth part and map the singular points to non-torsion points. Then we know that $\mathrm{Res}(X(M))$ is transverse to $\mathbb{T}_{p/q}$ for all but a finite number of p/q, see [4].

Consider the projection map $\pi : \mathbb{T}^2 \to X(\partial M)$ which is a 2-fold ramified covering. We may decompose $X(M)$ as a union of segments $[a_i, b_i]$ whose extremities contain all singular points. The restriction map Res can be lifted to $\mathbb{T}^2$ and the Chern–Simons invariant may be viewed as a map $\mathrm{CS} : [a_i, b_i] \to L$. Hence, we may apply to it the results of Proposition 2.5 and obtain Theorem 1.3.

We may comment that the flat sections $i_{1/0}^{k/\ell}$ of L^ℓ over the line $y = \frac{\pi k}{\ell}$ induces through the quotient $(x, y, \theta) \sim (-x, -y, -\theta)$ a flat section of L^ℓ that we denoted $\mathrm{CS}_{0/1}^{k/l}$ over the subvariety $\mathrm{Res}_{0/1}^{-1}(\frac{\pi k}{\ell})$.

3. Chern–Simons invariants of coverings

3.1. General setting

Beyond Dehn fillings, we can ask for the limit of the Chern–Simons measure of any sequence of 3-manifolds. A natural class to look at is the case of coverings of a same manifold M. Among that category, one can restrict to the family of cyclic coverings. One can even specify the problem to the following case.

QUESTION 3.1. — *Let $p : M \to \mathbb{T}$ be a fibration over the circle and M_n be the pull-back of the self-covering of $\mathbb{T}$ given by $z \mapsto z^n$. What is the asymptotic behaviour of μ_{M_n}?*

This problem can be formulated in the following way. Let Σ be the fiber of M and $f \in \mathrm{Mod}(\Sigma)$ be its monodromy. Any representation $\rho \in X(M)$ restricts to a representation $\mathrm{Res}(\rho) \in X(\Sigma)$ invariant by the action f_* of f on $X(\Sigma)$. Reciprocally, any irreducible representation $\rho \in X(\Sigma)$ fixed by f_* correspond to two irreducible representations in $X(M)$.

The Chern–Simons invariant corresponding to a fixed point may be computed in the following way: pick a path $\gamma : [0, 1] \to X(\Sigma)$ joining the trivial representation to ρ and consider the closed path obtained by composing γ with $f(\gamma)$ in the opposite direction. Then its holonomy along L is the Chern–Simons invariant of the corresponding representation.

Understanding the asymptotic behaviour of μ_{M_n} consists in understanding the fixed points of f_*^n on $X(\Sigma)$ and the distribution of Chern–Simons invariants of these fixed points, a problem which seems to be out of reach for the moment.

3.2. Torus bundles over the circle

In this elementary case, the computation can be done. Let $A \in \mathrm{SL}_2(\mathbb{Z})$ act on $\mathbb{R}^2/\mathbb{Z}^2$. Its fixed points form a group $G_A = \{v \in \mathbb{Q}^2, Av = v \bmod \mathbb{Z}^2\}/\mathbb{Z}^2$. If $\mathrm{Tr}(A) \neq 2$, which we suppose from now, G_A is isomorphic to $\mathrm{Coker}(A - \mathrm{Id})$ and has cardinality $|\det(A - \mathrm{Id})|$.

Following the construction explained above, the phase is a map $f: G_A \to \mathbb{Q}/\mathbb{Z}$ given by $f([v]) = \det(v, Av) \bmod \mathbb{Z}$. Hence, the measure we are trying to understand is the following:

$$\mu_A = \frac{1}{|\det(A - \mathrm{Id})|} \sum_{v \in G_A} \delta_{2\pi \det(v, Av)}.$$

Consider the ℓ-th moment μ_A^ℓ of μ_A. It is a kind of Gauss sum that can be computed explicitly. The map f is a quadratic form on G_A with values in $\mathbb{Q}/\mathbb{Z}$. Its associated bilinear form is $b(v,w) = \det(v, Aw) + \det(w, Av) = \det(v, (A - A^{-1})w)$. As $A + A^{-1} = \mathrm{Tr}(A)\,\mathrm{Id}$ and $\det(A - \mathrm{Id}) = 2 - \mathrm{Tr}(A)$ we get $b(v,w) = 2\det(v, (A - \mathrm{Id})w) \mod \mathbb{Z}$. Hence, if 2ℓ is invertible in G_A, then ℓb is non-degenerate and standard arguments (see [5] for instance) show that $|\mu_A^\ell| = |\det(A - \mathrm{Id})|^{-1/2}$. Hence we still get the same kind of asymptotic behaviour for the Chern–Simons measure of the torus bundles over the circle.

BIBLIOGRAPHY

[1] B. C. Berndt, R. J. Evans & K. S. Williams, *Gauss and Jacobi sums*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, 1998, xii+583 pages.

[2] L. Charles & J. Marché, "Knot state asymptotics II: Witten conjecture and irreducible representations", *Publ. Math., Inst. Hautes Étud. Sci.* **121** (2015), p. 323-361.

[3] P. A. Kirk & E. P. Klassen, "Chern-Simons invariants of 3-manifolds and representation spaces of knot groups", *Math. Ann.* **287** (1990), no. 2, p. 343-367.

[4] J. Marché & G. Maurin, "Singular intersections of subgroups and character varieties", https://arxiv.org/abs/1406.2862, 2014.

[5] V. Turaev, "Reciprocity for Gauss sums on finite abelian groups", *Math. Proc. Camb. Philos. Soc.* **124** (1998), no. 2, p. 205-214.

Manuscrit reçu le 31 octobre 2017,
révisé le 1er février 2018,
accepté le 13 mars 2018.

Julien MARCHÉ
Sorbonne Université
Institut de mathématiques de Jussieu - Paris Rive Gauche
4 place Jussieu
75005 Paris (France)
julien.marche@imj-prg.fr